LOCAL HOMOTOPY PROPERTIES OF TOPOLOGICAL EMBEDDINGS IN CODIMENSION TWO

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Abstract. Suppose $N$ is an $(n-2)$-manifold topologically embedded in an $n$-manifold $M$. Chapman and Quinn have shown that $N$ is locally flat in $M$ provided that the local homotopy groups of $M - N$ at points of $N$ are infinite cyclic and that all the higher local homotopy groups of $M - N$ at points of $N$ vanish. The main theorem in this paper asserts that it is only necessary to assume the homotopy conditions in dimensions below the middle dimension; i.e., if the local fundamental groups are good and if $M - N$ is locally $k$-connected at points of $N$ for $2 \leq k < n/2$, then $N$ is locally flat.

1. Introduction

Let $N^{n-2}$ be a topological $(n-2)$-submanifold of the topological $n$-manifold $M^n$. (All manifolds are without boundary.) In this paper we wish to study the problem of characterizing local flatness for $N$ in terms of local homotopy properties of the complement $M - N$. We begin with the relevant definitions.

Definitions. Let $x \in N \subset M$. We say that $N$ is locally flat at $x$ if there exists a neighborhood $U$ of $x$ in $M$ such that the pair $(U, U \cap N)$ is homeomorphic with the standard pair $(\mathbb{R}^n, \mathbb{R}^n - 2 \times \{0\})$. We say that $N$ is locally 1-arg at $x$ if for every neighborhood $U$ of $x$ in $M$ there exists a neighborhood $V$ of $x$ in $U$ such that the image of $\pi_1(V - N)$ in $\pi_1(U - N)$ is abelian. We say that $N$ is locally $k$-co-connected (abbreviated $k$-LCC) at $x$ if for every neighborhood $U$ of $x$ in $M$ there exists a neighborhood $V$ of $x$ in $U$ such that the image of $\pi_k(V - N)$ in $\pi_k(U - N)$ is trivial. We say that $N$ is locally homotopically unknotted at $x$ if $N$ is locally 1-arg at $x$ and $N$ is $k$-LCC at $x$ for every $k \geq 2$.

Remark. The “arg” in 1-arg stands for “abelian local groups.” Other authors (e.g. [FQ]) use the terminology “good local $\pi_1$” to mean the same thing. It is not difficult to see that if $N$ is locally 1-arg at $x$, then the image of $\pi_1(V - N)$ in $\pi_1(U - N)$ is infinite cyclic for sufficiently small $U$ and $V$. (See Lemma 2.1, below.)

The basic theorem relating these concepts is due to Chapman and Quinn. Chapman first proved that locally homotopically unknotted codimension two embeddings are locally flat as long as $n \geq 5$ [Ch] but Quinn later gave a different proof of this result [Q1] and also proved the theorem in case $n = 4$ [Q2]. The theorem is true in dimension 3 as well [Ca], but that case is not of interest to us here.

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Theorem [Chapman-Quinn]. Let $N^{n-2}$ be a topological $(n-2)$-submanifold of the $n$-manifold $M^n$. If $N$ is locally homotopically unknotted at $x$ for every $x \in N$, then $N$ is locally flat in $M$.

Note that the definition of locally homotopically unknotted involves assumptions about the local homotopy groups in all dimensions. In this paper we wish to investigate the question of whether or not it might be possible to weaken the hypotheses in the Chapman-Quinn Theorem and still conclude that the embedding is locally flat. In order to understand the question it is helpful to recall the analogous global problem for codimension two knots: If $\Sigma^{n-2} \subset S^n$ is a smooth knot and $\pi_i(S^n - \Sigma^{n-2}) \cong \pi_i(S^1)$ for $i < n/2$, then $\Sigma^{n-2}$ is unknotted [Le]. On the other hand, for each $k < n/2$ there exists a smooth slice knot $\Sigma^{n-2} \subset S^n$ such that $\pi_i(S^n - \Sigma^{n-2}) \cong \pi_i(S^1)$ for $i < k$ but $\pi_k(S^n - \Sigma^{n-2}) \not\cong \pi_k(S^1)$ [Su].

The analogy above leads us to conjecture that it should suffice to assume that $N$ is $k$-LCC for $k$ in the range $2 \leq k < n/2$. This is the best we can hope for because taking the suspension of one of the knots mentioned at the end of the previous paragraph gives the following example.

Example 1.1. For each $n$ and $k$ with $1 \leq k < (n-1)/2$ there exists a piecewise linear $(n-2)$-sphere $\Sigma \subset S^n$ such that $\Sigma$ is locally 1-alg and j-LCC at $x$ for every $x \in \Sigma$ and for every $j < k$ but there exists a point $x_0 \in \Sigma$ such that $\Sigma$ is not $k$-LCC at $x_0$. The embedding is locally flat except at two points.

But we must be careful with this analogy because we are dealing with topological embeddings, not piecewise linear or smooth ones. In the topological category we can have the following more surprising example.

Example 1.2 [Ve]. For each $n$ and $k$ with $1 \leq k \leq n-2$ there exists a topological $(n-2)$-sphere $\Sigma \subset S^n$ and a point $x \in \Sigma$ such that $\Sigma$ is locally 1-alg at $x$ and $\Sigma$ is j-LCC at $x$ for every $j < k$ but $\Sigma$ is not $k$-LCC at $x$. The embedding is locally flat except at a sequence of points; the sequence of nonlocally flat points has $x$ as its only limit point.

These examples show that we must be very careful about how we ask our question. In particular, Example 1.2 shows that if we focus our attention on a single point $x \in N$, all the $k$-LCC conditions are independent of each other in the range $2 \leq k \leq n-2$. So it is not the case that locally 1-alg at $x$ together with $k$-LCC at $x$ for $2 \leq k < n/2$ imply that $N$ is locally homotopically unknotted at $x$. However, the examples constructed in [Ve] do not have just one point at which they fail to be locally flat. If $k \geq n/2$, then the examples are forced to have a sequence of points converging to $x$ at which the codimension two sphere is not j-LCC for some $j < n/2$. Thus there is still hope for a positive answer to the question. In fact, if we assume the condition at all points in a neighborhood of $x$, then the answer to the question follows the expected pattern. That fact is the main result of this paper. Since the conclusion we seek is local we may as well assume that the open set on which the hypothesis holds is all of $N$.

Theorem 1.3. Let $N^{n-2}$ be a topological $(n-2)$-submanifold of the $n$-manifold $M^n$. If $N$ is locally 1-alg at $x$ for every $x \in N$ and $N$ is $k$-LCC at $x$ for every $x \in N$ and for $2 \leq k < n/2$, then $N$ is locally flat in $M$. 
Notice that the inequality $k < n/2$ in the theorem is a strict inequality. This is the real novelty of the theorem. The fact that it suffices to assume $k$-LCC for $k$ in the range $2 \leq k \leq n/2$ was apparently known to Hollingsworth and Rushing [HR]. Although they did not explicitly state a theorem about this, they incorporated that fact into their definition of locally homotopically unknotted [HR, page 393].

A particularly interesting special case of the theorem is the case $n = 4$. That is the case which inspired the work on the present paper. In the 4-dimensional case it is only necessary to assume a $\pi_1$ condition.

**Corollary.** Let $\Sigma$ be a surface (without boundary) topologically embedded in the 4-manifold $M^4$. If $\Sigma$ is locally 1-alg at $x$ for every $x \in \Sigma$, then $\Sigma$ is locally flat in $M$.

This result has already appeared as part of Theorem 9.3A in [FQ] but the proof given in [FQ] contains a gap. In particular, [FQ, Lemma 9.3B] purports to prove that, for a fixed $x \in \Sigma$, if $\Sigma$ is locally 1-alg at $x$, then $\Sigma$ is 2-LCC at $x$. Example 1.2 and the example in [LV] show that this is not the case.

