Solutions to HW #25.

I. Using partial fractions, we get that

\[ \frac{1}{1 + z^2} = \frac{a}{z + i} + \frac{b}{z - i} \Rightarrow a = \frac{1}{2} i, \ b = -\frac{1}{2} i. \]

Now

\[ \frac{a}{z + i} = \frac{a}{i + 1 - (z - 1)} = \frac{a}{1 + i} \cdot \frac{1}{1 - \frac{z - 1}{i - 1}} \]

\[ = \frac{a}{1 + i} \sum_{n=0}^{\infty} (-1)^n \left( \frac{z - 1}{1 + i} \right)^n \]

\[ = \sum_{n=0}^{\infty} (-1)^n \frac{a}{(1 + i)^{n+1}} (z - 1)^n. \]

Similarly,

\[ \frac{b}{z - i} = \frac{b}{1 - i + z - 1} = \frac{b}{1 - i} \cdot \frac{1}{1 - \frac{z - 1}{i - 1}} \]

\[ = \frac{b}{1 - i} \sum_{n=0}^{\infty} \left( \frac{z - 1}{i - 1} \right)^n \]

\[ = \sum_{n=0}^{\infty} (-1)^n \frac{b}{(1 - i)^{n+1}} (z - 1)^n. \]

Thus,

\[ \frac{1}{1 + z^2} = \sum_{n=0}^{\infty} (-1)^n \left[ \frac{a}{(1 + i)^{n+1}} + \frac{b}{(1 - i)^{n+1}} \right] (z - 1)^n \]

\[ = \sum_{n=0}^{\infty} \frac{(-1)^n}{2} \cdot \left( \frac{1}{(1 + i)^{n+1}} - \frac{1}{(1 - i)^{n+1}} \right) (z - 1)^n \]

\[ = \sum_{n=0}^{\infty} c_n (z - 1)^n, \]

where each

\[ c_n := \frac{(-1)^n}{2} \cdot \left( \frac{1}{(1 + i)^{n+1}} - \frac{1}{(1 - i)^{n+1}} \right). \]
One might guess the radius of convergence for this power series expression via a geometric argument: The complex number \( z = 1 \) is \( \sqrt{2} \) units away from \( z = \pm i \) (the complex numbers where \( f \) is undefined), so \( R = \sqrt{2} \). To prove this, we note from Thm. 3.47 that at each \( z \in \mathbb{C} \) where both

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{2(1+i)^{n+1}} (z-1)^n \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2(1-i)^{n+1}} (z-1)^n
\]

converge, the series \( \sum c_n(z-1)^n \) will converge. Thus, if \( R_1 \) and \( R_2 \) are the radii of convergence for these two series, then \( \sum c_n(z-1)^n \) converges at least on \( R = \min\{R_1, R_2\} \). (In fact, this \( R \) is the radius of convergence, not just a lower bound on it, because if the radius of convergence were larger than this \( R \), then \( \sum c_n(z-1)^n \) would converge absolutely for some \( z_0 \) with \( |z_0 - 1| > R \), and this would imply that both of the individual series converged at \( z_0 \) as well. ———) Thus, it suffices to determine the radii of convergence for the individual series. To make matters simpler (since multiplication by a constant does not change the radius of convergence for a series), we determine the radii of convergence for

\[
\sum_{n=0}^{\infty} (-1)^n \left( \frac{z-1}{1+i} \right)^n \quad \text{and} \quad \sum_{n=0}^{\infty} (-1)^{n+1} \left( \frac{z-1}{1-i} \right)^n.
\]

As these series are not centered at 0, I choose to go right back to the root test: For the first, we need

\[
1 > \lim_{n \to \infty} \sqrt[n]{\left| \frac{z-1}{1+i} \right|} = \frac{|z-1|}{|1+i|} = \frac{|z-1|}{\sqrt{2}} \quad \Rightarrow \quad |z-1| < \sqrt{2}.
\]

Thus, \( R_1 = \sqrt{2} \). Similarly, \( R_2 = \sqrt{2} \), and hence the radius of convergence \( R = \min\{R_1, R_2\} \) is \( \sqrt{2} \), just as we guessed.

II. We write \( A \cup B \) as the disjoint union

\[
A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A).
\]

Then

\[
\phi(A \cup B) + \phi(A \cap B) = \phi((A \setminus B) \cup (A \cap B) \cup (B \setminus A)) + \phi(A \cap B) = \phi((A \setminus B) \cup (A \cap B)) + \phi(B \setminus A) + \phi(A \cap B) \quad \text{(additivity of \( \phi \))}
\]

\[
= \phi(A) + \phi((B \setminus A) \cup (A \cap B)) \quad \text{(additivity again)}
\]

\[
= \phi(A) + \phi(B).
\]