Cell-like images and UV^m groups

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Abstract. Let X and A be compact metric spaces. The main problem studied in this paper is that of finding conditions under which a map $f : A \to X$ can be lifted to a cell-like space; i.e., conditions are sought under which there exist a cell-like continuum Z and continuous maps $g : Z \to X$ and $f' : A \to Z$ such that $g \circ f' = f$. A theorem is proved which spells out technical conditions on the embedding of A into X and on the homotopy pro-proups of X under which such a lifting exists. The main corollary asserts the following: If X is UV^{k+1}, $\dim A \leq k$, and $f : A \to X$ is continuous, then there exist a cell-like continuum Z and maps $g : Z \to X$ and $f' : A \to Z$ such that $g \circ f' = f$. An example is constructed which shows that the hypothesis cannot be weakened to UV^k. The theorem and example are both applied to the problem of calculating UV^m groups.

Introduction

Let X be a topological space and k a nonnegative integer. In [**Mr**$_2$], Mrozik defines a collection of variations on the k-th homotopy group of X. For each nonnegative integer m, he defines the k-th UV^m group $\pi_k^{(m)}(X)$ and also defines the k-th CE group $\pi_k^{\mathrm{CE}}(X)$. In this note we make some observations concerning those groups. We show that, for fixed X and k, these groups are nearly all isomorphic. Specifically, we show that the natural homomorphism $\pi_k^{\mathrm{CE}}(X) \to \pi_k^{(m)}(X)$ is an isomorphism for $m > k$. Mrozik has already shown [**Mr**$_2$, Proposition 2.8] that $\pi_k^{(m)}(X) = 0$ if $m < k$. Thus for each X and k there are at most two distinct groups: $\pi_k^{(k)}(X)$ and $\pi_k^{\mathrm{CE}}(X)$. We show by example that these two groups are, in general, different from each other and also different from either the ordinary homotopy group $\pi_k(X)$ or the k-th shape group $\check{\pi}_k(X)$. It should be noted that for LC^k spaces, all the groups are the same [**Mr**$_2$, Theorem 2.7].

The main tool used in this note is a lifting theorem for compacta. The theorem should be viewed as a relative version of Ferry's theorem in [**Fe**$_2$]. Ferry gives conditions under which a compactum X is the continuous image of a cell-like continuum Z. In our relative version of Ferry's theorem we begin with a compact subset A of X and ask whether or not A can be lifted to Z. That is, given $A \subset X$ we look for a cell-like continuum $Z \supset A$ and a map $g : Z \to X$ such that $g|A = id$. Obviously some restriction on the embedding of A into X is required. In particular, A must be contractible in any neighborhood of X. Before stating our theorem we give this property a name.

Definition: Let X be a continuum embedded in an ANR and $f : A \to X$ be a map. We say that f is *UV-inessential* if f is null-homotopic in U for every neighborhood U of X. It is easy to see that this definition depends only on f and X and is independent of the embedding of X. The *fundamental dimension* of X, denoted $\mathrm{Fd}(X)$ and also called the *shape dimension* of X, is defined to be the minimum of $\{\dim Y \mid Y$ has the shape of $X\}$.

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Theorem 1. *Suppose X is a continuum with pro-$\pi_1(X)$ profinite and k is a nonnegative integer such that pro-$\pi_i(X)$ is stable for $i \leq k$ and Mittag-Leffler for $i = k+1$. If A is a compactum with $Fd(A) \leq k$ and $f : A \to X$ is a UV-inessential map, then there exist a cell-like continuum Z and maps $g : Z \to X$ and $f' : A \to Z$ such that $g \circ f' = f$.*

Corollary. *If X is a UV^{k+1} continuum, $Fd(A) \leq k$, and $f : A \to X$, then there exist a cell-like continuum Z and maps $g : Z \to X$ and $f' : A \to Z$ such that $g \circ f' = f$.*

It is obvious that the UV-inessential hypothesis is necessary in Theorem 1, but the necessity of the condition on pro-π_{k+1} is less obvious. It might also seem at first that UV^k would suffice as a hypothesis in the Corollary since $\mathrm{Fd}(X) \leq k$ and X having property UV^k are enough to ensure that f is UV-inessential. But we show by example that the UV^k hypothesis is not enough and that some hypothesis on pro-π_{k+1} is needed.

