Math 333 Homework Problems #1
APPLIED PARTIAL DIFFERENTIAL EQUATIONS (2ND EDITION), by J.D. Logan

1. This problem is a variant of Problem 1.3.6 in the text. Consider the heat equation

\[ \partial_t u = \partial_x^2 u - r^2 u \]

\[ u(0, t) = 1, \quad u(1, t) = 1. \]

Here \( r > 0 \) is a constant which describes the rate at which the bar loses its heat across the lateral boundary. The steady-state temperature, \( U(x) \), is a time-independent solution to the heat equation, i.e., a solution to the ODE

\[ \frac{d^2}{dx^2} U - r^2 U = 0; \quad U(0) = U(1) = 1. \]

Graph the solution to this boundary value problem, and analyze the manner in which the heat is distributed in the bar.

2. Rewrite the following PDEs so that the interval \( 0 \leq y \leq L \) becomes the interval \( 0 \leq x \leq 1 \):

(a) \[ \partial_t u = 4 \partial_y^2 u + e^{-t} \cos(2y), \quad u(y, 0) = (8 - y)^2 \]

\[ u(0, t) + 2 \partial_y u(0, t) = 0, \quad 3u(8, t) - 4 \partial_y u(8, t) = 2 + \cos(t) \]

(b) \[ \partial_t u = 72 \partial_y^2 u - \text{sech}(y - 6)u + \frac{\cos(3y)}{1 + t^2}, \quad u(y, 0) = 3e^{-(y-12)} \]

\[ u(0, t) = 2 - e^{-t} \cos(4t), \quad 2u(12, t) + \partial_y u(12, t) = 0 \]

3. Rewrite the following PDEs so that the forcing at the boundaries is removed via the addition of a boundary condition forcing term:

(a) \[ \partial_t u = 2 \partial_y^2 u - u + 7e^{-(t-x)}, \quad u(x, 0) = 3 - \sin(4\pi x) \]

\[ u(0, t) + 2 \partial_y u(0, t) = 0, \quad 3u(10, t) - 4 \partial_y u(10, t) = 2 + \cos(t) \]

(b) \[ \partial_t u = \partial_y^2 u - \text{sech}(x - 1/2)u + 3e^{-2t} \sin(4\pi x), \quad u(x, 0) = 2 \]

\[ u(0, t) = 2 - e^{-t} \cos(4t), \quad 2u(1, t) + \partial_x u(1, t) = 0 \]

4. Rewrite the following PDEs so that the interval \( 0 \leq y \leq L \) becomes the interval \( 0 \leq x \leq 1 \), and the forcing at the boundaries is removed via the addition of a boundary condition forcing term:

(a) \[ \partial_t u = 20 \partial_y^2 u - 3u + \sin(y), \quad u(y, 0) = (y - 10)^4 \]

\[ u(0, t) + 2 \partial_y u(0, t) = 0, \quad u(10, t) - 2 \partial_y u(10, t) = 2 + \cos(t) \]

(c) \[ \partial_t u = 9 \partial_y^2 u - \text{sech}(y - 3)u + e^{-4t} \sin(2t), \quad u(y, 0) = 4e^{3(y-6)} \]

\[ u(0, t) = 5, \quad 3u(6, t) - 2 \partial_y u(6, t) = 0 \]