Problem 3.17

Statement. Suppose \( f : D \to \mathbb{R} \) and \( f(x) \geq 0 \) for all \( x \in D \). Show that if \( f \) is continuous at \( x_0 \), then \( \sqrt{f} \) is continuous at \( x_0 \).

There are several ways to approach this problem. Some advice: Don’t make it too hard; but do make sure you give reasons for your steps. Here is one example proof.

Proof. If \( x_0 \) is not an accumulation point of \( D \), then \( \sqrt{f} \) is continuous at \( x_0 \). (Why? Be sure you know.)

Suppose \( f \) is as described and continuous at \( x_0 \) and that \( x_0 \) is an accumulation point of \( D \). This means that \( \lim_{x \to x_0} f(x) = f(x_0) \) (Thm 3.1). We have already shown that for any nonnegative function \( f \), \( \lim_{x \to x_0} \sqrt{f(x)} = \sqrt{\lim_{x \to x_0} f(x)} \).

So \( \lim_{x \to x_0} \sqrt{f(x)} = \sqrt{f(x_0)} \). But this means that \( \sqrt{f} \) is continuous at \( x_0 \) (Thm 3.1 again).

Note: You may still use this problem for your portfolio if you give a proof different from the one above.

Problem 3.19b

Several of you were too quick to conclude that the proof for products would work just like the proof for sums. This is not the case. In fact, a product of uniformly continuous functions is not, in general, uniformly continuous. One example is \( f(x) = x \), \( g(x) = x \) and \( f(x)g(x) = x^2 \). \( f \) and \( g \) are uniformly continuous on \( \mathbb{R} \), but \( fg \) is not.
More interestingly, if you attempt the proof for the product case, you will discover that with one additional assumption about the functions \( f \) and \( g \), the proof will work: \( f \) and \( g \) must be bounded as well as uniformly continuous. Some of you actually did this proof but seemed to think that all functions are bounded.

**Problem 3.26**

The biggest problem here was getting off to a bad start. We need to show that \( E \) is closed (given some assumptions about \( E \)), so a good start for the proof is something like: “Let \( x_0 \) be an accumulation point of \( E \).” Of course, you then need to use the assumptions about \( E \) to show that \( x_0 \in E \).

A bad outline for the proof is one that tries to show something is an accumulation point. (If you did this, you probably got the comment “Backwards” written on your paper, since you are proving something that should be the hypothesis.)

**Problem 3.30**

Because I was running low on time, I did not grade this problem. I know it gave many of you difficulty, because I spoke to many of you about it. Here is a complete solution.

**Statement.** Suppose \( f : \mathbb{R} \rightarrow \mathbb{R} \) is continuous and let \( r_0 \in \mathbb{R} \). Then \( A = \{ x \in \mathbb{R} : f(x) \neq f(r_0) \} \) is an open set.

**Proof.** The direct way to show \( A \) is open is to show that every point in \( A \) is surrounded by a neighborhood (an interval) that is a subset \( A \).

So let \( x_0 \in A \), and let \( \varepsilon = | f(x_0) - f(r_0) | \). Then \( \varepsilon > 0 \) because \( f(x_0) \neq f(r_0) \). Since \( f \) is continuous, there is a positive real number \( \delta \) such that if \( | x - x_0 | < \delta \), then \( | f(x) - f(x_0) | < \varepsilon \). This means that for all \( x \in (x_0 - \delta, x_0 + \delta) \), \( | f(x) - f(x_0) | < \varepsilon \). But this means that \( f(x) \neq f(r_0) \) since \( | f(r_0) - f(x_0) | = \varepsilon \). \( \square \)

**Problem 3.36**

**Statement.** A finite union of compact sets is compact.
Proof. Suppose $E = \bigcup_{i=1}^{n} E_i$ and that each $E_i$ is compact. Let $\{G_\alpha : \alpha \in A\}$ be an open cover of $E$. (One of the most common mistakes in this proof was not starting with an arbitrary open cover of $E$. Remember, we need to prove something about ALL open covers of $E$.)

Since this open cover also covers each $E_i$, and each $E_i$ is compact, there are finite subcovers of $\{G_\alpha : \alpha \in A\}$ for each $E_i$. The union of these finite subcovers is a finite collection of open sets (a finite union of finite sets), and covers $E$ (because it covers each $E_i$. This shows that $E$ is compact. \[ \square \]

Many of you were at least on the write track here, but there were some difficulties writing things up and some difficulties with a clear understanding that you need to prove something about all covers of $E$. 