1. (a) Let \( a \in \mathbb{R} \). The set
\[
\{x \in X \mid g(x) > a\} = (\{x \mid f(x) > a\} \cap \{x \mid f(x) = g(x)\}) \cup (\{x \mid g(x) > a\} \cap \{x \mid g(x) \neq f(x)\})
\]
Since \( f \) is measurable, \( \{x \mid f(x) > a\} \in \mathcal{M} \). The space \( X \) is measurable and splits into the disjoint union
\[
X = \{x \mid f(x) = g(x)\} \cup \{x \mid f(x) \neq g(x)\}
\]
where the set \( \{x \mid f(x) \neq g(x)\} \) has (outer) measure zero, so it is measurable. (See Exercise 39.) Hence,
\[
\{x \mid f(x) = g(x)\} = X \setminus \{x \mid f(x) \neq g(x)\}
\]
and, as a result,
\[
\{x \mid f(x) > a\} \cap \{x \mid f(x) = g(x)\}
\]
are measurable. Moreover,
\[
\{x \mid g(x) > a\} \cap \{x \mid g(x) \neq f(x)\} \subset \{x \mid g(x) \neq f(x)\}
\]
Since the set on the right has outer measure zero, so does the one on the left, and hence it is measurable. Thus, \( \{x \mid g(x) > a\} \in \mathcal{M} \). Since \( a \in \mathbb{R} \) is arbitrary, we have that \( g \) is measurable.

(b) Let \( t \in X \) be such that \( f(t) < g(t) \). Then \( \exists q \in \mathbb{Q} \) such that \( f(t) < q < g(t) \), which shows that \( t \in \{x \mid f(x) < q\} \cap \{x \mid g(x) > q\} \). Now let \( \{q_j\}_{j=1}^{\infty} \) be an enumeration of \( \mathbb{Q} \), and set \( E_j = \{x \mid f(x) < q_j\} \cap \{x \mid g(x) > q_j\} \) for \( j = 1, 2, \ldots \). Since \( f, g \) are measurable, each \( E_j \in \mathcal{M} \). Moreover,
\[
\{x \mid f(x) < g(x)\} = \bigcup_{j=1}^{\infty} E_j,
\]
which is measurable since \( \mathcal{M} \) is closed under countable unions.

By symmetry of argument, the set \( \{x \mid g(x) < f(x)\} \) is also measurable. But then
\[
\{x \mid f(x) = g(x)\} = X \setminus (\{x \mid f(x) < g(x)\} \cup \{x \mid f(x) > g(x)\})
\]
is measurable as well, since \( \mathcal{M} \) is a ring.

2. Let \( A, B \in \mathcal{M} \), which means \( f^{-1}(A), f^{-1}(B) \in \mathcal{M} \). Then
\[
f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \in \mathcal{M}
\]
and
\[
f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B) \in \mathcal{M},
\]
showing that \( \mathcal{M} \) is a ring.

Now suppose \( (A_j) \) is a sequence of sets in \( \mathcal{M} \). It is the case that
\[
f^{-1} \left( \bigcup_j A_j \right) = \bigcup_j f^{-1}(A_j) \in \mathcal{M},
\]
since \( \mathcal{M} \) is closed under countable unions. Thus, \( \mathcal{M} \) is a \( \sigma \)-ring.

To get that \( \mathcal{M} \) is a \( \sigma \)-algebra, notice that all sets in \( \mathcal{M} \) are subsets of \([\infty, \infty]\), and \( f^{-1}([\infty, \infty]) = X \in \mathcal{M} \), which shows \([\infty, \infty] \in \mathcal{M}\).
3. For each open $\Omega \subset \mathbb{R}$, $\Omega$ may be written as the countable union of open intervals $I_j = (a_j, b_j)$ ($\mathbb{R}$ has a countable base of open intervals whose endpoints are rational numbers). Moreover, 

$$f^{-1}(\Omega) = f^{-1}\left(\bigcup_j I_j\right) = \bigcup_j f^{-1}(I_j) = \bigcup_j \left(\{x \mid f(x) < b_j\} \cap \{x \mid f(x) > a_j\}\right).$$

This shows $f^{-1}(\Omega) = \{x \mid f(x) \in \Omega\}$ is measurable for each open $\Omega \subset \mathbb{R}$.