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2. Homotopy properties of codimension two embeddings

Let $N^{n-2} \subset M^n$ be a pair of topological manifolds-without-boundary as in the Introduction. In this section we develop the basic homotopy properties enjoyed by such pairs if they have good local homotopy properties at every point. We begin by quoting a lemma of Chapman which allows us to specify exactly which neighborhoods of points of $N$ are good for $\pi_1$ of $M - N$. The proof is an application of Alexander Duality and can be found in [Ch, Lemma 3.1]. All homomorphisms of homotopy and homology groups are assumed to be inclusion-induced unless otherwise indicated.

**Lemma 2.1** [Ch]. Suppose $N$ is locally 1-alg at $x \in N$. Let $V_1 \supset V_2 \supset V_3$ be three neighborhoods of $x$ such that

1. $V_i \cap N \cong \mathbb{R}^{n-2}$ for each $i$,
2. $V_i$ is homotopic to a point in $V_{i-1}$ for $i = 2, 3$, and
3. $V_i$ satisfies the 1-alg hypothesis with respect to $V_{i-1}$ for $i = 2, 3$.

Then both $\text{im}[\pi_1(V_2 - N) \to \pi_1(V_1 - N)]$ and $\text{im}[\pi_1(V_3 - N) \to \pi_1(V_2 - N)]$ are infinite cyclic and the inclusion induces an isomorphism from $\text{im}[\pi_1(V_3 - N) \to \pi_1(V_2 - N)]$ to $\text{im}[\pi_1(V_2 - N) \to \pi_1(V_1 - N)]$.

We next prove three technical lemmas which show the key way in which our hypothesis of $k$-LCC at every point is used. The techniques of proof are very similar to those in [Ch] and [HR]. The first of these lemmas (Lemma 2.2) says that homotopies of $j$-dimensional polyhedra can be pushed off $N$. The second (Lemma 2.3) says roughly that if $N$ is locally $j$-co-connected, then it is globally $j$-co-connected. The final lemma (Lemma 2.4) says that $j$-dimensional polyhedra in the complement of $N$ can be pushed close to $N$ with homotopies supported in the complement of $N$. 
Lemma 2.2. Fix \( j \geq 1 \) and an open set \( V_0 \) such that \( V_0 \cap N \cong \mathbb{R}^{n-2} \). Suppose \( N \) is locally 1-alg at \( x \) for every \( x \in N \) and \( N \) is \( k \)-LCC at \( x \) for every \( x \in N \) and for all \( k \) in the range \( 2 \leq k \leq j \). For every positive function \( \epsilon \) defined on \( V_0 \) there exists a neighborhood \( V \) of \( V_0 \cap N \) such that if \( K^j \) is a compact \( j \)-dimensional polyhedron and \( f : K \times [0,1] \to V \) is a map with \( f(K \times \{0\}) \subset V - N \), then there is a map \( g : K \times [0,1] \to V_0 - N \) such that \( g(x,0) = f(x,0) \) for every \( x \in K \) and \( \text{dist}(f(x,t),g(x,t)) < \epsilon(f(x,t)) \) for every \( (x,t) \in K \times [0,1] \).

Proof. First consider the case \( j = 1 \). For each \( x \in N \cap V_0 \) there exists a neighborhood \( V_x \) of \( x \) such that any loop in \( V_x - N \) which is null-homologous in \( V_x - N \) bounds a disk of diameter less than \( \epsilon \) in \( V_0 - N \). Let \( V = \bigcup_{x \in V_0 \cap N} V_x \). Suppose \( f : K^1 \times [0,1] \to V \) with \( f(K \times \{0\}) \subset V - N \). Take a triangulation of \( K \) with small mesh and a partition \( 0 = t_0 < t_1 < \cdots < t_m = 1 \) such that for each 1-simplex \( \sigma \) in \( K \) and each \( i \), the diameter of \( f(\sigma \times [t_i, t_{i+1}]) \) is small relative to a Lebesgue number for the cover \( \{V_x\} \) of \( f(K \times [0,1]) \). Let \( K(0) \) denote the 0-skeleton of \( K \). Use the fact that \( N \) does not separate any open set to shift \( f(K(0) \times [0,1]) \) off \( N \) with a very small move. This defines \( g(K(0) \times [0,1]) \). Let \( \sigma^1 \) be a 1-simplex in \( K \). We next push each of the arcs \( f(\sigma^1 \times \{t_1\}) \) off \( N \) in such a way that each \( g(\partial(\sigma^1 \times [t_i, t_{i+1}])) \) is null-homologous in some \( V_x - N \). This is done by pushing them off one at a time, starting with \( f(\sigma^1 \times \{t_1\}) \). Push \( f(\sigma^1 \times \{t_1\}) \) off with a small move to define \( g(\sigma^1 \times \{t_1\}) \). If \( g(\partial(\sigma^1 \times [t_0, t_1])) \) links \( N \), add a small loop to \( g(\sigma^1 \times \{t_1\}) \) to make \( g(\partial(\sigma^1 \times [t_0, t_1])) \) null-homologous missing \( N \). Then proceed to \( f(\sigma^1 \times \{t_2\}) \). Push it off and add a loop to it, if necessary, to make \( g(\partial(\sigma^1 \times [t_1, t_2])) \) null-homologous. This process is continued inductively. Once \( g(\partial(\sigma^1 \times [t_i, t_{i+1}])) \cap N = \emptyset \) for every \( i \), we can use the choice of \( V_x \) to extend \( g \) to each of the 2-cells \( f(\sigma^1 \times [t_i, t_{i+1}]) \).

Next consider the case \( j = 2 \). For each \( x \in N \cap V_0 \) there exists a neighborhood \( V_x \) of \( x \) such that any singular 2-sphere in \( V_x - N \) bounds a singular 3-cell of diameter less than \( \epsilon/2 \) in \( V_0 - N \). Let \( \delta \) be a Lebesgue function for the cover \( \{V_x\} \). For each \( x \in V_0 \cap N \) we can use Lemma 2.1 to choose a neighborhood \( V_x' \) such that any loop in \( V_x' - N \) that is null-homologous in \( V_0 - N \) bounds a singular disk of diameter less than \( \delta(x)/2 \) in \( V_0 - N \). Let \( \delta' \) be a Lebesgue function for the cover \( \{V_x'\} \). Finally use the case \( j = 1 \) of the Lemma to choose a neighborhood \( V \) such that any homotopy of a 1-dimensional polyhedron in \( V \) is \((\delta'/2)\)-homotopic to a homotopy in the complement of \( N \). Now consider a map \( f : K^2 \times [0,1] \to V \) such that \( f(K^2 \times \{0\}) \subset V - N \). Take a triangulation of \( K \) with small mesh and a partition \( 0 = t_0 < t_1 < \cdots < t_m = 1 \) such that for each simplex \( \sigma \) in \( K \) and each \( i \), the diameter of \( f(\sigma \times [t_i, t_{i+1}]) \) is small relative to \( \delta'/2 \). Use the choice of \( V \) to define \( g(K(1) \times [0,1]) \). Consider a 2-simplex \( \sigma^2 \) in \( K \). For each \( i \), \( g((\partial\sigma^2) \times \{t_i\}) \) is null-homotopic in the complement of \( N \) (it bounds \( g(\sigma^2 \times \{0\} \cup (\partial\sigma^2) \times [0, t_i]) \), so the choice of \( V_x' \) allows us to extend \( g \) to \( \sigma^2 \times \{t_i\} \). Then the choice of \( V_x \) allows us to extend \( g \) to each of the 3-cells \( \sigma^2 \times [t_i, t_{i+1}] \).

