Preface

This is a textbook for an undergraduate course in axiomatic geometry. The course is aimed at mathematics majors who have completed the calculus sequence and perhaps a first course in linear algebra, but who have not yet encountered such upper-level mathematics courses as real analysis and abstract algebra. The course will normally be taken by students who are at the junior or senior level, but well-prepared sophomores also benefit from the course.

THE FOUNDATIONS OF GEOMETRY

The primary goal of the course is to study the foundations of geometry. That means returning to the beginnings of geometry, exposing exactly what is assumed there, and building the entire subject on those foundations. Such careful attention to the foundations has a long tradition in geometry, going back more than two thousand years to Euclid and the ancient Greeks. Over the years since Euclid wrote his famous *Elements*, there have been profound changes in the way in which the foundations have been understood. Most of those changes have been byproducts of efforts to understand the true place of Euclid's parallel postulate in the foundations, so the parallel postulate is one of the primary emphases of this book.

ORGANIZATION OF THE BOOK

The course begins with a quick look at Euclid's *Elements*, and Euclid's system of organization is used as motivation for the concept of an axiomatic system. A system of axioms for geometry is then carefully laid out. The axioms used here are based on the real numbers, in the spirit of Birkhoff, and their statements have been kept as close to those in contemporary high school textbooks as is possible.

After the axioms have been stated and certain foundational issues faced, neutral geometry, in which no parallel postulate is assumed, is extensively explored. Next both Euclidean and hyperbolic geometries are investigated from an axiomatic point of view. In order to get as quickly as possible to some of the interesting results of non-Euclidean geometry, the first part of the book focuses exclusively on results regarding lines, parallelism, and triangles. Only after those subjects have been treated separately in neutral, Euclidean, and hyperbolic geometries are results on area, circles, and construction introduced. While the treatment of these subjects does not exactly follow Euclid, it roughly parallels Euclid in the sense that Euclid collected most of his propositions about area in Book II and most of his propositions about circles in Books III and IV. The three chapters covering area, circles, and construction complete the coverage of the major theorems of Books I through VI of the *Elements*.

Next, the more modern notion of a transformation is introduced and some of the standard results regarding transformations of the plane are explored. A complete proof of the classification of the rigid motions of both the Euclidean and hyperbolic planes is included. There is a discussion of how the foundations of geometry can be reorganized to reflect the transformational point of view (as is common practice in contemporary high school geometry textbooks). Specifically, it is possible to replace the Side-Angle-Side Postulate with a postulate that asserts the existence of certain reflections.

The standard models for hyperbolic geometry are carefully constructed and the results of the chapter on transformations are used to verify their properties. The chapter on models can be relatively short because all the hard technical work involved in the constructions is done in the preceding chapter. The final chapter includes a study of some of the polygonal models that have recently been developed to help students understand what it means to say that hyperbolic space is negatively curved. The book ends with a discussion of the practical significance of non-Euclidean geometry and a brief look at the geometry of the real world.

PROOFS

An important secondary goal of the course is to teach the art of writing proofs. There is a growing recognition of the need for a course in which mathematics students learn how to write good proofs. Such a course should serve as a bridge between the lower-level mathematics courses, which are largely technique oriented, and the upper-level courses, which tend to be much more conceptual. This book uses geometry as the vehicle for helping students to write and appreciate proofs. The ability to write proofs is a skill that can only be acquired by actually practicing it, so most of the material on writing proofs is integrated into the course and the attention to proof permeates the entire text. This means that the book can also be used in classes where the students already have experience writing proofs; despite the emphasis on writing proofs, the book is still primarily a geometry text.

Having the geometry course serve as the introduction to proof represents a return to tradition in that the course in Euclidean geometry has for thousands of years been seen as the standard introduction to logic, rigor, and proof in mathematics. Using the geometry course this way makes historical sense because the axiomatic method was first introduced in geometry and geometry remains the branch of mathematics in which that method has had its greatest success. While proof and logical deduction are still emphasized in the standards for high school mathematics, most high school students no longer take a full-year course devoted exclusively to geometry in which there is a sustained introduction to proof. This makes it more important than ever that we teach a good college-level geometry course to all mathematics students. By doing so we can return geometry to its place as the subject in which students first learn to appreciate the importance of clearly spelling out assumptions and deducing results from those assumptions via careful logical reasoning.

The emphasis on proofs makes the course into a do-it-yourself course in that the reader will be asked to supply proofs for many of the key theorems. Students who diligently work the exercises come away from the course with a sense that they have an unusually deep understanding of the material. In this way the student should not only learn the mechanics of good proof writing style but should also come to more fully appreciate the important role proof plays in an understanding of mathematics.

