The Mathematica package for graph theory (DiscreteMath‘Combinatorica‘) provides three different operations defined on pairs of graphs. Complete the following definition, using Mathematica to generate enough examples so that you are sure you are defining these correctly.

1 Dfn. If \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) are graphs then

1. The *union* of \( G_1 \) and \( G_2 \), denoted \( G_1 \cup G_2 \), is the graph \( G = (V, E) \) where
2. The *join* of \( G_1 \) and \( G_2 \), denoted \( G_1 + G_2 \), is the graph \( G = (V, E) \) where
3. The *product* of \( G_1 \) and \( G_2 \), denoted \( G_1 \times G_2 \), is the graph \( G = (V, E) \) where

2 Question. Complete the following table. Here \( G_1 \) is a graph with \( n_1 \) vertices and \( m_1 \) edges and \( G_2 \) a graph with \( n_2 \) vertices and \( m_2 \) edges.

<table>
<thead>
<tr>
<th>Operation</th>
<th>no of vertices</th>
<th>no of edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>( G_1 \cup G_2 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( G_1 + G_2 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( G_1 \times G_2 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3 Conjecture. For all graphs \( G_1, G_2 \),

1. \( \overline{G_1 \cup G_2} = \overline{G_1} \cup \overline{G_2} \)
2. \( \overline{G_1 + G_2} = \overline{G_1} + \overline{G_2} \)
3. \( \overline{G_1 \times G_2} = \overline{G_1} \times \overline{G_2} \)

4 Conjecture. For all bipartite graphs \( G_1, G_2 \),

1. \( G_1 \cup G_2 \) is bipartite
2. \( G_1 + G_2 \) is bipartite
3. \( G_1 \times G_2 \) is bipartite

5 Conjecture. For all regular graphs \( G_1, G_2 \),

1. \( G_1 \cup G_2 \) is regular
2. \( G_1 + G_2 \) is regular
3. \( G_1 \times G_2 \) is regular