The cases \( j \geq 3 \) are handled inductively. They are a little easier than either of the cases \( j = 1 \) or \( j = 2 \) because there is no problem of linking in these higher dimensions. □

Lemma 2.3. Fix \( j \geq 1 \). Suppose \( N \) is locally 1-alg at \( x \) for every \( x \in N \) and \( N \) is \( k \)-LCC at \( x \) for every \( x \in N \) and for every \( k \) in the range \( 2 \leq k \leq j \). If \( V_1 \) is an open set such that \( V_1 \cap N \cong \mathbb{R}^{n-2} \), then there exists an open set \( V_2 \subset V_1 \) with \( V_1 \cap N = V_2 \cap N \) such that
any map $f : S^j \to V_2 - N$ is null-homotopic in $V_1 - N$. (If $j = 1$, we must also assume that $f$ is null homologous in $V_1 - N$.)

Proof. Consider the case $j = 1$. For each $x \in V_1 \cap N$ we can use Lemma 2.1 to find a positive number $\epsilon(x)$ such that any loop in the $\epsilon$ neighborhood of $x$ that misses $N$ and is null-homologous in $V_1 - N$ is null-homotopic in $V_1 - N$. Choose $V'$ using this $\epsilon$ and Lemma 2.2. Then choose $V_2 \subset V'$ so that $V_2$ strong deformation retracts to $N$ in $V'$. Suppose $f : S^1 \times \{0\} \to V_2 - N$ and $f$ is null-homologous in $V_1 - N$. The choice of $V_2$ allows us to extend $f$ to $f : S^1 \times [0, 1] \to V'$ so that $f(S^1 \times \{1\})$ is a single point. Using Lemma 2.2 we can find an $\epsilon$ approximation $g : S^1 \times [0, 1] \to V_1 - N$. Now $g(S^1 \times \{1\})$ has diameter less than $\epsilon$ and $g(S^1 \times \{1\})$ is null-homologous in $V_1 - N$ (since $g(S^1 \times \{0\})$ was), so the choice of $\epsilon$ allows us to find a singular 2-cell which $g(S^1 \times \{1\})$ bounds in $V_1 - N$. Thus $f(S^1 \times \{0\})$ bounds a singular 2-cell in $V_1 - N$.

The proof of the higher dimensional cases is similar. □

Lemma 2.4. Fix $j \geq 1$ and an open set $V_0$ such that $V_0 \cap N \cong \mathbb{R}^{n-2}$. Suppose $N$ is locally 1-alg at $x$ for every $x \in N$ and $N$ is $k$-LCC at $x$ for every $x \in N$ and for all $k$ in the range $2 \leq k \leq j$. For every neighborhood $U_1$ of $N \cap V_0$ in $V_0$ there exists a neighborhood $V_1$ of $N$ in $U_1$ having the following property: for every neighborhood $U_2$ of $N$ there exists a neighborhood $V_2$ of $N \cap V_0$ in $U_2$ such that if $f : K^j \to V_1 - N$ is a map of a $j$-dimensional polyhedron into $V_1 - N$, then $f$ is homotopic in $U_1 - N$ to a map of $K$ into $U_2 - N$. Furthermore, if $L$ is a subpolyhedron of $K$ and $f(L) \subset V_2 - N$, then the homotopy can be chosen to keep $L$ fixed.

Proof. Again we first consider the case $j = 1$. Given $U_1$, we choose $V_1$ in such a way that any loop in $V_1 - N$ that is null-homologous in $U_1 - N$ is null-homotopic in $U_1 - N$ (Lemma 2.3). Given $U_2$, we choose $V_2$ to be any connected neighborhood of $N$ contained in $U_2$. Consider a map $f : K^1 \to V_1 - N$. It is easy to pull the 0-skeleton in, so we may assume that $f(K^{(0)}) \subset V_2$. For each 1-simplex $\sigma^1$ in $K$, we choose an arc $\alpha$ in $V_2 - N$ joining the two points $f(\partial \sigma^1)$ and then add a loop to it, if necessary, so that $\alpha \cup f(\sigma^1)$ is null-homologous in $U_1 - N$. By the choice of $V_1$, this loop is null-homotopic in $U_1 - N$, so there is a homotopy from $f(\sigma^1)$ to $\alpha$ that keeps the endpoints fixed.

Now consider the case $j = 2$. Let $U_1$ be given. Use Lemma 2.3 to choose $U' \subset U_1$ such that any singular 2-sphere in $U' - N$ bounds a singular 3-cell in $U_1 - N$. Then use the previous case of this lemma to choose $V_1 \subset U'$. Let $U_2$ be given. Use Lemma 2.3 to choose $V' \subset U_2$ so that any loop in $V' - N$ that is null-homologous in $U_2 - N$ is null-homotopic in $U_2 - N$. Finally choose $V_1 \subset V'$ using the previous case of this lemma. Suppose $f : K^2 \to V_1 - N$. By the previous case of this lemma, we can pull the image of the 1-skeleton into $V' - N$ so we assume that $f(K^{(1)}) \subset V' - N$. For any 2-simplex $\sigma^2$ in $K$, the choice of $V'$ allows us to find a singular 2-cell $C \subset U_2 - N$ whose boundary is $f(\partial \sigma^2)$. But the choice of $U'$ says that the singular 2-sphere $f(\sigma^2) \cup C$ bounds in $U_1 - N$, so there is a homotopy of $f(\sigma^2)$ to $C$ that keeps $\partial \sigma^2$ fixed.

The higher dimensional cases are similar. □

3. Homology properties of codimension two embeddings

We begin this section by fixing some notation which will be assumed for the remainder
of this paper. For the remainder of this paper it will be assumed that \( N \subset M \) satisfies the hypotheses of Theorem 1.3, even though this will not be explicitly stated in the lemmas which follow. Specifically, we assume the following.

**Hypothesis.** \( N^{n-2} \) is a topological \((n-2)\)-submanifold of the \( n \)-manifold \( M^n \) such that \( N \) is locally 1-alg at \( x \) for every \( x \in N \) and \( N \) is \( k \)-LCC at \( x \) for every \( x \in N \) and for \( 2 \leq k < n/2 \).

Consider one point of \( x \in N \) and fix a neighborhood \( U_0 \) of \( x \). Let \( U_1 \subset U_0 \) be a second neighborhood of \( x \) such that \( U_1 \cap N \cong \mathbb{R}^{n-2} \) and \( U_1 \) satisfies the 1-alg condition relative to \( U_0 \). We will denote the infinite cyclic group \([\pi_1(U_1 - N) \to \pi_1(U_0 - N)] \) by \( \mathbb{J} \). We use \( t \) to denote a fixed generator of \( \mathbb{J} \) and write elements of \( \mathbb{J} \) multiplicatively; thus \( \mathbb{J} = \{t^i \mid i = 0, \pm1, \pm2, \ldots \} \). Let \( W = U_1 - N \) and let \( p : \tilde{W} \to W \) denote the infinite cyclic cover determined by the natural homomorphism of \( \pi_1(W) \) to \( \mathbb{J} \). We will use the same symbol \( t \) to denote the generator of the group of deck transformations of \( \tilde{W} \) which corresponds to \( t \). If \( P \) is any subset of \( W \), we use \( P^* \) to denote \( p^{-1}(P) \subset \tilde{W} \); we use the notations \( H_i(P^*; \mathbb{Z}) \) and \( H_i(P; \mathbb{Z}[\mathbb{J}]) \) interchangeably. From now on we will work with \( W \) as our ambient manifold; the reader should be certain to note that \( W \) is a deleted neighborhood of \( N \) and does not include any points of \( N \). The reason for doing this is so that we can take the \( \mathbb{J} \)-cover of \( W \).