Example 1. *Let Σ^{k+1} denote the k-fold suspension of the dyadic solenoid. There exists a map $f : S^k \to \Sigma^{k+1}$ for which no lift to a UV^{k+1} continuum is possible. Specifically, there do not exist a UV^{k+1} space Z and maps $g : Z \to \Sigma^{k+1}$ and $f' : S^k \to Z$ such that $g \circ f' = f$.*

The Corollary to Theorem 1 is used to prove our result about UV^m groups.

Theorem 2. *If X is a continuum and $m > k$, then the natural homomorphism $\pi_k^{\mathrm{CE}}(X) \to \pi_k^{(m)}(X)$ is an isomorphism.*

This means that, for fixed k, all the groups $\pi_k^{(m)}(X)$, $m > k$, are isomorphic to each other. But we show by example that $\pi_k^{(k)}(X)$ may be different.

Example 2. *If Σ^{k+1} is as in Example 1, then $\pi_k^{(k)}(\Sigma^{k+1}) = 0$ but $\pi_k^{(k+1)}(\Sigma^{k+1}) \neq 0$.*

1. Definitions and notation

The reader is referred to [**Mr**$_2$] for most definitions needed in this note. In particular, the definitions of LC^n and UV^n can be found there. A compactum is said to be *cell-like* (abbreviated CE) if it is contractible in any neighborhood of itself when embedded in an ANR. The kth homotopy pro-group of X is denoted pro-$\pi_k(X)$ and consists of the inverse system of homotopy groups of neighborhoods of X. The definitions of such properties as *stability* and *Mittag-Leffler* associated with these pro-groups can be found in [**MS**].

The UV^m groups and CE groups are defined in [**Mr**$_2$]; the reader should consult that reference for technical details. We merely give the intuitive idea here. Recall that an element of $\pi_k(X)$ is represented by a map $\beta : B^k \to X$ with the property that $\beta(S^{k-1})$ is a point. In a similar way, an element of $\pi_k^{(m)}(X)$ is represented by a triple (C, α, β) in which C is a UV^m compactum, α and β are maps with

$$S^{k-1} \xrightarrow{\alpha} C \xrightarrow{\beta} X,$$

and $\beta \circ \alpha$ is constant. An element of $\pi_k^{\mathrm{CE}}(X)$ is represented by the same kind of triple (C, α, β) in which C is cell-like. An equivalence relation is defined on these triples which involves moving across UV^m or CE sets. Thus the UV^m groups are more rigid than the shape groups in that each element is represented by an actual map into the space, but less confining than the homotopy groups because UV^m spaces are used as the domains instead of contractible ones.

2. LIFTINGS TO CELL-LIKE COMPACTA

In this section we present the proof of Theorem 1. The proof follows the outline of the proof of the main theorem in [**Fe**$_2$], but care must be taken to make sure that it is possible to relativize at each step. We begin by stating a proposition which is a relative version of Theorem $2'$ in [**Fe**$_1$]. This more technical version of the theorem is proved in the same way as the original, but extra care must be taken in the construction of the maps. This is particularly true of the part of the proof that corresponds to [**Fe**$_1$, Lemma 3.1].

PROPOSITION 1. *If $f : K \to L$ is a PL map between finite polyhedra and both K and L have finite fundamental groups, then for every n there exist a finite polyhedron $K' \supset K$, a CE-PL retraction $r : K' \to K$, and an AF^n map $f' : K' \to L$ such that $f \circ r$ is homotopic to f'. Moreover, if $L_0 \subset K \cap L, L_0 \hookrightarrow L$ is inessential, and $f|L_0 = id$, then we can choose f' so that $f'|L_0 = id$.*

The proof of Proposition 1 is based on two lemmas. The two lemmas correspond to Theorem $1'$ and Lemma 3.1 of [**Fe**$_1$].

