Solutions to Exam 2

I. Without considering the compactness of $F$, there is no direct comparison—that is, no way to link—$\mu(F)$ to the infinite sum $\sum_{n=1}^{\infty} \mu(A_n)$. The most natural link would be to write that

$$F \subset \bigcup_{n=1}^{\infty} A_n \implies \mu(F) \leq \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n),$$

but we cannot be sure that $\bigcup_{n=1}^{\infty} A_n$ is in $\mathcal{E}$ ($\mathcal{E}$ is a ring, but not a $\sigma$-ring), which means that $\mu$ need not be defined on this union. That is, this union is not necessarily in the domain of $\mu$.

II. Let $A_1, \ldots, A_n \subset [a, b]$ be a collection of intervals with $E \subset \bigcup_j A_j$. We claim that $\bigcup_j A_j \setminus (a, b)$ is a finite set. This is because the only other alternative is that $\bigcup_j A_j \setminus (a, b)$ contains an interval. That option is impossible, since the rationals are dense in $\mathbb{R}$, so every interval contains a rational, and all rationals within $[a, b]$ lie in some $A_j$.

Now, let $x_1, \ldots, x_m$ denote the points (if, indeed, there are any), ordered according to size, in $\bigcup_j A_j \setminus (a, b)$. For convenience, let $x_0 := a$ and $x_{m+1} := b$. Then

$$b - a = \sum_{i=1}^{m+1} (x_i - x_{i-1}) \leq \sum_{j=1}^{n} m(A_j).$$

Since the sum on the right is at least as large as $(b-a)$ for every collection (including the disjoint ones) of intervals whose union contains $[a, b]$, it follows that $(b-a) \leq \mu([a, b])$.

For the lower content question, we note that the only types of intervals wholly contained in $E$ are singleton sets (sets containing just one element, a rational number, in this case; this is because between every two rationals there exists an irrational) and the empty set. So, let $A_1, \ldots, A_n \subset [a, b]$ be a collection of intervals for which $\bigcup_n A_n \subset E$. By the above observation, each $A_j$ is a singleton or empty and, by virtue of the way we have defined $m$, $m(A_j) = 0$. Thus

$$\sum_j m(A_j) = 0.$$

Since the choice of intervals $A_j$ inside $E$ was arbitrary, we have $\mu(E) = 0$.

III. We assume that each $B_n \in \mathcal{M}_F(\mu)$, and that $d(B_n, A) \to 0$ as $n \to \infty$. By definition of $\mathcal{M}_F(\mu)$, for each (fixed) $n \in \mathbb{N}$, $\exists$ a sequence $(B_{nk})_{k=1}^{\infty}$ of sets in $\mathcal{E}$ s.t. $d(B_{nk}, B_n) \to 0$.
as \( k \to \infty \). So, choose \( k_1 \in \mathbb{N} \) s.t. \( d(B_{1k_1}, B_1) < 1 \). Similarly, choose

\[
\begin{align*}
k_2 & \in \mathbb{N} \text{ s.t. } d(B_{2k_2}, B_2) < 1/2, \\
k_3 & \in \mathbb{N} \text{ s.t. } d(B_{3k_3}, B_3) < 1/3, \\
& \vdots \\
k_n & \in \mathbb{N} \text{ s.t. } d(B_{nk_n}, B_n) < 1/n, \\
& \vdots
\end{align*}
\]

**Claim:** \( d(B_{nk_n}, A) \to 0 \) as \( n \to \infty \).

We have

\[
d(B_{nk_n}, A) \leq d(B_{nk_n}, B_n) + d(B_n, A) \quad \text{(by L.8(vii))}
\]

\[
< \frac{1}{n} + d(B_n, A).
\]

Since both terms on the right-hand side go to zero as \( n \to \infty \), the claim holds.

By the claim \( A \in \mathcal{M}_F(\mu) \), since each \( B_{nk_n} \in \mathcal{E} \).

IV. First, we prove the following claim.

**Claim:** Every subset of \( \mathbb{R}^p \) with zero outer measure is measurable.

Let \( A \subset \mathbb{R}^p \) satisfy \( \mu^*(A) = 0 \). For each \( n \in \mathbb{N} \), let \( A_n = \emptyset \). Then \( A_n \in \mathcal{E} \), and

\[
d(A_n, A) = \mu^*(A \cup A_n) = \mu^*(A) = 0, \quad \forall n.
\]

This more than satisfies the requirement for \( A \in \mathcal{M}_F(\mu) \subset \mathcal{M}(\mu) \) that \( d(A_n, A) \to 0 \) as \( n \to \infty \) for some sequence \( (A_n) \) in \( \mathcal{E} \). The claim is now proved.

Now, since we have already proved (in the proof of Thm. L.15) that (i) implies (ii) implies (iii) as well as (i) implies (ii) implies (iv) implies (v), it suffices to show (iii) iff (v) and (v) implies (i).