We are assuming that \( N \) is \( k \)-LCC at each point for \( 2 \leq k < n/2 \). We wish to prove that \( N \) is \( k \)-LCC at \( x \) for \( k \geq n/2 \). Fix \( x \in N \) and let \( B_2 \subset \text{int} B_1 \) be two concentric \( n \)-balls centered at \( x \). If we can show that it is possible to choose \( B_1 \) and \( B_2 \) in such a way that \( \pi_k(B_2 - N) \to \pi_k(B_1 - N) \) is the trivial homomorphism for \( k \geq n/2 \), then the proof will be complete. Since each map of \( S^k \) into \( B_2 - N \) has compact image, it suffices to show that there exist arbitrarily close neighborhoods \( W_1 \subset W_2 \) of \( N \) such that if \( P_i = B_i - W_i \), then \( \pi_k(P_2) \to \pi_k(P_1) \) is trivial. By the Eventual Hurewicz Theorem ([Fe, Proposition 3.1] or [Q1, Theorem 5.2]), it suffices to prove this in homology with local coefficients. In other words, we need to prove the following Proposition.

**Proposition 3.1.** Fix \( x \in N \). For each sufficiently small \( n \)-ball \( B_1 \) centered at \( x \) and contained in \( V_1 \) there exists a concentric \( n \)-ball \( B_2 \) such that there are arbitrarily close neighborhoods \( W_1 \subset W_2 \) of \( N \) for which the inclusion-induced homomorphism \( H_k(B_2 - W_2; \mathbb{Z}[\mathbb{J}]) \to H_k(B_1 - W_1; \mathbb{Z}[\mathbb{J}]) \) is trivial for \( k \geq n/2 \).

The proof of Proposition 3.1 is broken into three parts. First, in the remainder of this section we prove some lemmas about related homology inclusions that will be used later. Second, in §4 we will prove that there are arbitrarily small neighborhoods of \( x \) having cell decompositions with no cells above the middle dimension. This leaves only the middle dimensional case to be completed. In §5 we prove the middle dimensional case using coefficients in a field. Finally, we observe in the last section that the results of the preceding two sections suffice to complete the proof of Proposition 1.3.

Fix an open set \( V_0 \) such that \( V_0 \cap N = U_1 \cap N \) and there is a strong deformation retraction of \( V_0 \) to \( V_0 \cap N \) in \( U_1 \). The let \( V_1 \) be a neighborhood of \( N \cap U_1 \) such that polyhedra in \( V_1 - N \) of dimension less than \( n/2 \) can be pulled near \( N \) via homotopies whose tracks lie in \( V_0 - N \) (Lemma 2.4).
Lemma 3.2. Let $B_1$ be an $n$-ball centered at $x$ and contained in $V_1$, let $C$ be an $(n-2)$-ball in the interior of $N \cap B_1$, and let $B_2$ be a second $n$-ball centered at $x$ such that $B_2 \cap N$ is contained in the interior of $C$. There exist arbitrarily close neighborhoods $W_1 \subset W_2$ of $N$ such that if $P_i = B_i - W_i$, then

$$\ker[H_k(W - P_1; \mathbb{Z}[\mathcal{J}])] \to H_k(W; \mathbb{Z}[\mathcal{J}]) \supset \ker[H_k(W - P_2; \mathbb{Z}[\mathcal{J}])] \to H_k(W - P_2; \mathbb{Z}[\mathcal{J}])$$

for each $k < n/2$.

In other words, if a $k$-dimensional class in $W - P_1$ is null-homologous in $W$, then it is null-homologous missing $P_2$.

Proof. Let $W_2$ be any connected open neighborhood of $N$ in $V_1$. By Lemma 2.3 there exists a neighborhood $W'_1$ of $N$ in $W_2$ such that $\pi_k(p^{-1}(W'_1 - N)) \to \pi_k(p^{-1}(W_2 - N))$ is trivial for every $k < n/2$. Choose $\epsilon$ to be a positive number that is smaller than either $\frac{1}{2} \text{dist}(C, W - B_1)$ or $\frac{1}{2} \text{dist}(N, C - B_2)$. Then use $\epsilon$ as input to Lemma 2.2 to approximate by a homotopy in the complement of $N$ and is the identity outside the $2\epsilon$-neighborhood of $C$. The image of $L$ under that homotopy can be further pushed along $N$ out of the $(n-2)$-cell $C$ because $\dim L < \dim C$. We then use Lemma 2.2 to approximate by a homotopy in the complement of $N$. This last homotopy may be lifted to $\tilde{W}$, so it allows us to assume that $p \circ g : (K, L) \to (V_1 - N, V_1 - (N \cup B_2))$.

We first wish to push $p(g(L))$ out of $B_2$. The choice of $W_1$ allows us to find an $\epsilon$-homotopy of $p(g(L))$ that pushes the part of $p(g(L))$ in the $\epsilon$-neighborhood of $C$ into $N$ and is the identity outside the $2\epsilon$-neighborhood of $C$. The image of $L$ under that homotopy can be further pushed along $N$ out of the $(n-2)$-cell $C$ because $\dim L < \dim C$. We then use Lemma 2.2 to approximate by a homotopy in the complement of $N$. This last homotopy may be lifted to $\tilde{W}$, so it allows us to assume that $p \circ g : (K, L) \to (V_1 - N, V_1 - (N \cup B_2))$.

We next cut $p(g(K))$ off on $\partial B_2$. In other words, put $p \circ g$ in general position relative to $\partial B_2$. Then $(p \circ g)^{-1}(B_2)$ is a subpolyhedron of $K$. So $p \circ g(p \circ g)^{-1}(\partial B_2)$ can be extended (using the Tietze Extension Theorem) to a map of $(p \circ g)^{-1}(B_2)$ into $\partial B_2$. Thus there is a map $g' : K \to (V_1 - (N \cup B_2)) \cup \partial B_2$ that agrees with $p \circ g$ outside $(p \circ g)^{-1}(B_2)$.

By Lemma 2.4 and the choice of $V_1$, there exists a neighborhood $W'_2$ of $N$ such that $k$-dimensional polyhedra can be pulled into $W'_1$ keeping $W'_2$ fixed. Let $Q$ be a compact subpolyhedron of $K$ such that $g'(Q) \subset W'_2$, $g'(K - Q) \subset V_1 - (N \cup B_2)$, and $Q \cap L = \emptyset$. We may assume that $Q$ does not separate $K$. (If it does, adjust $g'$ slightly so that arcs in $Q$ which connect up components of $K - Q$ are moved off $N$ and then remove neighborhoods of the arcs from $Q$.) Then there exists a $k$-dimensional subpolyhedron $R$ of $K$ such that $K \setminus Q \cup R$. By the choice of $W'_2$ there is a homotopy that pulls $g'(R)$ into $W'_1$, keeping $g'(R \cap Q)$ fixed. This defines a new map $g'' : Q \cup R \to W'_1$ such that $g''|Q = g'|Q$ and $g''(R) \cap N = \emptyset$. Let $S$ denote a close neighborhood of $Q \cup R$ in $K$ and extend $g''$ to $S$. Then $p(g(L))$ is homologous to $g''(\partial S)$ since the difference bounds $g'(K - Q) \cup g''(S - Q)$. All those homologies are supported in $W$. Furthermore, the fact that $g' \simeq g''$ in $W$ allows us to lift the homologies to $\tilde{W}$.

