Math 333 Homework Problems #5
Applied Partial Differential Equations (2nd Edition), by J.D. Logan

1. Consider the heat equation
   \[ \partial_t u = 4 \partial_x^2 u, \quad u(x, 0) = u_0(x), \]
   with the Dirichlet boundary conditions
   \[ u(0, t) = u(1, t) = 0. \]
   Assume that \( u_0(x) \in C^2([0, 1]). \)
   (a) Find functions \( u_j(t) \) and \( v_j(x) \) so that the solution can be written
   \[ u(x, t) = \sum_{j=1}^{\infty} u_j(t)v_j(x). \]
   (b) Let the partial sum be denoted
   \[ u_M(x, t) := \sum_{j=1}^{M} u_j(t)v_j(x). \]
   Show that there is a number \( \lambda_M < 0 \) such that
   \[ \|u(x, t) - u_M(x, t)\| \leq e^{\lambda_M t}\|u_0\|. \]
   (c) Determine \( M \) so that for \( t > 0.02 \),
   \[ \|u(x, t) - u_M(x, t)\| \leq 10^{-4}\|u_0\|. \]

2. Solve the problem
   \[ \partial_t u = k \partial_x^2 u, \quad u(x, 0) = x(1-x), \]
   with the boundary conditions
   \[ u(0, t) = \sin(3t), \quad u(1, t) = 0. \]
   What is the solution as \( t \to +\infty? \)

3. Solve the problem
   \[ \partial_t u = k \partial_x^2 u, \quad u(x, 0) = 1 - x^2, \]
   with the boundary conditions
   \[ \partial_x u(0, t) = u(1, t) = 0. \]
4. Consider the following table of material properties.

<table>
<thead>
<tr>
<th></th>
<th>( c ) (kcal/g)</th>
<th>( \rho ) (g/cm(^3))</th>
<th>( K ) (kcal/cm/sec)</th>
<th>( k = K/(c\rho) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum (A)</td>
<td>2.15 \times 10^{-4}</td>
<td>2.70</td>
<td>4.90 \times 10^{-4}</td>
<td>0.84</td>
</tr>
<tr>
<td>Copper (C)</td>
<td>0.92 \times 10^{-4}</td>
<td>8.96</td>
<td>9.58 \times 10^{-4}</td>
<td>1.16</td>
</tr>
<tr>
<td>Gold (G)</td>
<td>1.08 \times 10^{-4}</td>
<td>19.32</td>
<td>1.91 \times 10^{-4}</td>
<td>0.09</td>
</tr>
<tr>
<td>Iron (I)</td>
<td>0.31 \times 10^{-4}</td>
<td>7.87</td>
<td>7.40 \times 10^{-4}</td>
<td>3.03</td>
</tr>
</tbody>
</table>

In class we used the Evans function to derive the following table:

<table>
<thead>
<tr>
<th></th>
<th>( \lambda_1 )</th>
<th>( M )</th>
<th>( \lambda_{M+1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A/C</td>
<td>-9.37</td>
<td>5</td>
<td>-357.6</td>
</tr>
<tr>
<td>A/G</td>
<td>-2.15</td>
<td>12</td>
<td>-346.5</td>
</tr>
<tr>
<td>A/I</td>
<td>-15.11</td>
<td>4</td>
<td>-349.2</td>
</tr>
</tbody>
</table>

Use the MATLAB Evans function evaluator to generate a table similar to the second for the material combinations of C/G, C/I, and G/I.

(a) For which material combination does the temperature most quickly settle to the total thermal energy of the initial data?

(b) For which material combination is the fewest number of terms needed in order to approximate the solution to the desired accuracy of \(10^{-4}\|u_0\|_w\) for \( t \geq 0.03\)?
5. Consider the heat equation

\[ c(x) \rho(x) \partial_t u = \partial_x [K(x) \partial_x u], \quad u(x,0) = u_0(x), \]

with the Dirichlet boundary conditions

\[ u(0,t) = u(1,t) = 0. \]

(a) What is the associated SL problem?

(b) Write the solution as a Fourier series,

\[ u(x,t) = \sum_{j=1}^{\infty} u_j(t)v_j(x). \]

Clearly identify the properties of the functions \( v_j(x) \).

(c) What is \( u_j(t) \) for each \( j \)?

(d) Show that

\[ \|u(x,t)\|_w \leq e^{\lambda_1 t} \|u_0\|_w. \]

What is the weight \( w(x) \). What is the number \( \lambda_1 \)?

(e) Denote the partial sum

\[ u_M(x,t) := \sum_{j=1}^{M} u_j(t)v_j(x). \]

Show that

\[ \|u(x,t) - u_M(x,t)\|_w \leq e^{\lambda_{M+1} t} \|u_0\|_w. \]

What is the number \( \lambda_{M+1} \)?

(f) Assuming that we want

\[ \|u(x,t) - u_M(x,t)\|_w \leq 10^{-5} \|u_0\|_w, \quad t \geq 0.02, \]

generate a table similar to that presented in Problem 4 for the material combinations A/C, A/G, A/I, C/G, C/I, and G/I.