Math 355 Homework Problems #1

1. Do the following form subspaces of \( \mathbb{F}_4[x] \)? Prove, or give a counter-example.
   
   (a) The subset of all polynomials in \( \mathbb{F}_4[x] \) of even degree.
   
   (b) The subset of all polynomials \( p(x) \) in \( \mathbb{F}_4[x] \) with \( p(0) = 0 \).
   
   (c) The subset of all polynomials \( p(x) \) in \( \mathbb{F}_4[x] \) with \( p'(0) \neq 0 \).

2. Which of each set below is a spanning set for \( \mathbb{F}_2[x] \)? Justify your answer in each case.
   
   (a) \( \{1, x - 1, x^2 + 1\} \).
   
   (b) \( \{x + 2, x - 2, x^2 - 2\} \).
   
   (c) \( \{1 + 2x + 3x^2, 4 + 5x + 6x^2, 7 + 8x + 9x^2\} \).

3. If \( A \in \mathcal{M}_n(\mathbb{F}) \), the transpose of \( A \), denoted \( A^T \), is the matrix where the \( j \)th column of \( A \) is the \( j \)th row of \( A^T \). Let \( \text{Sym}_n(\mathbb{F}) \subset \mathcal{M}_n(\mathbb{F}) \) denote the set of symmetric matrices. In other words, \( A \in \text{Sym}_n(\mathbb{F}) \) if and only if \( A = A^T \). Let \( \text{Skew}_n(\mathbb{F}) \subset \mathcal{M}_n(\mathbb{F}) \) denote the set of skew-symmetric matrices. In other words, \( A \in \text{Skew}_n(\mathbb{F}) \) if and only if \( A = -A^T \).
   
   (a) Show that \( \text{Sym}_n(\mathbb{F}) \) is a subspace.
   
   (b) Show that \( \text{Skew}_n(\mathbb{F}) \) is a subspace.
   
   (c) Show that \( \mathcal{M}_n(\mathbb{F}) = \text{Sym}_n(\mathbb{F}) \oplus \text{Skew}_n(\mathbb{F}) \).

4. Write \( \mathbb{F}_5[x] \) as the direct sum of 6 one-dimensional subspaces.

5. Write \( \mathcal{M}_2(\mathbb{F}) \) as the direct sum of 4 one-dimensional subspaces.

6. Let \( W_1, W_2, \ldots, W_n \) be a collection of subspaces of the vector space \( V \). Show that

\[
\bigcap_{j=1}^{n} W_j = W_1 \cap W_2 \cap \cdots \cap W_n
\]

is a subspace.