Let $\gamma''$ denote the homology class represented by a lift of $g''(\partial S)$. We have that $\gamma'' \sim \gamma$ in $\tilde{W} - P_2^*$ and $\gamma''$ is represented by a map into $p^{-1}(W'_1 - N)$. The choice of $W'_1$ now gives $\gamma'' \sim 0$ in $p^{-1}(W_2 - N)$, so the proof is complete. $\square$
Note: The entire construction described in the proof above takes place in $V_0$, so we may assume that the homology classes we have constructed are supported in $p^{-1}(V_0 - N)$. This observation will be important in the proof of Lemma 3.4, below.

**Lemma 3.3.** If $P_1$ and $P_2$ are as in Lemma 3.2, then

$$H_k(W, W - P_1; \mathbb{Z}) \to H_k(W, W - P_2; \mathbb{Z})$$

is the trivial homomorphism for $k < n/2$.

**Proof.** Suppose $(c, \partial c)$ is a class in $H_k(W, W - P_1; \mathbb{Z})$. We use the notation $|c|$ to denote the support of a representative of $c$. Lemma 3.2 says $\partial c = \partial d$, where $|d| \subset \widetilde{W} - P_2^*$. But Lemma 2.4 says that $c - d$ is homologous to $e$ where $|e| \subset W_2$. Hence $c$ is homologous to $d + e$ and the proof is complete because $|d| \cup |e| \subset \widetilde{W} - P_2^*$. □

Downstairs we can improve the dimension by 1. The reason for this is the fact that the local homology of the complement downstairs is always trivial.

**Lemma 3.4.** If $P_1$ and $P_2$ are as in Lemma 3.2 and $P_1$ is sufficiently small, then

$$H_k(W, W - P_1; \mathbb{Z}) \to H_k(W, W - P_2; \mathbb{Z})$$

is the trivial homomorphism for $k \leq n/2$.

**Proof.** (The reader should probably review the definitions of $V_0$ and $V_1$ at this point; those definitions can be found just above the statement of Lemma 3.2) By Alexander Duality, $H_{k+1}(U_1, U_1 - N) \cong \tilde{H}_c^{n-k-1}(U_1 \cap N)$ and $\tilde{H}_c^{n-k-1}(U_1 \cap N) \cong \tilde{H}_c^{n-k-1}(\mathbb{R}^{n-2}) = 0$ since $k \geq 2$. The following diagram with exact rows shows that $H_k(V_0 - N) \to H_k(U_1 - N)$ is trivial.

$$
\begin{array}{ccc}
H_{k+1}(V_0, V_0 - N) & \longrightarrow & H_k(V_0 - N) \\
\downarrow & & \downarrow \\
0 = H_{k+1}(U_1, U_1 - N) & \longrightarrow & H_k(U_1 - N) \\
& & \downarrow 0 \\
& & H_k(U_1)
\end{array}
$$

Now suppose $(c, \partial c)$ is a class in $H_k(W, W - P_1; \mathbb{Z})$. We can trim off part of $|c|$ so that $|c| \subset V_1$. Then the proof of Lemma 3.2 shows that $\partial c = \partial d$, where $|d| \subset V_1 - P_2$. Since $\partial(c - d) = 0$, $c - d$ represents a class in $H_k(V_0 - N)$ and $c - d$ is null-homologous in $W$ by the paragraph above. Thus $c = d$ in $H_k(W, W - P_1; \mathbb{Z})$ and the proof is complete. □

4. **The structure of neighborhoods**

In this section we use the homology results of the previous section to construct neighborhoods of $x$ having improved homotopy properties. Before we can do that we need a result about $\pi_1$. 


**Lemma 4.1.** Fix \( x \in N \) and suppose \( n \geq 5 \). For each sufficiently small \( n \)-ball \( B_1 \) centered at \( x \) there exists a concentric \( n \)-ball \( B_2 \) in \( B_1 \) such that for sufficiently small \( n \)-balls \( B_3 \subset B_2 \) there are arbitrarily small neighborhoods \( W_1 \subset W_2 \subset W_3 \) of \( N \) such that if \( P_i = B_i - W_i \), then the inclusion-induced homomorphism \( \pi_1(\partial P_2^*) \to \pi_1(P_1^* - P_3^*) \) is trivial.

**Proof.** The outline of the proof is as follows. Step 1 consists of choosing \( B_2 \) and \( B_3 \). Step 2 is the construction of \( W_2 \). Step 3 is the proof that \( \pi_1(\partial P_2^*) \to \pi_1((B_1 - N)^* - P_3^*) \) is trivial. The final step is the observation that \( W_1 \) can be chosen so that the conclusion of the Lemma holds.

Let \( B_1 \) be given. We first explain how to choose \( B_2 \) and \( B_3 \). Begin by choosing an \((n-2)\)-cell \( C_1 \) in \( N \) such that \( x \in \text{int} \, C_1 \) and \( C_1 \subset \text{int} \, B_1 \). Choose \( B_2 \) to be an \( n \)-cell centered at \( x \) such that \( B_2 \cap N \subset \text{int} \, C_1 \) and the inclusion \( B_2 - N \hookrightarrow B_1 - N \) satisfies the 1-alg hypothesis. Let \( C_2 \) be an \((n-2)\)-cell in \( N \) such that \( x \in \text{int} \, C_2 \) and \( C_2 \subset \text{int} \, B_2 \). Let \( U \) be a neighborhood of \( C_1 - C_2 \) in \( B_1 \) whose closure misses \( x \). For each \( y \in C_1 - C_2 \), choose a neighborhood \( V_y \) such that any loop in \( V_y - N \) which is null-homologous in \( W \) is null-homotopic in \( U - N \). Let \( \epsilon \) be a Lebesgue number for this cover. Also choose \( \epsilon \) so that \( \epsilon < \frac{1}{3} \min \{ \text{dist}(B_2, N - C_1), \text{dist}(\partial B_2, C_2) \} \). Use this \( \epsilon \) and Lemma 2.2 to determine a neighborhood \( V \) of \( N \). Choose another neighborhood \( V' \) such that \( V' \)-deforms to \( N \) in \( V \). Let \( U' \) be the preimage of \( \overline{C_1 - C_2} \) under this deformation. Define \( Q = \partial B_2 - V' \) and \( \partial Q = Q \cap \partial V' \). We may assume that both \( Q \) and \( \partial Q \) are finite polyhedra. Let \( \ell_1, \ell_2, \ldots, \ell_q \) be a set of generators for \( \pi_1(Q, \partial Q) \). For each \( \ell_i \) there exists an arc \( m_i \) in \( U' - N \) such that \( \ell_i \cup m_i \) is null-homologous in \( W \). Hence \( \ell_i \cup m_i \) bounds a singular 2-cell \( \Delta_i \subset B_1 - N \) by the choice of \( B_1 \). Choose \( B_3 \) to be an \( n \)-ball centered at \( x \) which misses \( (\cup_{i=1}^q \Delta_i) \cup \overline{U} \).

We next explain how to choose \( W_2 \). Let \( W_3 \) be any connected neighborhood of \( N \) which is contained in \( V' \). Use Lemma 2.2 to choose a neighborhood \( W_2 \) of \( N \) such that homotopies in \( W_2 \) can be pushed off \( N \) staying in \( W_3 \). Then choose \( W_2 \) so that \( W_2 \) deforms to \( N \) in \( W_2' \).