Now let $c \in \mathbb{R}$. Since $(c, \infty)$ is an open interval and $g$ is continuous, we have $\Omega_c := g^{-1}((c, \infty))$ is open in $\mathbb{R}$. Thus,

$$\{x \mid (g \circ f)(x) > c\} = f^{-1}(\Omega_c) \in \mathcal{M}.$$ 

Since this set is measurable for each $c \in \mathbb{R}$, we have $g \circ f$ is measurable.

4. (a) Suppose $x = 0.a_1a_2a_3a_4 \ldots = 0.\gamma_1\gamma_2\gamma_3\gamma_4 \ldots$, where each $a_j$, $\gamma_j \in \{0, 1, 2\}$ and some $a_j \neq \gamma_j$. Then one of these ternary expansions—let us suppose it is the one with $\gamma$’s—ends in a string of 2’s. That is, there is a natural number $M$ for which $\gamma_j = 2$ for each $j > M$, but $\gamma_M \neq 2$. With what we know about expansions, we have that $a_j = \gamma_j$ for $j = 1, \ldots, M$, $a_M = 1 + \gamma_M$, and $a_j = 0$ for $j > M$.

Now let the number $N_a \in \mathbb{N}$ be such that $a_{N_a} = 1$ and $a_j \neq 1$ for $j < N_a$, if such a number $N_a$ exists; otherwise, let $N_a = \infty$. Choose the $b_j$, $j < N_a$ as described above. Working from the expansion $0.\gamma_1\gamma_2\gamma_3 \ldots$ in place of $0.a_1a_2a_3 \ldots$, choose $N_\varepsilon$ analogously to $N_a$ and numbers $\beta_j$, $j < N_\varepsilon$ analogously to the $b_j$. Notice that, since $a_j = 0$ for $j > M$, we must have $N_a \leq M$. We consider two cases:

- **Case: $N_a < M$**
  Since the $a_j = \gamma_j$ for $j < M$, it is clear that the $N_a = N_\varepsilon$, and the $b_j = \beta_j$, $j = 1, \ldots, N_a$. Thus, $\sum_{n=1}^{N_a} b_n 2^{-n} = \sum_{j=1}^{N_a} \beta_j 2^{-j}$.

- **Case: $N_a = M$**
  Since $\gamma_M \neq a_M = 1$ and $\gamma_j = 2$ for $j > M$, it follows that $N_\varepsilon = \infty$, $\beta_M = 0$ and $\beta_j = 1$ for $j > M$. Obviously it is still true that $b_j = \beta_j$ for $j < M$, and $b_M = 1$. Thus,

$$\sum_{n=1}^{\infty} \frac{\beta_n}{2^n} = \sum_{n=1}^{M-1} \frac{\beta_n}{2^n} + \sum_{n=M+1}^{\infty} \frac{\beta_n}{2^n} = \sum_{n=1}^{M-1} \frac{b_n}{2^n} + \sum_{n=M+1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{M-1} \frac{b_n}{2^n} + \frac{1}{2} \sum_{n=0}^{M-1} \left(\frac{1}{2}\right)^n.$$

(b) As we know, the set $[0,1] \setminus C$ is made up of countably-many disjoint open intervals $I_k^n$, $n = 0, 1, 2, \ldots$, $k = 1, \ldots, 2^n$, where

- $I_0^n = (\frac{1}{3}, \frac{2}{3})$,
- $I_1^n = (\frac{1}{3}, \frac{2}{3})$, $I_2^n = (\frac{7}{32}, \frac{8}{32})$,
- $I_3^n = (\frac{1}{3}, \frac{2}{3})$, $I_2^n = (\frac{7}{32}, \frac{8}{32})$, $I_3^n = (\frac{19}{32}, \frac{20}{32})$, $I_4^n = (\frac{25}{32}, \frac{26}{32})$,

etc.
Fix $n$ and $k$, and let $x, y \in \mathbb{T}^k_n$. This means that $x, y$ have ternary expansions $0.a_1a_2a_3 \ldots, 0.c_1c_2c_3 \ldots$ respectively such that $a_j = c_j \in \{0, 2\}$ for $j = 1, \ldots, n$, and $a_k = c_k = 1$. Thus,
\[
f(x) = \frac{1}{2^k} + \sum_{m=1}^{k-1} a_m \frac{1}{2^{m+1}} = \frac{1}{2^k} + \sum_{m=1}^{k-1} c_m \frac{1}{2^{m+1}} = f(y),
\]
where a sum $\sum_{m=1}^{k-1}$ should be taken as zero if $k = 1$.

