Rejection region for 100 coin flips

From the command

```r
qbinom(.025, 100, .5)
```

## [1] 40

we learn that, for a binomial random variable \( X \sim \text{Binom}(100, .5) \), the cumulative probability up to but not including \( X = 40 \) is 0.025. Actually, that is not quite true, since

```r
pbinom(40, 100, .5)
```

## [1] 0.02844397

```r
pbinom(39, 100, .5)
```

## [1] 0.0176001

which shows \( P(X \leq 39) = 0.0176 \), while \( P(X \leq 40) = 0.0284 \); we cannot hit 0.025 exactly. Since, the two-tailed area

\[
P(X \leq 39 \text{ or } X \geq 61) = 2 \cdot P(X \leq 39) \approx 0.0352,
\]

while the two-tailed area

\[
P(X \leq 40 \text{ or } X \geq 60) = 2 \cdot P(X \leq 39) \approx 0.05688,
\]

the former is the appropriate rejection region for an hypothesis test involving the count of heads in 100 flips from a coin with hypotheses

\[
H_0: \pi = 0.5, \quad H_a: \pi \neq 0.5
\]

and significance level \( \alpha = 0.05 \). We display the null distribution along with the rejection region in red:

```r
plotDist("binom", params=c(100, .5), col=c("red","forestgreen"),
    groups=abs(x-50) <= 10)
```
Computing $\beta$, the probability of Type II Error

Suppose our coin actually has a probability of landing “heads” equaling 0.75. Then, counter to what is hypothesized in $H_0$, $X \sim \text{Binom}(100, 0.75)$. We overlay this distribution (displayed in gray) with the null distribution.

```r
plotDist("binom", params=c(100, .5), col=c("red","forestgreen"),
          groups=abs(x-50) <= 10, xlim=c(30,80), ylim=c(0,0.1))
plotDist("binom", params=c(100, .75), col="gray60", add=TRUE)
```

The probability of making a Type II error, $\beta$, should be small, as the likelihood of values from our coin (with $\pi_a = 0.75$) falling in the green region (where the null hypothesis is not rejected) appears to be small. We can find its actual value with commands like

```r
sum(dbinom(40:60, 100, 0.75))
```

```
## [1] 0.0006665922
```
or

\[ \text{pbinom}(60, 100, .75) - \text{pbinom}(39, 100, .75) \]

## [1] 0.0006865922

Now, if our coin has a probability of “heads” equaling 0.55, the likelihood of Type II error should rise. The gray distribution (corresponding to how the coin actually behaves) has a lot more of its probability lying inside the nonrejection region.

\[ \text{plotDist}("\text{binom}", \text{params}=c(100, .5), \text{col}=\text{c("red","forestgreen")}, \]
\[ \text{groups}=\text{abs}(x-50) <= 10, \text{xlim}=c(30,80), \text{ylim}=c(0,0.1)) \]
\[ \text{plotDist}("\text{binom}", \text{params}=c(100, .55), \text{col}=\text{"gray60"}, \text{add=TRUE}) \]

We compute \( \beta \) as before, seeing (as predicted) it is much larger than before.

\[ \text{pbinom}(60, 100, .55) - \text{pbinom}(39, 100, .55) \]

## [1] 0.8648077

So, \( \beta \) can only be calculated when we make a presumption about \( \pi_a \), the probability our coin produces a “head”. Not only does its value depend on how far away the true value of \( \pi \) is from what is hypothesized, but it also depends on the choice of significance level \( \alpha \).

**Power**

Power is defined as the probability a false null hypothesis is rejected. So

\[
\text{power} = 1 - P(\text{not rejecting a false } H_0) = 1 - \beta.
\]

Like \( \beta \), it relies on \( \alpha \) and knowledge of \( \pi_a \), making it difficult to calculate. We may illustrate how the power of a binomial test changes as \( \pi_a \) changes.
\begin{verbatim}
piAlt = seq(0, 1, .02)
myBeta = pbinom(60, 100, piAlt) - pbinom(39, 100, piAlt)
xyplot(1-myBeta ~ piAlt, type="l", xlab="probability of success", ylab="Power")

enn = 1:2000
critical = qbinom(.025, enn, .5)
beta = pbinom(enn-critical,enn,.55) - pbinom(critical-1,enn,.55)
xyplot(1-beta ~ enn, type="l", lwd=0.5, xlab="n", ylab="power")
\end{verbatim}

You can, in fact, increase the power of a binomial test at any fixed value of $\pi_a$ and $\alpha$ by increasing the sample size $n$. Our next plot gives power for different choices of $n$, assuming that $\pi_a = 0.55$ and $\alpha = 0.05$. 