Numerical approximations to definite integrals

- Riemann (rectangle) sums already give us approximations
  
  Main types: left-hand, right-hand and midpoint rules

- Question: Why rectangles?
  
  - trapezoids
    
    * Area of a trapezoid with bases $b_1$, $b_2$, height $h$

    * Approximation to $\int_a^b f(x) \, dx$ using $n$ steps all of width $\Delta x = (b - a)/n$ (Trapezoid Rule)

  - Remarkable fact: Trapezoid rule does not improve over midpoint rule.

  - parabolic arcs

    * $\int_{-h}^h g(x) \, dx$, when $g(x) = Ax^2 + Bx + C$
      is chosen to pass through $(-h, y_0)$, $(0, y_1)$
      and $(h, y_2)$

    * Approximation to $\int_a^b f(x) \, dx$ using $n$ (even) steps all of width $\Delta x$ (Simpson’s Rule)
• Error bounds
  
  – No such thing available for a general integrand $f$
  – Formulas (available when $f$ is sufficiently differentiable)

  * **Trapezoid Rule.** Suppose $f''$ is continuous throughout $[a, b]$, and $|f''(x)| \leq M$ for all $x \in [a, b]$. Then the error $E_T$ in using the Trapezoid rule with $n$ steps to approximate $\int_a^b f(x) \, dx$ satisfies

    $$|E_T| \leq \frac{M(b-a)^3}{12n^2}.$$ 

  * **Simpson’s Rule.** Suppose $f^{(4)}$ is continuous throughout $[a, b]$, and $|f^{(4)}(x)| \leq M$ for all $x \in [a, b]$. Then the error $E_S$ in using Simpson’s rule with $n$ steps to approximate $\int_a^b f(x) \, dx$ satisfies

    $$|E_S| \leq \frac{M(b-a)^5}{180n^4}.$$ 

  – Use

    * For a given $n$, gives an upper bound on your error
    * If a desired upper bound on error is sought, may be used to determine *a priori* how many steps to use