(iii) \( \Rightarrow \) (v) Let \( A \subset \mathbb{R}^p \), and let \( (G_n) \) be a sequence of open subsets of \( \mathbb{R}^p \) such that \( A^c \subset (\bigcap_n G_n) \) and \( \mu^*((\bigcap_n G_n) \setminus A^c) = 0 \). For each \( n \in \mathbb{N} \) we take \( F_n := G_n^c \subset A \), where \( F_n \) is a closed set. Thus, \( (\bigcup_n F_n) \subset A \). And, since \( (\bigcap_n G_n) \setminus A^c = A \setminus (\bigcup_n F_n) \), we have \( \mu(A \setminus (\bigcup_n F_n)) = 0 \).

(v) \( \Rightarrow \) (iii) The argument is essentially the reverse of (iii) \( \Rightarrow \) (v).

(v) \( \Rightarrow \) (i) Let \( A \subset \mathbb{R}^p \) and \( (F_n) \) be a sequence of closed subsets of \( \mathbb{R}^p \) satisfying condition (v). Since each \( F_n \) is a Borel set, \( \bigcup_n F_n \) is measurable. Moreover, since \( A \setminus (\bigcup_n F_n) \) has outer measure 0, this is a measurable set as well. Finally,

\[
A = \left( A \setminus \left( \bigcup_n F_n \right) \right) \cup \left( \bigcup_n F_n \right),
\]
showing that $A$ is the union of two measurable sets. Since $\mathcal{M}(\mu)$ is a ring, this shows that $A$ is measurable.

V. Let $A \subset \mathbb{R}^p$ with $m^*(A) > 0$. Define the sets $S$, $S_i$, and the sequence $(q_i)$ as in Thm. L.22. In that theorem, we proved that $S$ was nonmeasurable. This means that each $S_i$ is nonmeasurable as well, by Lemma L.20, since $S$ and $S_i$ are just translates of one another. For each $i$ set $A_i := A \cap S_i$. Notice that, since $S_i \cap S_j = \emptyset$ whenever $i \neq j$, it follows that the $A_i$ are disjoint. Also, since $A \subset (0, 1) \subset \bigcup_i S_i$, it follows that $\bigcup_i A_i = A$.

Suppose that each $A_i$ were measurable. Then as the countable union of measurable sets, $A$ is measurable as well, and the $\sigma$-additivity of Lebesgue measure on sets in $\mathcal{M}(m)$ implies
\[
\sum_{i=1}^{\infty} m(A_i) = m \left( \bigcup_i A_i \right) = m(A) > 0.
\]
Thus, $m(A_{i_0}) > 0$ for some $i_0$. But $A_{i_0} \subset S_{i_0}$, and the measurability of $A_{i_0}$ implies that $m(A_{i_0}) = 0$ (by Lemma L.20 and a problem in HW 33).

11.3 Case: For each $x \in X$, $(f_n(x))$ is a bounded sequence.

Let
\[
g(x) := \limsup_{n \to \infty} f_n(x) \quad \text{and} \quad h(x) := \liminf_{n \to \infty} f_n(x).
\]
Since at each $x$, $(f_n(x))$ is bounded, we get that $g(x), h(x) \in (-\infty, \infty)$, $\forall x \in X$. By Thm. 11.17, both $g$ and $h$ are measurable, which means that $g - h$ is measurable (by Thm. 11.18). (Note: $g - h$ is well-defined $\forall x$, since it produces no instances of $(\infty - \infty)$.)

Now by Thm. S.7, the set on which $(f_n)$ converges is
\[
\{x \mid (g - h)(x) = 0\} = \{x \mid (g - h)(x) \geq 0\} \cap \{x \mid (g - h)(x) \leq 0\},
\]
and is measurable since each of the latter sets is measurable.

General case:

First, we note that the argument for the case above works perfectly well in the general case so long as $\{x \mid g(x) = h(x) = -\infty\}$ and $\{x \mid g(x) = h(x) = +\infty\}$ are empty sets.

To generalize the argument, then, for each $n \in \mathbb{N}$ we define
\[
h_n(x) := \max\{\min\{h, n\}, -n\},
\]
which we know to be measurable for each $n$ by Coro. L.25 and the first problem in this PS 35. Note that each $h_n$ is a bounded function (having range in $[-n, n]$), so
\[
\{x \mid g(x) = h_n(x) = -\infty\} \quad \text{and} \quad \{x \mid g(x) = h_n(x) = +\infty\}
\]
are empty. Thus, the argument above may be applied to get that the set
\[
\{ x \mid g(x) - h_n(x) = 0 \}
\]
is measurable \( \forall n \). Now \( f_n(x) \) converges at \( x \in X \) if and only if \( g(x) = h(x) \in (-\infty, \infty) \),
and this happens if and only if \( \exists N_x \in \mathbb{N} \) s.t. \( g(x) = h_n(x), \forall n \geq N \). Thus,
\[
\{ x \mid f_n(x) \text{ converges} \} = \bigcup_{N=1}^{\infty} \bigcap_{n \geq N} \{ x \mid g(x) - h_n(x) = 0 \},
\]
a measurable set.