An elementary fact about expansions is the following: If $x, y \in [0, 1]$ with $x \neq y$, and $0.x_1x_2x_3 \ldots, 0.y_1y_2y_3 \ldots$ are base-$b$ ($b \in \mathbb{N} \setminus \{1\}$) expansions of $x$ and $y$ (each $x_j, y_j \in \{0, 1, \ldots, b-1\}$), then taking $M_b$ be the smallest positive integer $j$ such that $x_j \neq y_j$ (a well-defined number), we have $x < y$ if and only if $x_{M_b} < y_{M_b}$. Thus, let us suppose $x < y$, that they have ternary expansions $0.a_1a_2a_3 \ldots, 0.c_1c_2c_3 \ldots$, respectively, and that $M = M_3$ is a positive integer such that $a_j = c_j$ for $j < M$ but $a_M < c_M$. Let $N$ be the smallest $j$ for which $a_j = 1$. ($N = \infty$ if this criterion cannot be met.) If $N > M$, then since $c_M > a_M$, it follows that $a_M = 0$. Now, if $c_M = 1$, then
\[
f(y) = \frac{a_1}{2^2} + \frac{a_2}{2^3} + \cdots + \frac{a_{M-1}}{2^M} + \frac{1}{2^M},
\]
while if $c_M = 2$,
\[
f(y) = \frac{a_1}{2^2} + \frac{a_2}{2^3} + \cdots + \frac{a_{M-1}}{2^M} + \frac{2}{2^M+1} + \text{other nonnegative terms}.
\]

Either way
\[
f(x) \leq \frac{a_1}{2^2} + \cdots + \frac{a_1}{2^2} + \sum_{j=M+1}^{\infty} \frac{2}{2^{j+1}} = \frac{a_1}{2^2} + \cdots + \frac{a_1}{2^2} + \frac{1}{2^M} \leq f(y).
\]

If $N \leq M$, it is even easier to see that $f(x) \leq f(y)$.

Next note that every real number in $[0, 1]$ has a binary expansion $0.b_1b_2b_3 \ldots$, each $b_j \in \{0, 1\}$. To find an $x$ such that
\[
f(x) = \sum_{j=1}^{\infty} \frac{b_j}{2^j},
\]
take $x$ to be the number with ternary expansion $0.a_1a_2a_3 \ldots$ where each $a_j = 2b_j$. By inspection, $f(0) = 0$ and $f(1) = 1$.

Thus far, we have shown that $f$ maps $[0, 1]$ onto $[0, 1]$, and is monotone increasing. Being monotone, the only kind of discontinuity $f$ can have is a jump discontinuity. But if it had any such discontinuities, there would be numbers in $[0, 1]$ absent from the range of $f$, and this is not the case. Thus, $f$ is continuous. To finish, note that we showed $f$ is constant not just on each open interval $I_k^n$ in $[0, 1] \setminus C$, but on the closure $\overline{I_k^n}$. The endpoints of each $I_k^n$ are points of $C$, and hence there are no members of the range of $f$ not found in the image of $C$ under $f$.