LEMMA 1. *If $f : K \to L$ is a $(k+1)$-connected map of finite polyhedra, then there exist a finite polyhedron $K' \supset K$, a CE-PL retraction $r : K' \to K$, and a UV^k-PL map $f' : K' \to L$ such that $f \circ r$ is homotopic to f'. Moreover, if $L_0 \subset K \cap L$ and $f|L_0 = id$, then we can choose f' so that $f'|L_0 = id$.*

PROOF: This is simply a more technical statement of what Ferry actually proves in [**Fe**$_1$]. He constructs a diagram

$$\begin{tikzcd} K' \arrow[r, "g"] \arrow[d, "r"'] & M(f) \arrow[d, "c"] \\ K \arrow[r, "f"'] & L \end{tikzcd}$$

in which $K \subset K'$, r is a CE-PL retraction, g is UV^k, c is the mapping cylinder retraction, and the upper triangle commutes exactly on K. [Here $M(f)$ denotes the mapping cylinder of f.] ∎

LEMMA 2. *Let B be a connected finite PL cell complex and let $p : E \to B$ be a Hurewicz fibration from an ANR to B with fiber $\mathcal{F}$. If $\mathcal{F}$ has finite skeleta, then for every n there* *exist a finite polyhedron K, a PL-AF^n map $g : K \to B$ and an n-connected map $h : K \to E$ such that $p \circ h = g$. Moreover, if L_0 is a connected subcomplex of B such that $L_0 \hookrightarrow B$ is inessential, then K can be chosen so that $L_0 \subset K$ and $g|L_0 = id$.*

PROOF: Since $L_0 \hookrightarrow B$ is inessential, p is trivial over L_0. Thus there exists a commutative diagram

$$\begin{tikzcd} L_0 \times \mathcal{F} \arrow[r, "h''"] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

in which $h''|\{x\} \times \mathcal{F}$ is a homotopy equivalence from $\{x\} \times \mathcal{F}$ to $p^{-1}(x)$ for every $x \in L_0$. Let F be the $(n+1)$-skeleton of $\mathcal{F}$, $K' = L_0 \times F$, and $h' = h''|K'$. Then we have

$$\begin{tikzcd} K' \arrow[r, "h''"] & E|L_0 \arrow[d, "p|"] \\ & B \end{tikzcd}$$

in which $p' = p''|K'$ is AF^n and the restriction of h' to any fiber is n-connected.

The proof is completed by inductively extending to the cells of $B - L_0$ as in [**Fe**$_1$]. (See the top of p. 374 in [**Fe**$_1$].) ▮

PROOF OF PROPOSITION 1: The proof parallels that of [**Fe**$_1$, Theorem 2$'$], as explained in the middle of p. 374 of [**Fe**$_1$]. The map $f : K \to L$ is homotopy equivalent to a fibration $p : E \to L$ with fiber $\mathcal{F}$. By [**Fe**$_1$, Lemma 3.2], $\mathcal{F}$ has finite skeleta so we can apply Lemma 2 to this fibration. With $N = \max\{n+2, \dim K + 2\}$, this produces

$$\begin{tikzcd} K \arrow[r, "\simeq"] & E \arrow[d, "p"'] & K'' \arrow[l, "h''"'] \\ & L & \end{tikzcd}$$

in which $K \to E$ is a homotopy equivalence, h'' is N-connected, and p'' is AF^N. Since $K \to E$ is a homotopy equivalence and h'' is N-connected, there is a map $d : K'' \to K$ which is N-connected. Now $N > \dim K + 1$, so there exists $s : K \to K''$ such that $d \circ s$ is homotopic to id. Notice that s will be $(N-1)$-connected. Furthermore, we can choose s to be an extension of the identity map from $L_0 \subset K$ to $L_0 \subset K''$.

