Solutions to HWs #18, #19 and #20.

I. To see that \( f \) is a function, all we need is the definition of injectivity (one-to-oneness): that \( f(b) = f(a) \Rightarrow b = a \). Thus, \( f^{-1}(E) \) where \( E \) is any singleton set (i.e., set comprised of one element) is again a singleton set. It may also be observed that the surjectivity of \( f \) implies that the domain of \( f^{-1} \) is all of \( Y \). Like \( f \), \( f^{-1} \) is surjective as well, a natural by-product of the fact that \( f \) is defined on all of \( X \). Another fact about \( f^{-1} \) is that it is injective (otherwise \( f \) would not be a function.)

To see that \( f^{-1} \) is continuous, let \( E \subset X \) be closed. Since \( X \) is compact, \( E \) is compact (Thm. 2.35) and, by Thm. 4.14, \( F(E) \) is compact (by continuity of \( f \)). By Thm. 2.34, \( f(E) \) is closed. But, \( f(E) = f^{-1}(E) \), and thus we have shown that the preimage under \( f^{-1} \) of an arbitrary closed set is closed.

5.17 By Taylor’s Thm. \( \exists t \in (-1, 0) \) s.t.
\[
f(-1) = \frac{f(0)}{0!} (-1)^0 + \frac{f'(0)}{1!} (-1)^1 + \frac{f''(0)}{2!} (-1)^2 + \frac{f'''(t)}{3!} (-1)^3.
\]
Using the values supplied, this expression becomes
\[
f^{(3)}(t) = 3f''(0). \quad (1)
\]
Another application of Taylor’s Thm., this time on the right side of \( x = 0 \), gives that \( \exists s \in (0, 1) \) s.t.
\[
f(1) = \frac{f(0)}{0!} (1)^0 + \frac{f'(0)}{1!} (1)^1 + \frac{f''(0)}{2!} (1)^2 + \frac{f'''(s)}{3!} (1)^3.
\]
Again using the values supplied, we have
\[
-f^{(3)}(s) = 3f''(0) - 6. \quad (2)
\]
Subtracting equation (2) from (1), we have
\[
f^{(3)}(t) + f^{(3)}(s) = 6,
\]
which means that at least one of the numbers \( f^{(3)}(t), f^{(3)}(s) \) is \( \geq 3 \).

4.4 That \( f(E) \) is dense in \( f(X) \) is a corollary to Thm. S.19. To see this, let \( a \in X \). We will show that \( f(a) \in \overline{f(E)} \). \( E \) is dense in \( X \) \( \Rightarrow \exists \) a sequence \( (x_n) \) in \( E \) s.t. \( x_n \to a \). By S.19, the continuity of \( f \) at \( a \) implies \( f(x_n) \to f(a) \). But each \( f(x_n) \in f(E) \), so \( f(a) \in \overline{f(E)} \).
We already have that \( f(x) = g(x) \) for \( x \in E \). Let \( a \in E^c \), which means \( a \in E' \) (denseness of \( E \)). Let \((x_n)\) be a sequence in \( E \) with \( x_n \to a \). Then

\[
\begin{align*}
g(a) &= \lim_n g(x_n) \quad \text{(continuity of } g) \\
      &= \lim_n f(x_n) \quad \text{(since } f = g \text{ on } E) \\
      &= f(a) \quad \text{(continuity of } f) .
\end{align*}
\]