(c) By part (b), $g$ is continuous and, for $y > x$, $g(y) = f(y) + y \geq f(x) + y > f(x) + x = g(x)$, so $g$ is injective (one-to-one). Since $g(0) = 0$ and $g(1) = 2$, the intermediate value theorem tells us that $g$ maps $[0, 1]$ onto $[0, 2]$, making it a one-to-one correspondence between the compact sets $[0, 1]$ and $[0, 2]$. By Exercise *23, $g$ is an homeomorphism.
(d) Let $I^n_k$ be the open intervals contained in $[0,1]$ defined in the proof of part (b). Since $f = c$, $c$ a constant on each $I^n_k$ (the constant changes for different $I^n_k$), the set $g(I^n_k) = \{g(x) \mid x \in I^n_k\} = c + I^n_k$. Thus, by Lemma L.22, $m(g(I^n_k)) = m(I^n_k) = 3^{-(n+1)}$. We now have
\[ [0,2] = g(C) \cup g \left( \bigcup_{n=0}^{\infty} \bigcup_{k=1}^{2^n} I^n_k \right) = g(C) \cup \left( \bigcup_{n=0}^{\infty} g(I^n_k) \right). \]
Thus,
\[ g(C) = [0,2] \setminus \left( \bigcup_{n=0}^{\infty} \bigcup_{k=1}^{2^n} g(I^n_k) \right) \in \mathcal{M}, \]
and, since the $g(I^n_k) \in [0,2]$ are pairwise disjoint,
\[ m(g(C)) = m([0,2]) - m \left( \bigcup_{n,k} g(I^n_k) \right) = 2 - \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} m(g(I^n_k)) = 2 - \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} 3^{-(n+1)} = 2 - \frac{1}{3} \sum_{n=0}^{\infty} \left( \frac{2}{3} \right)^n = 2 - \frac{1}{3} \cdot \frac{1}{1 - 2/3} = 1. \]
Since $g(C)$ has positive measure, it contains a subset $B$ which is nonmeasurable. Now take $A = h(B) \subset C$. Since the Cantor set $C$ has measure zero, $m(A) = 0$, and hence $A \in \mathcal{M}$. However, $B = h^{-1}(A)$ is nonmeasurable.

(e) If we take $p = \chi_A$ (as defined in the part (d)), and $q = h = g^{-1}$, then $p \circ q$ is not measurable. To see this, suppose $\Omega \subset \mathbb{R}$ is an open set with $1 \in \Omega$ but $0 \not\in \Omega$. If $\chi_A \circ h$ were measurable, then it would be the case that $(\chi_A \circ h)^{-1}(\Omega)$ would be measurable (this because $\Omega$ is a countable union of sets of the form $[-\infty,b) \cap (a,\infty]$, and the preimage of such sets under a measurable function is measurable). But,
\[ (\chi_A \circ h)^{-1}(\Omega) = \{x \mid (\chi_A \circ h)(x) \in \Omega\} = g(A) \not\in \mathcal{M}. \]

(f) The set we have that fits these criteria is $A$ (from part (d)). We have already established that $A$ is measurable.

To see that it is not a Borel set, we first make a few observations. We have that $h = g^{-1}$ is continuous, and hence a measurable function. We argued in a previous problem (Problem 2 on the in-class portion of this exam) that $h^{-1}(\Omega) \in \mathcal{M}$ for each open $\Omega$. Thus, the $\sigma$-algebra $\mathcal{M}$ (associated with $h$) from Problem 2 (of this take-home portion of the exam) contains all open sets. The definition of the Borel algebra $\mathcal{B}$ is that it is the smallest $\sigma$-algebra containing every open set, and hence $\mathcal{B} \subset \mathcal{M}$.

Now suppose (to get a contradiction) $A \in \mathcal{B}$. Thus, $A \in \mathcal{M}$, meaning that it is one of the sets for which $h^{-1}(A) = g(A) \in \mathcal{M}$. 

5. (a) Consider the sets $A_n = E_1 \setminus E_n$, $n \in \mathbb{N}$. Since $E_{n+1} \subset E_n$ for each $n$, we have $A_n \subset A_{n+1}$. Set $A = \bigcup_n A_n$. We have
\[ m(A) = \lim_n m(A_n) = \lim_n m(E_1 \setminus E_n) = \lim_n [m(E_1) - m(E_n)] = m(E_1) - \lim_n m(E_n), \]
where the first equality holds by Theorem 11.3, and our assertion $m(E_1 \setminus E_n) = m(E_1) - m(E_n)$ uses the assumption that $m(E_1) < \infty$. But,

$$A = \bigcup_n A_n = \bigcup_n (E_1 \setminus E_n) = E_1 \setminus \left( \bigcap_n E_n \right),$$

so

$$m(A) = m(E_1) - m\left( \bigcap_n E_n \right).$$

Thus, we have

$$m(E_1) - m\left( \bigcap_n E_n \right) = m(E_1) - \lim_n m(E_n),$$

which yields the result.