The final step in the proof is to apply Lemma 1 to $s : K \to K''$. This gives

$$\begin{tikzcd} K' \arrow[r, "s'"] \arrow[d, "r"'] & K'' \arrow[d, "p''"] \\ K \arrow[r, "f"'] & L \end{tikzcd}$$

in which r is a CE-PL retraction and s' is UV^n. Then $f' = p'' \circ s'$ is the composition of a UV^n and an AF^n map and is therefore AF^n. All maps are the identity an L_0. ∎

Proof of Theorem 1: We observe that, without loss of generality, we may assume in Theorem 1 that $f : A \to X$ is an inclusion map. This is so because if the conclusion of the theorem holds for the inclusion $A \hookrightarrow M(f)$, then it holds for $f : A \to X$. We write $X = \varprojlim\{K_i, \alpha_i\}$ where each K_i is a finite polyhedron, $\alpha_i : K_i \to K_{i-1}$ is a PL map, and the induced homomorphisms $\pi_m(K_i) \to \pi_m(K_{i-1})$ are isomorphisms for $m \leq k$ and epimorphisms for $m = k + 1$ (see [**Fe**$_1$, p. 382]). We further assume that each α_i is the composition of an inclusion map followed by a projection of the form

$$\alpha_i : K_i \xrightarrow{\text{inclusion}} K_{i-1} \times I^{n_i} \xrightarrow{\text{projection}} K_{i-1}.$$

It is obvious that each α_i can be approximated by a composition of that sort (for n_i sufficiently large), and Brown's Theorem [**Br**, Theorem 2] allows us to replace α_i with such an approximating map. Let L_i be a subpolyhedron of K_i such that $A = \varprojlim\{L_i, \beta_i\}$ with $\beta_i = \alpha_i|L_i$. Since $\mathrm{Fd}(A) \leq k$, we may assume that $\dim L_i \leq k$ for each i.

Theorem 1 will be proved by constructing a diagram analogous to that on p. 269 of [**Fe**$_2$], but with the additional feature that each L_i can be lifted. Since f is UV-inessential, the inclusion $L_0 \hookrightarrow K_0$ extends to a map $cL_0 \to K_0$, where cL_0 is the cone on L_0. We apply Proposition 1 to the map $cL_0 \to K_0$. This gives us a homotopy commuting diagram of the form

$$\begin{tikzcd}
D_0 \arrow[d] & \\
cL_0 \arrow[r] & K_0
\end{tikzcd}$$

in which v_0 is a PL-AF^{k+1} map, $D_0 \to cL_0$ is a PL-CE retraction, and $v_0|L_0 = id$.

Recall that we can factor α_1 as

$$K_1 \xrightarrow{\subset} K_0 \times I^{n_1} \xrightarrow{\text{proj}} K_0.$$

Define $P_1 = (v_0 \times id)^{-1}(K_1) \subset D_0 \times I^{n_1}$ and $v_0' = (v_0 \times id)|P_1$. Then we have a commutative diagram

$$\begin{tikzcd}
D_0 \arrow[d, "v_0"'] & D_0 \times I^{n_1} \arrow[l, "\text{proj}"'] \arrow[d, "v_0 \times id"'] & P_1 \arrow[l, "\supset"'] \arrow[d, "v_0'"] \\
K_0 & K_0 \times I^{n_1} \arrow[l, "\text{proj}"'] & K_1. \arrow[l, "\supset"']
\end{tikzcd}$$

We claim that P_1 is k-connected. In order to check this, let $g : S^m \to P_1$ be a map, $m \leq k$. Since D_0 is contractible, g extends to $G : D^{m+1} \to D_0 \times I^{n_1}$. Now the sequence

$$\pi_{j+1}(K_1) \to \pi_{j+1}(K_0 \times I^{n_1}) \to \pi_{j+1}(K_0 \times I^{n_1}, K_1) \to \pi_j(K_1) \to \pi_j(K_0 \times I^{n_1})$$

is exact, so $\pi_{j+1}(K_0 \times I^{n_1}, K_1) = 0$ for $j \leq k$ and thus there is a homotopy of $(v_0 \times id) \circ G$ which pulls D^{m+1} into K_1 and keeps S^m fixed. Since v_0 is an AF^{k+1} map, this homotopy can be lifted to a homotopy in $D_0 \times I^{n_1}$ that pulls D^{m+1} into a small neighborhood of P_1, keeping S^m fixed. Since P_1 is a polyhedron, this is enough to conclude that g is null-homotopic in P_1.

