Math 355 Homework Problems #9

1. Let $A \in \mathcal{M}_n(\mathbb{F})$ be semi-simple. For each eigenvalue $\lambda_j$ let $v_j$ be an associated eigenvector, and let $w_j$ be an associated adjoint eigenvector. Assume the adjoint eigenvectors are scaled so that

$$\langle w_j, v_k \rangle = \begin{cases} 0, & j \neq k \\ 1, & j = k. \end{cases}$$

For each $j$ set $P_j = v_j w_j^\dagger$ to be the rank one spectral projection matrix. For a given $1 \leq \ell \leq n$ set

$$Q = P_1 + P_2 + \cdots + P_{\ell}.$$

(a) What is a basis for $\text{Ran}(Q)$?

(b) What is a basis for $\ker(Q)$?

(c) Show that $Q^2 = Q$.

(d) Show that $QA = AQ$.

(e) Show that $QA^D = A^D Q$.

2. In Problem 1 suppose that $\ell = \text{rank}(A)$. Show that $AA^D = A^D A = Q$.

3. Let $A = \begin{pmatrix} 0 & 2 \\ 6 & 1 \end{pmatrix}$.

(a) Find the spectral decomposition of $A$.

(b) Find the spectral decomposition of $A^{-1}$.

(c) Find the spectral decomposition of $e^{A^t}$.

4. Let $A \in \mathcal{M}_n(\mathbb{F})$ be simple. Prove the following properties of the Drazin inverse, $A^D$:

(a) $(A^D)^D = A^2 A^D = A$.

(b) $(A^D)^n = (A^n)^D$ for any positive integer $n$.

(c) $A^D = A$ if and only if $A^3 = A$.

5. Let $A, B \in \mathcal{M}_n(\mathbb{F})$ be simple and similar, i.e., there is an invertible matrix $P$ such that $A = PBP^{-1}$. Show that $A^D$ is similar to $B^D$.

6. Compute the Drazin inverse of $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$.