

A NEW PROOF OF THE TRIVIAL RANGE COMPLEMENT THEOREM

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June, 1988

In this note we give a new proof of the following theorem.

Theorem. *Suppose M^n is a compact PL n -manifold and X and Y are compact subsets of the interior of M^n which have fundamental dimension less than or equal to k , $2k + 2 \leq n$, and which satisfy the inessential loops condition. If $M - X$ and $M - Y$ are homeomorphic, then X and Y have the same shape.*

For $n \geq 5$, this is one half of a theorem which was first stated in [4]. The proof given here is simpler than the previous proofs of the theorem and also improves on those proofs in that it applies in all dimensions; in particular, it is valid when $n = 4$. The reason that the standard proof for the theorem is only valid for $n \geq 5$ is that it makes use of the converse of the theorem (same shape implies homeomorphic complements) and the converse is still not known to hold in dimension 4. In the proof given here, we depart from the standard pattern (which dates back to Chapman's original proof of the complement theorem) of beginning the proof by re-embedding the compacta in a hyperplane. The reader is referred to [2] for an explanation of the standard proof as well as for a complete history of the finite dimensional complement theorem in shape theory. All relevant terminology and notation are also explained in [2]. It should be pointed out that Peter Mrozik [1, Corollary 4.6] has recently given an independent proof of the theorem for the case $M^n = S^n$ which is also valid in all dimensions.

Remark. The assumption that $M - X$ and $M - Y$ are homeomorphic can be replaced by the weaker hypothesis that $M - X$ and $M - Y$ have the same weak proper homotopy type. The proof of the stronger theorem is essentially the same as the one given below.

Proof of the Theorem. Let $h : M - X \rightarrow M - Y$ be a homeomorphism. Since we know that $\text{Fd}(X) \leq k$, $2k + 2 \leq n$, and X satisfies ILC, we can apply [3] and [4] to see that $\pi_q(U, U - X) = 0$ for every neighborhood U of X and for all $q \leq k + 1$. The fact that $\text{Fd}(X) \leq k$ also implies that for every neighborhood U of X there exists a smaller neighborhood U' of X and a k -dimensional polyhedron K in U

such that the inclusion map $U' \hookrightarrow U$ is homotopic in U to a map $\alpha : U' \rightarrow K$ [4, Proposition 2.1]. The fact that $\pi_q(U, U - X) = 0$ for $q \leq k$ means we may assume that $K \subset U - X$. Similar properties hold for Y . We can therefore construct defining sequences $\{U_i\}_{i=0}^\infty$ and $\{V_i\}_{i=0}^\infty$ of neighborhoods of X and Y respectively such that the following properties hold for each $i \geq 1$.

- (1) $U_i \subset \text{Int } U_{i-1}$ and $V_i \subset \text{Int } V_{i-1}$.
- (2) $V_i - Y = h(U_i - X)$.
- (3) There exists a k -dimensional polyhedron $K_i \subset U_{i-1} - X$ and a map $\alpha_i : U_i \rightarrow K_i$ such that α_i is homotopic in U_{i-1} to the inclusion map $U_i \hookrightarrow U_{i-1}$.
- (4) There exists a k -dimensional polyhedron $L_i \subset V_{i-1} - Y$ and a map $\beta_i : V_i \rightarrow L_i$ such that β_i is homotopic in V_{i-1} to the inclusion map $V_i \hookrightarrow V_{i-1}$.

Define $f_i : U_i \rightarrow V_{i-1}$ by $f_i = h \circ \alpha_i$ and define $g_i : V_i \rightarrow U_{i-1}$ by $g_i = h^{-1} \circ \beta_i$. The sequences $\{f_i\}_{i=1}^\infty$ and $\{g_i\}_{i=1}^\infty$ determine shape morphisms from X to Y and Y to X , respectively. We claim that these morphisms are inverse to one another. In order to prove the claim, it suffices to show that $g_{i-1} \circ f_i : U_i \rightarrow U_{i-2}$ and $f_{i-1} \circ g_i : V_i \rightarrow V_{i-2}$ are homotopic to the inclusion maps.

Consider $g_{i-1} \circ f_i = h^{-1} \circ \beta_{i-1} \circ h \circ \alpha_i$. We know that there is a homotopy from $h|_{K_i}$ to $\beta_{i-1} \circ h|_{K_i}$ which takes place in V_{i-2} . Since both $h(K_i)$ and $\beta_{i-1}(h(K_i))$ lie in $V_{i-2} - Y$, we can use the fact that $\pi_q(V_{i-2}, V_{i-2} - Y) = 0$ for $q \leq k + 1$ to push this homotopy off Y (and into the domain of h^{-1}). Thus

$$g_{i-1} \circ f_i = h^{-1} \circ \beta_{i-1} \circ h \circ \alpha_i \simeq h^{-1} \circ h \circ \alpha_i = \alpha_i \simeq \text{inclusion}.$$

The proof for $f_{i-1} \circ g_i$ is similar.

REFERENCES

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2. R. Sher, *Complement theorems in shape theory*, Shape Theory and Geometric Topology, Springer-Verlag, Berlin and New York, 1981, pp. 150–168.
3. G. Venema, *Embeddings of compacta with shape dimension in the trivial range*, Proc. Amer. Math. Soc. **55** (1976), 443–448.
4. ———, *Neighborhoods of compacta in euclidean space*, Fundamenta Math. **109** (1980), 71–78.