Now $L_1 \subset K_1 \cap (L_0 \times I^{n_1})$ and $v_0|L_0 = id$. Thus $L_1 \subset P_1$ and by the previous paragraph we can extend the inclusion $L_1 \hookrightarrow P_1$ to a map $cL_1 \to P_1$. We apply Proposition 1 to the map $cL_1 \to P_1$ to produce a contractible polyhedron D_1, a PL-CE retraction $D_1 \to cL_1$ and an AF^{k+1} map $\mu_1 : D_1 \to P_1$. If we define v_1 to be $v_0' \circ \mu_1$, we have a commutative diagram of the form

$$\begin{tikzcd}
D_0 \arrow[d, "v_0"'] & D_1 \arrow[l, "\gamma_1"'] \arrow[d, "v_1"] \\
K_0 & K_1 \arrow[l, "\alpha_1"']
\end{tikzcd}$$

with the property that $L_0 \subset D_0$, $L_1 \subset D_1$, and $\gamma_1|L_1 = \beta_1 : L_1 \to L_0$. The construction above is repeated recursively to produce an infinite diagram

$$\begin{tikzcd}
D_0 \arrow[d, "v_0"'] & D_1 \arrow[l, "\gamma_1"'] \arrow[d, "v_1"'] & D_2 \arrow[l, "\gamma_2"'] \arrow[d, "v_2"'] & D_3 \arrow[l, "\gamma_3"'] \arrow[d, "v_3"'] & \cdots \arrow[l] \\
K_0 & K_1 \arrow[l, "\alpha_1"'] & K_2 \arrow[l, "\alpha_2"'] & K_3 \arrow[l, "\alpha_3"'] & \cdots \arrow[l]
\end{tikzcd}$$

To complete the proof we define $Z = \varprojlim\{D_i, \gamma_i\}$ and $g = \varprojlim\{v_i\}$. Since Z is the inverse limit of a sequence of contractible polyhedra, it is cell-like. We have constructed D_i in such a way that $L_i \subset D_i$, so we have the desired lift $f' : A \to Z$.

3. RELATIONSHIP BETWEEN UV^m GROUPS AND CE GROUPS

In this section we prove Theorem 2. There is a natural homomorphism $h_m : \pi_k^{\mathrm{CE}}(X) \to \pi_k^{(m)}(X)$ defined as follows. An element of $\pi_k^{\mathrm{CE}}(X)$ is an equivalence class of a triple (C, α, β) in which C is cell-like, $\alpha : S^{k-1} \to C$, and $\beta : C \to X$. Since every cell-like space is UV^m, the triple (C, α, β) also represents an element of $\pi_k^{(m)}(X)$. Define h_m by $h_m([C, \alpha, \beta]) = [C, \alpha, \beta] \in \pi_k^{(m)}(X)$.

It is clear that h_m is a well-defined homomorphism. We show that, for $m > k$, h_m is one-to-one and onto by constructing a function $H_m : \pi_k^{(m)}(X) \to \pi_k^{\mathrm{CE}}(X)$ which is a 2-sided inverse for h_m. One application of the Corollary to Theorem 1 is required to define H_m and a second is required to prove that H_m in well-defined.

Let $[C, \alpha, \beta]$ be an element of $\pi_k^{(m)}(X)$. Then C is UV^m and

$$S^{k-1} \xrightarrow{\alpha} C \xrightarrow{\beta} X.$$

By the Corollary to Theorem 1, there exists a cell-like compactum C' and maps $\alpha' : S^{k-1} \to C'$ and $f : C' \to C$ such that $f \circ \alpha' = \alpha$. If we define $\beta' = \beta \circ f$, we have a commutative diagram

$$\begin{tikzcd}
& C' \arrow[d, "f"'] & \\
S^{k-1} \arrow[r, "\alpha"] & C \arrow[r, "\beta"] & X,
\end{tikzcd}$$

so $[C', \alpha', \beta'] = [C, \alpha, \beta]$ in $\pi_k^{(m)}(X)$. Define $H_m([C, \alpha, \beta]) = [C', \alpha', \beta'] \in \pi_k^{\mathrm{CE}}(X)$. We must show that H_m is well-defined. Once that is done the proof is complete since it is obvious that H_m is inverse to h_m.

