Solutions to HW #28.

First, we show that $m(R) = \mu(R)$ when $R \subset \mathbb{R}^2$ is a rectangle. Let $R_1, \ldots, R_n$ be disjoint rectangles such that $R \subset \bigcup_{j=1}^n R_j$. Since $m$ is a nonnegative additive set function on $\mathcal{E}$, it is monotone on $\mathcal{E}$ (L.2(i), point 4), so

$$m(R) \leq m\left(\bigcup_{j=1}^n R_j\right) = \sum_{j=1}^n m(R_j).$$

The choice of disjoint rectangles $R_j$ containing $R$ was arbitrary, so $m(R) \leq \mu(R)$. But since $R$ is a rectangle containing itself, it should be clear that $\mu(R) \leq m(R)$, and hence $m(R) = \mu(R)$.

To show $m(R) = \mu(R)$, assume $S_1, \ldots, S_\ell$ are disjoint rectangles such that $\bigcup_{j=1}^\ell S_j \subset R$. Once again, by the monotonicity of $m$ on $\mathcal{E}$,

$$\sum_{j=1}^\ell m(S_j) = m\left(\bigcup_{j=1}^\ell R_j\right) \leq m(R).$$

The choice of rectangles $S_j$ was arbitrary, so $\mu(R) \leq m(R)$. Once again, the other inequality $\mu(R) \geq m(R)$ is obvious, so $m(R) = \mu(R)$.

Thus $\mu(R) = m(R) = \mu(R)$, which shows both that $R$ has content, and that it is equal to $m(R)$. 