I. Let $A \subset \mathbb{R}^p$ with $m^*(A) > 0$. Define the sets $S, S_i$, and the sequence $(q_i)$ as in Thm. L.22. In that theorem, we proved that $S$ was nonmeasurable. This means that each $S_i$ is nonmeasurable as well, by Lemma L.20, since $S$ and $S_i$ are just translates of one another. For each $i$ set $A_i := A \cap S_i$. Notice that, since $S_i \cap S_j = \emptyset$ whenever $i \neq j$, it follows that the $A_i$ are disjoint. Also, since $A \subset (0, 1) \subset \bigcup_i S_i$, it follows that $\bigcup_i A_i = A$.

Suppose that each $A_i$ were measurable. Then as the countable union of measurable sets, $A$ is measurable as well, and the $\sigma$-additivity of Lebesgue measure on sets in $\mathcal{M}(m)$ implies

$$\sum_{i=1}^{\infty} m(A_i) = m \left( \bigcup_i A_i \right) = m(A) > 0.$$ 

Thus, $m(A_{i_0}) > 0$ for some $i_0$. But $A_{i_0} \subset S_{i_0}$, and the measurability of $A_{i_0}$ implies that $m(A_{i_0}) = 0$ (by Lemma L.20 and a problem in HW 33).

II. Let $f : X \to [-\infty, \infty]$ be a constant function—that is, $f(x) = b, \forall x$. Let $a \in \mathbb{R}$. Then

$$f^{-1}((a, \infty]) = \begin{cases} \emptyset, & \text{if } b \leq a, \\ X, & \text{if } b > a. \end{cases}$$

Since $\mathcal{M}$ is a $\sigma$-algebra, it contains both $\emptyset$ and $X$. Thus, the criterion of Defn. 11.13 for the measurability of $f$ is met.

III. We assume that each $A_i \in \mathcal{M}$. Then

$$\chi_{A_i}^{-1}((a, \infty]) = \begin{cases} \emptyset, & \text{if } a \geq 1, \\ A_i, & \text{if } 0 \leq a < 1, \\ X, & \text{if } a < 0. \end{cases}$$

Thus, $\forall a \in \mathbb{R}$, $\{x \in X \mid a < \chi_{A_i}(x)\}$ is a measurable set, showing $\chi_{A_i}$ to be a measurable function. We have just shown that constant functions are measurable, so each $y_i\chi_{A_i}$ is a measurable function by Thm. 11.18, which also may be applied inductively to guarantee that finite sums

$$\sum_{i=1}^{n} y_i\chi_{A_i}$$

are measurable.
11.3 Case: For each \( x \in X \), \((f_n(x))\) is a bounded sequence.

Let
\[
g(x) := \limsup_{n \to \infty} f_n(x) \quad \text{and} \quad h(x) := \liminf_{n \to \infty} f_n(x).
\]

Since at each \( x \), \((f_n(x))\) is bounded, we get that \( g(x), h(x) \in (-\infty, \infty) \), \( \forall x \in X \).

By Thm. 11.17, both \( g \) and \( h \) are measurable, which means that \( g - h \) is measurable (by Thm. 11.18). (Note: \( g - h \) is well-defined \( \forall x \), since it produces no instances of \( (\infty - \infty) \).) Now by Thm. S.7, the set on which \((f_n)\) converges is
\[
\{ x \mid (g - h)(x) = 0 \} = \{ x \mid (g - h)(x) \geq 0 \} \cap \{ x \mid (g - h)(x) \leq 0 \},
\]
and is measurable since each of the latter sets is measurable.

**General case:**

First, we note that the argument for the case above works perfectly well in the general case so long as \( \{ x \mid g(x) = h(x) = -\infty \} \) and \( \{ x \mid g(x) = h(x) = +\infty \} \) are empty sets.

To generalize the argument, then, for each \( n \in \mathbb{N} \) we define
\[
h_n(x) := \max\{\min\{h, n\}, -n\},
\]
which we know to be measurable for each \( n \) by Coro. L.25 and the first problem in this PS 35. Note that each \( h_n \) is a bounded function (having range in \([-n, n]\)), so
\[
\{ x \mid g(x) = h_n(x) = -\infty \} \quad \text{and} \quad \{ x \mid g(x) = h_n(x) = +\infty \}
\]
are empty. Thus, the argument above may be applied to get that the set
\[
\{ x \mid g(x) - h_n(x) = 0 \}
\]
is measurable \( \forall n \). Now \( f_n(x) \) converges at \( x \in X \) if and only if \( g(x) = h(x) \in (-\infty, \infty) \), and this happens if and only if \( \exists N_x \in \mathbb{N} \) s.t. \( g(x) = h_n(x), \forall n \geq N \). Thus,
\[
\{ x \mid f_n(x) \text{ converges} \} = \bigcup_{N=1}^{\infty} \bigcap_{n \geq N} \{ x \mid g(x) - h_n(x) = 0 \},
\]
a measurable set.

11.14 \( \Rightarrow \): Assume that \( f = u + iv \) is measurable. Then, by definition, \( u, v \) are measurable.

Let \( V \) be an open set in \( \mathbb{C} \). There exists a countable collection of open rectangles
\[
R_n := \{ \alpha + i\beta \in \mathbb{C} \mid a_n < \alpha < b_n, c_n < \beta < d_n \} \quad \text{s.t.} \quad V = \bigcup_{n=1}^{\infty} R_n.
\]
Now, for each $n$,

$$f^{-1}(R_n) = \{ x \in X \mid f(x) \in R_n \}$$

$$= u^{-1}((a_n, b_n)) \cap v^{-1}((c_n, d_n))$$

$$= \left( u^{-1}((a_n, \infty]) \cap u^{-1}([-\infty, b_n)) \right) \cap \left( v^{-1}((c_n, \infty]) \cap v^{-1}([-\infty, d_n)) \right),$$

which is measurable by Thm. 11.15 and the fact that $\mathcal{M}$ is a $\sigma$-algebra. Finally, the result follows from noting that

$$f^{-1}(V) = f^{-1}\left( \bigcup_n R_n \right) = \bigcup_n f^{-1}(R_n),$$

which is a countable union of measurable sets.

$\Leftarrow$: Assume that $f^{-1}(V)$ is measurable for each open $V \subset \mathbb{C}$. Consider open sets of the form

$$\Omega_a := \{ \alpha + i\beta \in \mathbb{C} \mid \alpha > a, \beta \in \mathbb{R} \} \quad \text{and} \quad G_b := \{ \alpha + i\beta \in \mathbb{C} \mid \alpha \in \mathbb{R}, \beta > b \}.$$

Since the $\Omega_a, G_b$ are open, we have

$$u^{-1}((a, \infty]) = f^{-1}(\Omega_a) \in \mathcal{M},$$

and

$$v^{-1}((b, \infty]) = f^{-1}(G_b) \in \mathcal{M},$$

for each real $a, b$. Thus, $u$ and $v$ are measurable functions, which means $f$ is as well.