NATIONAL STANDARDS

A third major goal of the course is to implement the recommendations in the recent report on “The Mathematical Education of Teachers” (MET) [12]. Those recommendations, in turn, are based on the “Principles and Standards for School Mathematics” of the National Council of Teachers of Mathematics [54]. The basic recommendation is that “Prospective teachers need mathematics courses that develop a deep understanding of the mathematics they will teach” [12, Part I, page 7]. This course is designed to do precisely that in the area of geometry. A second basic recommendation in MET is that courses for prospective mathematics teachers should make explicit connections with high school mathematics. Again, this book attempts to implement that recommendation in geometry. The goal is to implement the basic recommendations of MET, not necessarily to cover every geometric topic that future teachers need to see; some geometric topics will be included in other courses, such as linear algebra.

An example of the way in which connections with high school geometry have influenced the design of the course is the choice of the axioms that are used as the starting point. The axioms on which the development of the geometry in the text is based are almost exactly those that are used in high school textbooks. While most high school textbooks still include an axiomatic treatment of geometry, there is no standard set of axioms that is common to all high school geometry courses. Therefore, various axiom systems are considered in the text and the merits and advantages of each are discussed. The axioms on which the course is ultimately based are as close as possible to those in the Geometry textbook in the University of Chicago School Mathematics Project (UCSMP) series [71], a textbook that is in wide use at the present time. One of the main goals of the course is to help preservice teachers understand the logical foundations of the geometry course they will teach and that can best be accomplished in the context of axioms that are like the ones they will encounter later in the classroom. There are many other connections with high school geometry that are brought in as the course progresses.

Some of the newer high school mathematics curricula present mathematics in an integrated way that emphasizes connections between the branches of mathematics. There is no separate course in geometry, but rather a geometry thread is woven into all the high school mathematics courses. In order to teach such a course well, the teacher herself needs to have an understanding of the structure of geometry as a coherent subject. This book is intended to provide such an understanding.

One of the recurring themes in “The Mathematical Education of Teachers” [12] is the recommendation that prospective teachers must acquire an understanding of high school mathematics that goes well beyond that of a typical high school graduate. One way in which such understanding of geometry is often measured is in terms of the van Hiele model of geometric thought. This model is described in

Appendix D. The goal of most high school courses is to develop student thinking to Level 3. A goal of this course is to bring students to Level 4 (or to Level 5, depending on whether the first level is numbered 0 or 1). It is recognized, however, that not all students entering the course are already at Level 3 and so the early part of the course is designed to ensure that students are brought to that level first.

HISTORICAL AND PHILOSOPHICAL PERSPECTIVE

A final goal of the course is to present a historical perspective on geometry. Geometry is a dynamic subject that has changed over time. It is a part of human culture that was created and developed by people who were very much products of their time and place. The foundations of geometry have been challenged and reformulated over the years, and beliefs about the relationship between geometry and the real world have been challenged as well.

The material in the book is presented in a way that is sensitive to such historical and philosophical issues. This does not mean that the material is presented in a strictly historical order or that there are lengthy historical discussions but rather that geometry is presented in such a way that the reader can understand and appreciate the historical development of the subject and so that it would be natural to investigate the history of the subject while learning it. Many chapters include suggested readings on the history of geometry that can be used to enrich the course.

Throughout the book there are references to philosophical issues that arise in geometry. For example, one question that naturally occurs to anyone studying non-Euclidean geometry is this: What is the connection between the abstract entities that are studied in a course on the foundations of geometry and properties of physical space? The book does not present dogmatic answers to such questions, but instead simply raises them in an effort to promote student thinking. The hope is that this will serve to counter the common perception that mathematics is a subject in which every question has a single correct answer and in which there is no room for creative ideas or opinions.

TECHNOLOGY

In recent years powerful computer software has been developed that can be used to explore geometry. The study of geometry from this book can be greatly enhanced by such dynamic software and the reader is encouraged to find appropriate ways in which to incorporate this technology into the geometry course. While software can enrich the experience of learning geometry from this book, its use is not required. The book can be read and studied quite profitably without it.

There are several commercially available pieces of software designed for use in a course such as this, and any one of them will serve the purpose. *Geometer's Sketchpad*™ (Key Curriculum Press) is widely used and readily available; it is probably the natural choice if you are just starting out. *Cabri Geometry*™ (Texas Instruments) is less commonly used in college-level courses, but it is also completely adequate. It has some predefined tools, such as an inversion tool and a test for collinearity, that are not included in Sketchpad. *Cinderella*™ (Springer-Verlag) is a newer piece of software that is also very good. It has the advantage that it allows diagrams to be drawn in all three two-dimensional geometries: Euclidean, hyperbolic, and spherical. Another advantage is that it allows diagrams to be easily exported as Java applets. A program called NonEuclid is freely available on the World Wide Web and it can be used to enhance the non-Euclidean geometry in the course. New software is being produced all the time, so you may find that other products are available to you.

This is a course in the foundations of axiomatic geometry, and software will necessarily play a more limited role in such a course than it might in other kinds of geometry courses. Nonetheless, there is an appropriate role for software in a course such as this and the author hopes that the book will demonstrate that. There is no reason for those who love the proofs of Euclid to resist the use of technology. After all, Euclid himself made use of the limited technology that was available to him, namely the compass and straightedge. In the same way we can make good use of modern technology in our study of geometry. It is especially important that future high school teachers learn to understand and appreciate the appropriate use of technology.