Define \( P_2 = B_2 - W_2 \) and \( P_3 = B_3 - W_3 \). Our next goal is to show that \( \pi_1(\partial P_2^*) \to \pi_1((B_1 - N)^* - P_3^*) \) is trivial. Let \( \gamma \) be a loop in \( \partial P_2^* \) and define \( \gamma' = p(\gamma) \). We show that \( \gamma' \) is null-homotopic in \( (B_1 - N) - P_3 \) and then lift this homotopy to \( \tilde{W} \) to obtain the desired result. We first use the choice of \( W_2 \) to push \( \gamma' \) to the boundary of \( C_2 \) and out of \( B_3 \) in exactly the same way we pushed out of \( B_2 \) in the proof of Lemma 3.2. Next put \( \gamma' \) in general position with respect to \( \partial Q \). Then \( \gamma' \cap Q \) consists of a finite number \( \gamma_1, \gamma_2, \ldots, \gamma_m \) of subarcs. The choice of \( B_3 \) allows us to find an arc \( \mu_i \) in \( U' \) joining the endpoints of \( \gamma_i \) such that \( \gamma_i + \mu_i \) is null-homotopic missing \( B_3 \cup N \). This means that \( \gamma' \) may be replaced by a loop \( \gamma'' \) in \( U' - N \). There exists an \( \epsilon \)-deformation of \( \gamma'' \) to \( \overline{C_1 - C_2} \). Since \( n \geq 5 \), \( C_1 - C_2 \) is simply connected. So \( \gamma'' \) is homotopic to a point in the \( \epsilon \) neighborhood of \( C_1 - C_2 \). We can use the choice of \( \epsilon \) and \( V \) along with Lemma 2.2 to \( \epsilon \)-push this homotopy into the complement of \( N \). This means that \( \gamma'' \) is homotopic to \( \gamma''' \) with \( \text{diam}(\gamma''') < \epsilon \). The choice of \( \epsilon \) then shows us that \( \gamma''' \) is null-homotopic in \( U \). So \( \gamma' \) is null-homotopic in \( (B_1 - N) - B_3 \).

Let \( g_1, g_2, \ldots, g_n, t \) be a set of generators for \( \pi_1(\partial P_2) \). We may assume that each \( g_i \) lifts to \( \partial P_2^* \). (If not, replace \( g_i \) with \( gt^\epsilon \) for some \( \ell \).) Then each element of \( \pi_1(\partial P_2^*) \) can be written in terms of \( \{ t^j \tilde{g}_1 \}_{i,j} \). We fix a homotopy shrinking each \( g_i \) in \( (B_1 - N) - P_3 \) and choose \( W_1 \) so close to \( N \) that it misses the tracks of all these homotopies. Then we will
have \( \pi_1(\partial P_1) \to \pi_1(P_1 - P_3) \) trivial. \( \square \)

**Lemma 4.2.** Fix \( x \in N \) and suppose \( n \geq 5 \). For each \( q > 0 \) there exists a sequence \( B_1 \supset B_2 \supset \cdots \supset B_q \) of \( n \)-balls centered at \( x \) such that there are arbitrarily close neighborhoods \( W_1 \subset W_2 \subset \cdots \subset W_q \) so that if \( P_i = B_i - W_i \) then the inclusion induced homomorphisms \( \pi_1(P_i^*) \to \pi_1(P_{i-1}^*) \) and \( \pi_1(P_i^* - P_{i+1}^*) \to \pi_1(P_{i-1}^* - P_{i+2}^*) \) are trivial.

**Proof.** Construct the sequence \( B_1 \supset B_2 \supset \cdots \supset B_q \) inductively, using Lemma 4.1 to construct \( B_{i+1} \) from \( B_i \). Using the 1-\( \text{alg} \) hypothesis we can also arrange that \( \pi_1(p^{-1}(B_i - N)) \to \pi_1(p^{-1}(B_{i-1} - N)) \) is trivial. Choose \( W_1 \subset W_2 \subset \cdots \subset W_q \) to satisfy the conclusion of Lemma 4.1 as well.

Consider a loop \( \ell \subset P_i^* \). There exists a map \( f : \Delta^2 \to p^{-1}(B_{i-1} - N) \) such that \( f|\partial \Delta^2 = \ell \). Put \( f \) in general position with respect to \( \partial P_i^* \). Then \( f^{-1}(\partial \Delta^2) \) consists of a finite collection of circles in \( \text{int} \Delta^2 \). Each outermost circle shrinks in \( P_i^* - P_{i+1}^* \) by Lemma 4.1. Thus we can redefine \( f \) to map the interior of this circle into \( P_{i-1}^* - P_{i+1}^* \). We do this for each outermost circle. The result is a map \( f' : \Delta^2 \to P_{i-1}^* \) such that \( f'|\partial \Delta^2 = \ell \). So \( \pi_1(P_i^*) \to \pi_1(P_{i-1}^*) \) is the trivial homomorphism.

The proof of the second part of the Lemma is similar. We start with a loop in \( P_i^* - P_{i+1}^* \). It bounds a singular disk in \( p^{-1}(B_{i-1} - N) \). We use Lemma 4.1 to cut this disk off on \( \partial P_i^* \cup \partial P_{i+1}^* \). The result is a singular disk in \( P_{i-1}^* - P_{i+2}^* \). \( \square \)

**Proposition 4.3.** Fix \( x \in N \). For each sufficiently small \( n \)-ball \( B_1 \) centered at \( x \) there exists a concentric \( n \)-ball \( B_2 \) and arbitrarily small neighborhoods \( W_1 \subset W_2 \subset P \subset B_1 - W_1 \) and \( P \) has a handle decomposition with no handles of index greater than \( n/2 \).

**Proof.** First suppose \( n \geq 5 \). Fix \( q \gg 2^n \). Let \( P_1 \supset P_2 \supset \cdots \supset P_q \) be a sequence as in Lemma 4.2. We may also assume that each \( P_{i+1} \subset P_i \) satisfies the hypotheses of Lemma 3.2. We may then apply the Relative Eventual Hurewicz Theorem [Q1, Theorem 5.2] to the inclusions \( (P_i, P_i - P_{i+j}) \to (P_{i-1}, P_{i-1} - P_{i+j+1}) \) since Lemma 4.2 gives us exactly the \( \pi_1 \)-conditions we need and Lemma 3.3 gives us the higher homology hypotheses we need. Thus there is an integer \( s \) such that for all \( k < n/2 \), \( \pi_k(P_i, P_i - P_{i+j}) \to \pi_k(P_{i-s}, P_{i-s} - P_{i+s+1}) \) is trivial. (The number \( s \) is independent of the particular \( P_i \) used; it depends only on \( k \). In fact \( s = \lceil \sqrt{k+2} \rceil \) will work; see [Q1, Theorem 5.2].) This means that relative \( k \)-dimensional polyhedra in \( (P_i, P_i - P_{i+1}) \) can be pushed off \( P_{i+s+1} \) with a homotopy and hence (using engulfing) can be pushed off some \( P_{i+t} \) with an ambient isotopy whose support lies in the interior of \( P_i \).

The PL manifold \( P \) can now be constructed as follows. Start with \( P_1 \) and form a handle decomposition of \( P_1 \). Each \( j \)-dimensional handle has an \( (n-j) \)-dimensional cocore. If \( j > n/2 \), then \( n-j < n/2 \), so the paragraph above implies that the cocore of each \( j \)-handle can be isotoped off \( P_{i+t+1} \) keeping \( P_1 - P_i \) fixed. We construct \( P \) by starting with \( P_1 \), pushing the cocore of each \( j \)-handle off \( P_1 \) for \( j > n/2 \), and then removing a small regular neighborhood of the cocore of each of the \( j \)-handles, \( j > n/2 \). The remaining manifold has only handles of index \( n/2 \) or less.

