**Problem 0.18**

**Theorem:** If $f : A \rightarrow B$ is 1-1 and $\text{im} f = B$ (i.e., $f$ is also onto), then

- $(f^{-1} \circ f)(a) = a$ for each $a \in A$.
- $(f \circ f^{-1})(b) = b$ for each $b \in B$.

**Proof:** Since $f$ is 1-1 and onto, the inverse function $f^{-1} : B \rightarrow A$ exists. Let $a$ be an arbitrary element of $A$, and let $b = f(a)$. Then $f^{-1}(b) = a$, so $f^{-1}(f(a)) = f^{-1}(b) = a$. This demonstrates the first claim.

Now let $b$ be an arbitrary element of $B$. Since $f$ is onto, there is an element $a \in A$ such that $f(a) = b$. $f^{-1}(b) = a$ by the definition of an inverse, so we have $f(f^{-1}(b)) = f(a) = b$. This demonstrates the second claim.
Problems 0.19, 0.20 and 0.24

1. If you use notation like $P(n)$, be sure to define it. In 0.20, for example, it is fine to say something like: Let $P(n)$ be the statement “$1 + 3 + 5 + \cdots + 2n - 1 = n^2$”. This will let you say things like: $P(n)$ is true. But don’t say things like $P(n) = n^2$ as this is just plain silly. A statement can’t equal $n^2$.

2. Be sure that the flow of the argument in the inductive part of your proof is clear.
   - Don’t assume the conclusion. (Or make it appear that you are doing so.)
   - Make it clear what you are assuming in the inductive step and how you make use of the assumption.
   - Be sure it is clear just which form of induction you are using. Make sure the form you choose is strong enough for your proof. In Problem 0.24, for example, you need to use a stronger form of induction (something like the one from Problem 0.28).

   If you have some comment about “flow” by your proof, be sure you understand what you did incorrectly.

3. Generally, most of you seem to have the basic idea of inductive proofs but need to pay attention to some details in writing them up.
Problem 0.28

Statement of problem omitted, see text.

Proof 1: Assume for sake of contradiction that (a) and (b) hold, but that there is some $n \in \mathbb{Z}$ such that $n \geq n_0$ and $P(n)$ is false. Let $N$ be the least such integer (this exists by the modified version of the well-ordering principle, page 15). So $P(N)$ is false, but if $n_0 \leq i < N$, then $P(i)$ is true.

By (a), $N > m$. And since $N > m$, and for all $i$ with $n_0 \leq i < N$, $P(i)$ is true, by (b) we see that $P(N)$ must be true. But this contradicts the fact that $P(N)$ is false. Thus whenever (a) and (b) hold, it must be that $P(i)$ is true for all integers $i \geq n_0$.

Proof 2: Let $Q(n)$ be the statement “$P(i)$ is true for all $i$ with $n_0 \leq i < m + n$”. Then (a) implies that $Q(1)$ is true. And (b) says that if $k \in \mathbb{J}$ and $Q(k)$ is true, then $Q(k + 1)$ is true. So by the first form of induction, $Q(n)$ is true for all $n \in \mathbb{J}$.

To make sure you understand this proof, it might be good to write down statements $Q(1), Q(2),$ and $Q(10)$. 