3.8.2 Find the Jacobian matrix $J_F(x, y)$ at the point $(-1, 4)$ for the function $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by

$$F(x, y) = \begin{bmatrix} F_1(x, y) \\ F_2(x, y) \end{bmatrix} = \begin{bmatrix} x^3 - y^2 + y \\ xy + x^2 \end{bmatrix}.$$ 

3.8.5 Solve the nonlinear system of equations

$$10 - x + \sin(x + y) - 1 = 0$$
$$8y - \cos^2(z - y) - 1 = 0$$
$$12z + \sin z - 1 = 0$$

using Newton’s method with $x_0 = 0.1, y_0 = 0.25$ and $z_0 = 0.08$.

3.6.6 Consider the system of two equations

$$f_1(x, y) = x^2 + ay^2 - 1 = 0$$
$$f_2(x, y) = (x - 1)^2 + y^2 - 1 = 0$$

where $a$ is a parameter.

(a) Write down two explicit formulas for Newton’s iterations for this system. First, write it in the form

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - J^{-1}(x_n, y_n) \begin{bmatrix} f_1(x_n, y_n) \\ f_2(x_n, y_n) \end{bmatrix},$$

where you explicitly exhibit the Jacobian $J$ and its inverse $J^{-1}$. Second, evaluate this expression explicitly—i.e., multiply it out and simplify.

(b) Using these formulas, write a (very short) Octave/Matlab program to implement Newton iteration just for this example. Try it for $a = 1$.

⋆16 Let $S \subset \mathbb{R}^n$ be open and convex (convexity means that for each pair of vectors $x, y \in S$, the entire line segment $x + t(y - x), 0 \leq t \leq 1$ lies in $S$), and suppose $F: S \rightarrow S$,
given by

\[ F(x) = F(x_1, \ldots, x_n) = \begin{bmatrix} F_1(x_1, \ldots, x_n) \\ F_2(x_1, \ldots, x_n) \\ \vdots \\ F_n(x_1, \ldots, x_n) \end{bmatrix}, \]

is differentiable in \( S \). Let \( x, y \in S \), and define the function \( g: [0, 1] \to S \) by

\[ g(t) = F(x + t(y - x)). \]

Show that

\[ g'(t) = J(x + t(y - x))(y - x), \]

where \( J(x) \) is the Jacobian matrix given by

\[ J(x) = J(x_1, \ldots, x_n) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(x) & \cdots & \frac{\partial F_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1}(x) & \cdots & \frac{\partial F_n}{\partial x_n}(x) \end{bmatrix}. \]

Note that this is an exercise in using the chain rule.