In case \( n = 4 \) the proof is much easier. Each \( P_i \) has a handle decomposition with handles of dimensions \( \leq 3 \), so we need to eliminate the 3-handles. It is enough to show that \( \pi_1(P_i, P_i - P_{i-1}) \to \pi_1(P_{i+1}, P_{i+1}) \) is trivial. The proof proceeds as follows: Given
a path $\alpha$ in $P_i$ with endpoints in $P_i - P_{i-1}$, use the fact that $P_{i+1}$ does not separate to join the endpoints with a path $\beta$ in $P_i - P_{i-1}$. Add a loop, if necessary, so that $\alpha \cup \beta$ is null-homotopic in $P_{i+1}$. □

**Corollary 4.4.** Proposition 3.1 holds for $k > n/2$.

**Proof.** This is clear since the inclusion factors through a space with trivial homology in dimensions $k > n/2$. □

5. A duality principle for homology with coefficients in a field

In this section we prove a weak version of Proposition 3.1. It is weaker than the version we need in that it only works for coefficients in a field. It uses a duality principle for infinite cyclic covers that is similar to the one used by Milnor in [Mi]. Let $F$ be a field and let $\Lambda$ denote the group algebra $F[\mathbb{Z}]$. Notice that for any $U \subset W$ the homology group $H_k(\tilde{W}, U^*; F)$ can be thought of either as a vector space over $F$ or as a module over $\Lambda$. It is important to distinguish between “finitely generated over $F$” and “finitely generated over $\Lambda$.” We use $H_k(W, U; \Lambda)$ as a shorthand for $H_k(\tilde{W}, U^*; F)$.

Fix a sequence of three PL manifolds $P_3 \subset P_2 \subset P_1$ such that each of the inclusions $P_3 \subset P_2$ and $P_2 \subset P_1$ satisfies the hypotheses of Lemmas 3.2 and 3.4. Fix $k \leq n/2$ and let $\alpha_i : H_k(W, W - P_i; \Lambda) \to H_k(W, W - P_{i+1}; \Lambda)$ denote the inclusion-induced homomorphism.

**Lemma 5.1.** $\text{im}(\alpha_2) = \text{im}(\alpha_2 \circ \alpha_1)$.

**Proof.** Consider the following commutative diagram in which each row is exact.

$$
\begin{array}{cccc}
H_k(W; \Lambda) & \xrightarrow{\beta_1} & H_k(W, W - P_1; \Lambda) & \longrightarrow & H_{k-1}(W - P_1; \Lambda) \\
\downarrow & & \downarrow & & \downarrow \\
H_k(W; \Lambda) & \xrightarrow{\beta_2} & H_k(W, W - P_2; \Lambda) & \longrightarrow & H_{k-1}(W - P_2; \Lambda) \\
\downarrow & & \downarrow & & \downarrow \\
H_k(W; \Lambda) & \xrightarrow{\beta_3} & H_k(W, W - P_3; \Lambda) & \longrightarrow & H_{k-1}(W - P_3; \Lambda)
\end{array}
$$

By Lemma 3.2, we have $\delta \circ \gamma = 0$, so $\text{im}(\alpha_2) \subset \text{im}(\beta_3)$. On the other hand, $\text{im}(\alpha_2) \supset \text{im}(\beta_3)$ is obvious so we have $\text{im}(\alpha_2) = \text{im}(\beta_3)$. Now the inclusion $P_1 \subset P_3$ has all the same properties that the inclusion $P_2 \subset P_3$ has, so we have $\text{im}(\alpha_2 \circ \alpha_1) = \text{im}(\beta_3)$ as well. □

**Lemma 5.2.** $\text{im}(\alpha_2)$ is finite dimensional over $F$.

**Proof.** For each $i$ the exact sequence

$$0 \to C_*(W, W - P_i; \Lambda) \xrightarrow{t_i^{-1}} C_*(W, W - P_i; \Lambda) \xrightarrow{p_i} C_*(W, W - P_i; F) \to 0$$

of chain complexes gives rise to a long exact sequence

$$\cdots \to H_k(W, W - P_i; \Lambda) \xrightarrow{t_i^{-1}} H_k(W, W - P_i; \Lambda) \xrightarrow{p_i} H_k(W, W - P_i; F) \to \cdots$$
of homology groups. Therefore we have the following commutative diagram with exact rows.

\[
\begin{array}{ccc}
H_k(W, W - P_1; \Lambda) & \xrightarrow{t-1} & H_k(W, W - P_1; \Lambda) \\
\alpha_1 \downarrow & & \downarrow \alpha_1 \\
H_k(W, W - P_2; \Lambda) & \xrightarrow{t-1} & H_k(W, W - P_2; \Lambda) \\
\alpha_2 \downarrow & & \downarrow \alpha_2 \\
H_k(W, W - P_3; \Lambda) & \xrightarrow{t-1} & H_k(W, W - P_3; \Lambda) \\
\end{array}
\xrightarrow{p^*} \begin{array}{ccc}
H_k(W, W - P_1; F) & & H_k(W, W - P_1; F) \\
\alpha_1 \downarrow & & \downarrow 0 \\
H_k(W, W - P_2; F) & & H_k(W, W - P_2; F) \\
\alpha_2 \downarrow & & \downarrow 0 \\
H_k(W, W - P_3; F) & & H_k(W, W - P_3; F)
\end{array}
\]

The vertical arrows on the right are 0 by Lemma 3.4. The upper half of the diagram shows that \(\text{im}(\alpha_1) \subset (t-1)(H_k(W, W - P_2; \Lambda)) = \text{im}(t-1)\). Thus \(\text{im}(\alpha_2) = \text{im}(\alpha_2 \circ \alpha_1) \subset \text{im}(\alpha_2 \circ (t-1)) = (t-1) \text{im}(\alpha_2)\). It follows that \(\text{im}(\alpha_2) = (t-1) \text{im}(\alpha_2)\). Now \(\text{im}(\alpha_2)\) is a subspace of \(H_k(W, W - P_3; \Lambda)\) which is finitely generated over \(\Lambda\). Since \(\text{im}(\alpha_2)\) is finitely generated over \(\Lambda\) and \(\text{im}(\alpha_2) = (t-1) \text{im}(\alpha_2)\), \(\text{im}(\alpha_2)\) is a finitely generated \(\Lambda\)-module whose order ideal in non-trivial. It follows that \(\text{im}(\alpha_2)\) is finite dimensional over \(F\). (See the proof of Assertion 5, page 118, of [Mi]. This is the point in the proof at which it is important to be working over a field.) \(\square\)

Let \(K\) be a connected subpolyhedron of \(\tilde{W}\) such that \(p(K) = W\) and for each compact subpolyhedron \(L\) of \(W\), \(p^{-1}(L) \cap K\) is compact. Notice that, for each compact \(C \subset \tilde{W}\), \(C \cap t^i(K) = \emptyset\) for all but finitely many \(i\). Define

\[
N_q = \bigcup_{i \geq q} t^i(K), \quad \text{and} \quad N'_q = \bigcup_{i \leq -q} t^i(K).
\]

**Lemma 5.3.** \(\text{im}[H_{k-1}(N_0, N_0 - P^*_i; F) \to H_{k-1}(N_0, N_0 - P^*_{i+1}; F)]\) is finite dimensional over \(F\) for \(i = 1, 2\).

**Proof.** Define \(N = N_0 \cap N'_0\). Consider the following commutative diagram in which each row is part of an exact Mayer-Vietoris sequence. (Coefficients in \(F\) are understood.)

\[
\begin{array}{ccc}
H_{k-1}(N, N - P^*_i) & \to & H_{k-1}(N_0, N_0 - P^*_i) \oplus H_{k-1}(N'_0, N'_0 - P^*_i) \to H_{k-1}(\tilde{W}, \tilde{W} - P^*_i) \\
\downarrow & & \downarrow \gamma \\
H_{k-1}(N, N - P^*_{i+1}) & \to & H_{k-1}(N_0, N_0 - P^*_{i+1}) \oplus H_{k-1}(N'_0, N'_0 - P^*_{i+1}) \to H_{k-1}(\tilde{W}, \tilde{W} - P^*_{i+1}) \\
\end{array}
\xrightarrow{\beta}
\]

Since \(H_{k-1}(N, N - P^*_i; F)\) is finite dimensional (by excision and the choice of \(K\)) and the image of \(\beta\) is finite dimensional, it follows that the image of \(\gamma\) is finite dimensional. \(\square\)
Lemma 5.4. \( \text{im}[H_k(\tilde{W}, N_0 \cup (\tilde{W} - P_1^*); F) \to H_k(\tilde{W}, N_0 \cup (\tilde{W} - P_3^*); F)] \) is finite dimensional over \( F \).