(b) For each $n$ let $E_n = [n, \infty) \subset \mathbb{R}$. Each $E_{n+1} \subset E_n$ and $m(E_n) = \infty$, $\forall n$, so $\lim_n m(E_n) = \infty$. But $\bigcap_n E_n = \emptyset$.

(c) Take $\epsilon, \delta > 0$ to be given. For each $n \in \mathbb{N}$ let

$$G_n := \{ x \in E : |f_n(x) - f(x)| \geq \epsilon \}, \quad \text{and set} \quad E_k := \bigcup_{n=k}^{\infty} G_n,$$

for $k = 1, 2, \ldots$. The measurability of each $f_n$ implies that the pointwise limit function $f$ is measurable as well. Since each $f_n$ is real-valued, $f_n - f$ is a well-defined function (no instances of $\infty - \infty$) and is measurable, so $|f_n - f|$ is measurable, too. Thus, the sets $G_n$ and $E_k$ are measurable. Note that $E_{k+1} \subset E_k$ for each $k \in \mathbb{N}$, and that $\bigcap_k E_k = \emptyset$ by virtue of the fact that $f_n \to f$ pointwise on $E$. Finally, $E_1 \subset E$ and so $m(E_1) \leq m(E) < \infty$. Applying part (a), we have

$$\lim_{k \to \infty} m(E_k) = m \left( \bigcap_{k=1}^{\infty} E_k \right) = m(\emptyset) = 0.$$

Thus, $\exists N \in \mathbb{N}$ such that $k \geq N$ implies $m(E_k) < \delta$. By definition, $x \in E_N$ means $x \in G_n$ for some $n \geq N$, and hence there is some $n \geq N$ for which $|f_n(x) - f(x)| \geq \epsilon$. By taking $A = E_N$, we get that for $x \notin A$, $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$.

(d) The first issue is “What $f$ should we take to be the limit function?” Let $S = \{ x \in E \mid (f_n(x)) \text{ is a convergent sequence} \}$. We may define

$$f(x) := \begin{cases} \lim_n f_n(x), & \text{if } x \in S, \\ 0, & \text{if } x \in E \setminus S. \end{cases}$$

One can show that $f$ is measurable (reliant on the fact that $m(E \setminus S) = 0$). By problem 1(a), no matter how $f$ is defined on $E \setminus S$, it will still be measurable.

We may define the sets $G_n, E_k$ as in the previous proof, and argue they are measurable just as before. Once again, $E_{k+1} \subset E_k$ for $k = 1, 2, \ldots$, but this time $\bigcap_k E_k = E \setminus S$, so we get

$$\lim_{k \to \infty} m(E_k) = m \left( \bigcap_{k=1}^{\infty} E_k \right) = m(E \setminus S) = 0.$$

The rest of the proof is identical.
(e) Let $\eta > 0$ be given. For $n = 1, 2, \ldots$, let $A_n \subset E$ be chosen as in part (d) so that $m(A_n) < 2^{-n}\eta$ and $\exists N_n \in \mathbb{N}$ such that $k \geq N_n$ and $x \not\in A_n$ imply $|f_k(x) - f(x)| < \frac{1}{n}$. Let $A = \bigcup_n A_n$. Then since each $A_n \in \mathcal{M}$, $A$ is measurable and

$$m(A) \leq \sum_{n=1}^{\infty} m(A_n) \leq \eta \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \eta.$$ 

To see that $f_n \to f$ uniformly on $E \setminus A = E \setminus \bigcup_n A_n = \bigcap_n (E \setminus A_n)$, let $\epsilon > 0$. For some $N \in \mathbb{N}$ it is the case $0 < \frac{1}{N} < \epsilon$. The set $E \setminus A \subset E \setminus A_N$, so $k \geq N$ implies $|f_k(x) - f(x)| < \epsilon$ for every $x \in E \setminus A$. 
