Transitive Closure

Math 156

Closure of a Relation

Let $R$ be a relation and $P$ a property that relations might have (e.g., reflexive, symmetric, transitive, etc.). The $P$-closure of $R$ is the smallest relation that contains $R$ and has property $P$.

Reflexive and Symmetric Closures

The reflexive and symmetric closures are fairly simply because we can simply add to our relation the things that are forced by the reflexive or symmetric properties. In terms of matrices, if $M$ is the matrix for relation $R$, then

- The matrix for the reflexive closure is:

- The matrix for the symmetric closure is:

Transitive Closure

Small Example

Transitive closure is a bit trickier. Consider the following small example of a relation on $A = \{a, b, c, d\}$:

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $aRb$ and $bRc$, we must add in $a Rc$. And since $b Rc$ and $c Rd$, we must add in $b Rd$.

$$M_2 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

But the resulting relation is not transitive. Now we can get from $a$ to $c$ and from $c$ to $d$, so we need to be able to go directly from $a$ to $d$.

$$M_3 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we have a transitive relation, but for a larger relation, it could take more steps.
Paths

Let $R^k$ be the relation such that

- $(a, b) \in R^k \iff$ there is a path of length exactly $k$ from $a$ to $b$.

Notice that

- $R^1 =$

And

- $R$ is transitive $\iff$

We can check this using matrices since if $M_R$ is the matrix representing $R$, then the matrix representing $R^k$ is

- $M_{R^k} =$

So the matrix check that $R$ is transitive is

- $R$ is transitive $\iff$

Connectivity

Now let $R^*$ be the relation such that

- $(a, b) \in R^* \iff$ there is a path of any length from $a$ to $b$.

This is sometimes called the connectivity relation associated with $R$.

Although paths can be of any length, the length of the shortest path between $a$ and $b$ is limited. If $|A| = n$ and $R$ is a relation on $A$, then by the ______________________________

- $R^* = R^1 \cup R^2 \cup R^3 \cup \cdots \cup R^n$ because

So

- $M_{R^*} =$

Connectivity and Transitivity

We can use $R^*$ to check whether $R$ is transitive:

- $R$ is transitive $\iff$

Furthermore, $R^*$ is always transitive:

- If $(a, b) \in R^*$ and $(b, c) \in R^*$, then
Is $R^*$ the Transitive Closure of $R$?

$R^*$ is transitive and $R \subseteq R^*$. So $R^*$ is the transitive closure of $R$ unless

Suppose $S$ is transitive and $R \subseteq S$. If we can show

then we will know that $R^*$ is the transitive closure of $R$.

We can show this as follows:

- $R \subseteq S \Rightarrow$
- $S$ transitive $\Rightarrow$
- So

Matrix Algorithm 1 for Transitive Closure

We can compute $R^*$ using matrices since

- $M_{R^*}$ =

If $|A| = n$, how efficient is this? (How many bit operations?)

- bitops to get each $M_{R^k} = M_R^{[k]}$:
- total bitops to get $M_{R^*}$:

Warshall’s algorithm does better

Warshall’s algorithm builds up to the transitive closure a different way.

- $W_0 = M_R$

- $W_k =$

- So $W_n = M_{R^*}$

Efficiency:

- How many bit operations does it take to compute $W_{k+1}$ from $W_k$?
- Overall efficiency:
Examples

Find the transitive closure of the relations represented by these matrices using (a) Algorithm 1 and (b) Warshall's Algorithm.