In the first part of the course (Chapters 1 through 6), the objective is to carefully expose all the assumptions that form the foundations of geometry and to understand for ourselves how the basic results of geometry are built on those foundations. For most users the software is a black box in the sense that we either don't know what assumptions are built into it or we have only the authors' description of what went into the software. As a result, software is of limited use in this part of the course and it will not be mentioned explicitly in the first six chapters of the book. But you should be using it to draw diagrams and to experiment with what happens when you vary the data in the theorems. During that phase of the course the main function of the software is to illustrate the relationships being studied.

It is in the second half of the course that the software comes into its own. Computer software is ideal for experimenting, exploring, and discovering new relationships. In order to illustrate that, several of the later chapters include sections in which the software is used to explore ideas that go beyond those that are presented in detail and to discover new relationships. In particular, there are such exploration sections in the chapters on Euclidean geometry and circles. The entire chapter on constructions is written as an exploration with only a limited number of proofs or hints provided in the text.

The exploratory sections of the text have been expanded into a laboratory manual entitled *Exploring Advanced Euclidean Geometry with Geometer's Sketchpad*. This document may be downloaded, free of charge, from the textbook's website at `http://calvin.edu/~venema/geometrybook.html`. The manual includes complete instructions on the use of Geometer's Sketchpad. Laboratory exercises guide students as they discover and explore many of the theorems of advanced Euclidean geometry, including the theorems of Ceva and Menelaus as well as the nine-point circle theorem. Written exercises outline the proofs of the major theorems. The final chapter explores constructions in the Poincaré disk model of hyperbolic geometry.

DESIGNING A COURSE

A full-year course will cover essentially all the material in the text. There can be some variation based on instructor and student interest, but most or all of every chapter should be included.

An instructor teaching a one-semester or one-quarter course will be forced to pick and choose. It is important that this be done carefully so that the course reaches some of the interesting and useful material that is to be found in the second half of the book.

Chapters 1 and 2 set the stage for what comes later, so they should definitely be covered in some way. But they can be treated lightly in order to free up time for other things. Topics from Chapters 3 and 4 should be covered as needed, depending on student background. The basic coverage of geometry begins with Chapter 5. Chapters 5 and 6 form the heart of a one-semester course. Those chapters should be included in any course taught from the book. At least some of Chapter 7 should also be included in any course. Starting with Chapter 8, the chapters are largely independent of each other and an instructor can select material from them based on the interests and needs of the class.

Several sample course outlines are included below. Many other variations are possible. It should be noted that the suggested outlines are ambitious and many instructors will choose to cover less.

The suggested course for future high school teachers includes just a brief introduction to each of the topics in later chapters. The idea is that the course should provide enough background so that students can study these topics in more depth later if they need to. It is hoped that this book can later serve as a valuable reference for those who go on to teach geometry courses. The book could be a resource that provides information about rigorous treatments of such topics as parallel lines, area, circles, constructions, transformations, and so on, that are part of the high school curriculum.

POSSIBLE ONE-SEMESTER COURSE OUTLINES

A course emphasizing Euclidean Geometry

Chapter	Topic	Number of weeks
1–4	Preliminaries	$\leqslant 2$
5	Axioms	2
6	Neutral geometry	3
7	Euclidean geometry	2
9	Area	1–2
10	Circles	1–2
12	Transformations	2

A course emphasizing non-Euclidean Geometry

Chapter	Topic	Number of weeks
1–4	Preliminaries	$\leqslant 2$
5	Axioms	2
6	Neutral geometry	3
7	Euclidean geometry	1
8	Hyperbolic geometry	2
9	Area	1–2
13	Models	1–2
14	Geometry of space	1

A course for future high school teachers

Chapter	Topic	Number of weeks
1–4	Preliminaries	$\leqslant 2$
5	Axioms	2
6	Neutral geometry	3
7	Euclidean geometry	1
8	Hyperbolic geometry	1
9	Area	1
10	Circles	1
12	Transformations	1
13	Models	1
14	Geometry of space	1

SUPPLEMENTS

Two supplements are available: an *Instructors' Manual* and a computer laboratory manual entitled *Exploring Advanced Euclidean Geometry with Geometer's Sketchpad*.

The *Instructors' Manual* contains information about how to teach from the book. In particular, it includes suggestions about what a one-semester course should cover from each chapter and what can be omitted. There is also a table that shows dependencies among the various sections. The *Instructors' Manual* contains complete solutions to all of the exercises. Instructors should contact their local Prentice Hall sales representative or the Prentice Hall offices in Upper Saddle River, New Jersey, to obtain a copy.

The laboratory manual was described earlier in the Preface, under Technology. It may be downloaded, free of charge, from the website

`http://calvin.edu/~venema/geometrybook.html.`

The website contains supplementary material, more information about the book, and errata.

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