Axioms for the Real Number System

Math 361

Fall 2003

The Real Number System

The real number system consists of four parts:

1. A set ($\mathbb{R}$). We will call the elements of this set real numbers, or reals.
2. A relation $<$ on $\mathbb{R}$. This is the order relation.
3. A function $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. This is the addition operation.
4. A function $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. This is the multipliation operation.

We will state 12 axioms that describe how the real number system behaves. The first eleven will say that the real number system forms an *ordered field*. The final axiom will require a little discussion.

Operation Axioms

For all x, y, and z

1. Associative laws:
   $\forall x \forall y \forall z \ [(x + y) + z = x + (y + z) \text{ and } (x \cdot y) \cdot z = x \cdot (y \cdot z)]$
2. Commutative laws:
   $\forall x \forall y \ [x + y = y + x \text{ and } x \cdot y = y \cdot x]$
3. Distributive law:
   $\forall x \forall y \forall z \ [x \cdot (y + z) = x \cdot y + x \cdot z]$

Identity and Inverse Axioms

4. Additive identity:
   There is an element (called 0) such that $\forall x \ [0 + x = x]$. [Uniqueness can be proved.]

5. Additive inverse:

   $\forall x \exists y \; [x + y = 0]$. [We write $y = -x$; uniqueness can be proved.]

6. Multiplicative identity:

   There is an element (called 1) such that $0 \neq 1$ and $\forall x \; 1 \cdot x = x$. [Uniqueness can be proved.]

7. Multiplicative inverse:

   $\forall x \;\; [x \neq 0 \Longrightarrow \exists y \, (x \cdot y = 1)]$. [We write $y = \frac{1}{x}$. Uniqueness can be proved.]

Order Axioms

8. Translation invariance of order:

   $\forall x \forall y \; [x < y \Longrightarrow x + z < y + z]$.

9. Transitivity of order:

   $\forall x \forall y \; [(x < y \text{ and } y < z) \Longrightarrow x < z]$.

10. Trichotomy:

    $\forall x \forall y$ exactly one of the following is true: $x < y$, $y < x$, or $x = y$.

11. Scaling and order:

    $\forall x \forall y \forall z \; [(x < y \text{ and } z > 0) \Longrightarrow xy < yz]$

Any number system that satisfies Axioms 1–11 is called an *ordered field.* Examples: $\mathbb{Q}$ and $\mathbb{R}$ are both ordered fields.

The Completeness Axiom

12. Every non-empty subset that is bounded above has a *least* upper bound.

Basic Results about $\mathbb{R}$

Theorem 0.19: Let x, y, and z be real numbers. Then

(a) If $x < y$, then $-y < -x$.
(b) $(-1) \cdot (-1) = 1$
(c) If $0 < 1$.
(d) If $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{z}$.
(e) If $x < y$ and $z < 0$, then $yz < xz$.
(f) If $x^2 \geq 0$.

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Theorem 0.20: If S is a nonempty set of reals that is bounded from below, then S has a greatest lower bound.

Betweenness Results

Theorem 0.21: Let x be any real number. Then there is an integer n such that $n \leq x < n + 1$.

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Theorem 0.23: If p is any positive real number, there is a positive real number x such that $x^2 = p$.

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Theorem 0.23: If p is any positive real number, there is a positive real number x such that $x^2 = p$.

Theorem 0.24: [The irrationals are dense in the reals.]

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Absolute Value

Theorem 0.25: Let a and b be any real numbers. Then

(a) $|ab| = |a| \cdot |b|$

(b) If $\varepsilon > 0$, then $|a| \leq \varepsilon$ iff $-\varepsilon \leq a \leq \varepsilon$.

(c) (Triangle Inequality) $|a+b| \leq |a| + |b|$.

(d) $||a| - |b|| \leq |a - b|$.