DO NOT start Mathematica.
If you already have, please exit Mathematica before continuing.

1 The definition

Recall the definition of limit: \( \lim_{x \to a} f(x) = L \) means

- for any \( \varepsilon > 0 \), no matter what positive number Alice picks \( \varepsilon \)
- one can find a number \( \delta > 0 \), Bob can find a positive number \( \delta \)
- such that
- if \( 0 < |x - a| < \delta \), such that
- then \( |f(x) - L| < \varepsilon \).
- if \( x \) is within \( \delta \) of \( a \) (but \( x \neq a \))
- then \( f(x) \) is as close to \( L \) as Alice specified.

Note: The Greek letters used above are called epsilon (\( \varepsilon \)) and delta (\( \delta \)).

One can think of the definition of limits in terms of a game between two players, Alice and Bob. The game works as follows: Bob proposes a value of the limit \( L \), Alice then challenges Bob to find an interval near \( a \) such that \( f(x) \) is within some small distance \( \varepsilon \) from the limit \( L \), that is Alice chooses \( \varepsilon \) and challenges Bob to make sure that \( f(x) \) should be in the interval \( (L - \varepsilon, L + \varepsilon) \). Bob must then specify the interval of \( x \) values by providing the number \( \delta \), i.e., the interval \( (x - \delta, x + \delta) \). Bob wins if every \( x \) in the interval \( (x - \delta, x + \delta) \) (except possibly \( x = a \)) satisfies \( f(x) \in (L - \varepsilon, L + \varepsilon) \). Alice wins if there is some \( x \) in the interval \( (x - \delta, x + \delta) \) (but not \( x = a \)) such that \( |f(x) - L| > \varepsilon \).

If the limit is indeed \( L \), Bob will be able to find a \( \delta \) for any such \( \varepsilon \) chosen by Alice. That is, Bob can always win. If there is some \( \varepsilon \) that Alice could pick for which Bob has no winning \( \delta \), then Alice can win and the limit is not \( L \).

We can picture these intervals on a graph like the one below:

2 Applying the Definition

The Epsilon-Delta Applet provides an interactive version of this picture that we will use in this lab to explore the definition of limit a bit further. Load the applet using Internet Explorer (not Netscape). Select “Epsilon-Delta Applet” from the list at

http://www.calvin.edu/~rpruim/courses/m161/F01/java/

Each function below has already been entered in the examples menu of the applet. Remember: Mathematica should NOT be running.

1. Let \( f(x) = 6x - x^2 \). Consider \( \lim_{x \to 2} f(x) \).

   a) What is \( \lim_{x \to 2} f(x) \)? How does this show up on the graph?
   b) If Alice picks \( \varepsilon = 0.3 \) and Bob picks \( \delta = 0.1 \), who wins?
   c) If Alice picks \( \varepsilon = 0.03 \) and Bob picks \( \delta = 0.01 \), who wins?
   d) If Alice picks \( \varepsilon = 0.003 \) and Bob picks \( \delta = 0.001 \), who wins?
   e) Based on the results above and the graphs you have looked at, if Alice picks some \( \varepsilon > 0 \), what do you think Bob should pick for \( \delta \)? (You don’t have to prove that this works, but it does as long as \( \varepsilon \) is small.)
2. Let \( f(x) = x^2 - 2x - 1 \). Consider \( \lim_{x \to 3} f(x) \).

a) Since \( f \) is a \( \frac{x^2 - 2x - 1}{x-3} \), \( L = \lim_{x \to 3} f(x) \) is easy to compute, namely \( L = \underline{\text{_____}} \). Enter this value on the second variable input line where it says “test limit \( L = \)”. (The value of \( a = 3 \) has already been set correctly.)

b) If Alice picks \( \varepsilon = \frac{1}{2} \) and Bob picks \( \delta = \frac{1}{4} \), who wins? Explain how you know in terms of the graph given by the applet. (You may need to zoom in.)

c) If Alice picks \( \varepsilon = 1 \) and Bob picks \( \delta = \frac{1}{5} \), who wins? Explain how you know in terms of the graph given by the applet. (You may need to zoom in.)

d) If Alice picks \( \varepsilon = 1 \), what is the largest value Bob can pick for \( \delta \) and still win? (Approximate this as well as you can using the graph.)

3. Let \( f(x) = x^2 + 1 \). Consider \( \lim_{x \to 0} f(x) \).

a) What is \( \lim_{x \to 0} f(x) \)?

b) If Alice picks \( \varepsilon = 0.25 \), what should Bob choose for \( \delta \)?

c) If Alice picks \( \varepsilon = 0.09 \), what should Bob choose for \( \delta \)?

d) Show that if Alice picks \( \varepsilon \) and Bob picks \( \delta = \sqrt{\varepsilon} \), then Bob wins. (This proves that the limit is what Bob claims it is.)

4. Let \( f(x) \) be the function in Example 4 of the applet. \( \lim_{x \to 2} f(x) \) does not exist. This means Alice should always be able to win the game. How should Alice pick \( \varepsilon \) in order to win?

5. Let \( f(x) \) be the function in Example 5 of the applet. Does \( \lim_{x \to 1} f(x) \) exist? Explain.

6. Let \( f(x) = 2x \sin(1/x) \).

a) What is \( \lim_{x \to 0} f(x) \)?

b) If Alice picks \( \varepsilon = 0.5 \), what should Bob choose for \( \delta \)?

c) If Alice picks \( \varepsilon = 0.1 \), what should Bob choose for \( \delta \)?

d) If Alice picks \( \varepsilon = 0.01 \), what should Bob choose for \( \delta \)?

e) If Alice picks some \( \varepsilon > 0 \), what should Bob choose for \( \delta \)? Prove that your choice works.