Proof. Consider the following diagram in which each row is part of the long exact sequence of a triple.

\[
\begin{array}{ccc}
H_k(\tilde{W}, \tilde{W} - P_1^*; F) & \to & H_k(\tilde{W}, N_0 \cup (\tilde{W} - P_1^*); F) \\
\downarrow & & \downarrow \delta_1 \\
H_k(\tilde{W}, \tilde{W} - P_2^*; F) & \to & H_k(\tilde{W}, N_0 \cup (\tilde{W} - P_2^*); F) \\
\alpha_2 \downarrow & & \downarrow \delta_2 \\
H_k(\tilde{W}, \tilde{W} - P_3^*; F) & \to & H_k(\tilde{W}, N_0 \cup (\tilde{W} - P_3^*); F)
\end{array}
\]

Since both \( \beta_1 \) and \( \alpha_2 \) have finite dimensional images (by Lemmas 5.2 and 5.3), an elementary linear algebra argument shows that \( \delta_2 \circ \delta_1 \) has finite dimensional image as well. \( \square \)

Lemma 5.5. There exists an integer \( s \geq 0 \) such that

\[ H_k(\tilde{W}, N_s \cup (\tilde{W} - P_1^*); F) \to H_k(\tilde{W}, N_0 \cup (\tilde{W} - P_3^*); F) \]

is the trivial homomorphism.

Proof. By Lemma 5.4, the image of \( H_k(\tilde{W}, N_0 \cup (\tilde{W} - P_1^*); F) \) in \( H_k(\tilde{W}, N_0 \cup (\tilde{W} - P_3^*); F) \) is finitely generated. We choose \( s \) large enough so that \( N_{-s} \) contains a representative of each of these generators. Then the homomorphism \( H_k(\tilde{W}, N_0 \cup (\tilde{W} - P_1^*); F) \to H_k(\tilde{W}, N_{-s} \cup (\tilde{W} - P_3^*); F) \) is trivial. We shift everything by \( t^s \) in order to reach the desired conclusion. \( \square \)

We end this section with a proof of a weak version of Proposition 3.1. It assumes coefficients in a field, but is valid in the crucial middle dimension.

Proposition 5.6. If the sequence of PL manifolds \( P_3 \subset P_4 \subset P_3 \subset P_2 \subset P_1 \) is such that each of the inclusions \( P_{i+1} \subset P_i \) satisfies the hypotheses of Lemmas 3.2 and 3.4, then the inclusion-induced homomorphism \( H_k(P_1; \Lambda) \to H_k(P_3; \Lambda) \) is trivial for \( k \leq n/2 \)

Proof. The cases \( k < n/2 \) are covered by Corollary 4.4, so we focus on the case \( k = n/2 \). By Poincaré Duality, \( H_k(P_i; \Lambda) \cong H^k(P_i, \partial P_i; \Lambda) \) and, by excision, \( H^k(P_i, \partial P_i; \Lambda) \cong H^k(W, W - P_i; \Lambda) \). Furthermore, the inclusion induced homomorphism \( H_k(P_1; \Lambda) \to H_k(P_3; \Lambda) \) is dual to \( H^k(W, W - P_1; \Lambda) \to H^k(W, W - P_3; \Lambda) \), so we show that this last homomorphism is trivial.

Note that \( H^k(W, W - P_i; \Lambda) = H^k(\widetilde{W}, \tilde{W} - P_i^*; F) \) and that

\[ H^k(\tilde{W}, \tilde{W} - P_i^*; F) = \lim_{q \to \infty} H^k(\tilde{W}, (\tilde{W} - P_i) \cup N_q \cup N_q'; F). \]
Let $s$ be the integer given by Lemma 5.5 and consider the following diagram. (Each row is part of an exact Mayer-Vietoris sequence; coefficients in $\Lambda$ are understood; $E_q = N_q \cup N'_q$.)

$$
\begin{array}{c}
H^{k-1}(\bar{W}, \bar{W} - P^*_1) \rightarrow H^k(\bar{W}, (\bar{W} - P^*_1) \cup E_q) \rightarrow H^k(\bar{W}, (\bar{W} - P^*_1) \cup N_q) \\
0 \downarrow \quad \downarrow \quad \downarrow 0 \\
H^{k-1}(\bar{W}, \bar{W} - P^*_3) \rightarrow H^k(\bar{W}, (\bar{W} - P^*_3) \cup E_{q+s}) \rightarrow H^k(\bar{W}, (\bar{W} - P^*_3) \cup N_{q+s}) \\
0 \downarrow \quad \downarrow \quad \downarrow 0 \\
H^{k-1}(\bar{W}, \bar{W} - P^*_5) \rightarrow H^k(\bar{W}, (\bar{W} - P^*_5) \cup E_{q+2s}) \rightarrow H^k(\bar{W}, (\bar{W} - P^*_5) \cup N_{q+2s})
\end{array}
$$

The vertical arrows on the right are 0 by Lemma 5.5 (translated by $t^q$ and $t^{q+s}$). The vertical arrows on the left are 0 by Lemma 3.3. It follows that the composite vertical arrow in the center is 0 as well. Taking a limit as $q \rightarrow \infty$ gives the desired conclusion. \hfill \Box

6. Proof of Theorem 1.3

As observed in §3, it suffices to prove Proposition 3.1. Most cases of Proposition 3.1 were proved in §4. The only remaining case is that in which $n$ is even and $k = n/2$.

Use Propositions 4.3 and 5.6 to choose PL manifolds $P_2 \subset P_1$ satisfying the hypotheses of Lemmas 3.2 and 3.4 such that

1. $P_1$ has a cell decomposition with no cells of dimension greater than $k$, and
2. $H_k(P^*_2; \mathbb{Q}) \rightarrow H_k(P^*_1; \mathbb{Q})$ is trivial.

Since $H^{k+1}(P^*_1; \mathbb{Z}) = 0$, the Universal Coefficient Theorem for Cohomology shows that $\text{Ext}(H_k(P^*_1; \mathbb{Z})) = 0$. Thus $H_k(P^*_1; \mathbb{Z})$ is torsion-free [HS, Theorem III.6.1]. Consider the commutative diagram

$$
\begin{array}{c}
H_k(P^*_2; \mathbb{Z}) \xrightarrow{\alpha} H_k(P^*_1; \mathbb{Z}) \\
\downarrow \otimes \mathbb{Q} \quad \downarrow \otimes \mathbb{Q} \\
H_k(P^*_2; \mathbb{Q}) \rightarrow H_k(P^*_1; \mathbb{Q}).
\end{array}
$$

Let $x \in H_k(P^*_1; \mathbb{Z})$. If $\alpha(x) \in H_k(P^*_1; \mathbb{Z})$ is not 0, then its image in $H_k(P^*_1; \mathbb{Q})$ is not 0 either by [HS, Exercise III.7.5]. But the bottom arrow is the trivial homomorphism by the choice of $P_1$ and $P_2$. Hence $\alpha = 0$ and the proof is complete.

References


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