To show H_m well-defined, we must prove the following: *If C_1 and C_2 are UV^m spaces, Z_1 and Z_2 are cell-like spaces, and maps $\alpha_1, \beta_1, f_1, g, \alpha_2, \beta_2,$ and f_2 are defined so that the diagram*

$$\begin{tikzcd}
& Z_1 \arrow[r, "f_1"] & C_1 \arrow[d, "g"'] & \\
S^{k-1} \arrow[r, "\alpha_2"] & Z_2 \arrow[r, "f_2"] & C_2 \arrow[r, "\beta_2"] & X
\end{tikzcd}$$

commutes, then $[Z_1, \alpha_1, \beta_1 \circ f_1] = [Z_2, \alpha_2, \beta_2 \circ f_2]$ in $\pi_k^{\mathrm{CE}}(X)$.

We may assume that both α_1 and α_2 are collared embeddings. (Otherwise, we just replace Z_i with $M(\alpha_i)$.) Form a continuum A from the disjoint union of Z_1 and Z_2 by identifying each point $\alpha_1(x)$ with $\alpha_2(x)$, $x \in S^{k-1}$. Notice that A has the shape of S^k (since there is a cell-like map from A to the suspension of S^{k-1}) and therefore $\mathrm{Fd}(X) = k < m$. Also, there exists a map $f : A \to C_2$ defined by $f|Z_1 = g \circ f_1$ and $f|Z_2 = f_2$. Applying the Corollary to the map $f : A \to C_2$ gives a cell-like space Z_0 and maps $f' : A \to Z_0$ and $g' : Z_0 \to C_2$ such that $g' \circ f' = f$.

The diagram

$$\begin{tikzcd}
& Z_1 \arrow[d] & & & \\
S^{k-1} \arrow[r, "\alpha"] & A \arrow[r, "f'"] & Z_0 \arrow[r, "g'"] & C_2 \arrow[r, "\beta_2"] & X \\
& Z_2 \arrow[u] & & &
\end{tikzcd}$$

shows that $[Z_1, \alpha_1, \beta_1 \circ f_1] = [Z_0, f' \circ \alpha, \beta_2 \circ g'] = [Z_2, \alpha_2, \beta_2 \circ f_2]$.

4. Examples

We begin this section with two simple examples which illustrate the point that the UV^m groups are different from either the homotopy groups or the shape groups. Let C denote the $(k+1)$-dimensional example constructed by Daverman and Venema in [**DV**, §3]. The 2-dimensional version is pictured below.

A 2-dimensional example

If $k > 1$, then $\check{\pi}_k(C) \cong \pi_k(S^1) = 0$ while $\pi_k^{(k)}(C)$ and $\pi_k^{\mathrm{CE}}(C)$ are uncountable [**Mr**$_2$, Theorem 4.1 and Lemma 5.3]. This shows that the UV^k and CE groups are, in general, different from the shape groups. On the other hand, if W is the Warsaw Circle, then $\check{\pi}_1(W) \cong \pi_1^{(1)}(W) \cong \pi_1^{\mathrm{CE}}(W) \cong \mathbf{Z}$, but $\pi_1(W) = 0$, so these groups are also different from the ordinary homotopy groups.

The main purpose of this section is to show by example that the two groups $\pi_k^{(k)}(X)$ and $\pi_k^{(k+1)}(X) \cong \pi_k^{\mathrm{CE}}(X)$ are not always the same (as they are for the two examples described above and for all the examples constructed in [**Mr**$_2$]). Let Σ^1 denote the dyadic solenoid. Then $\Sigma^1 = \varprojlim\{S_i^1, \alpha_i^1\}$, where each $S_i^1 = S^1$ and $\alpha_i^1 : S_i^1 \to S_{i-1}^1 : z \mapsto z^2$ for $i \geq 1$. We use $\alpha_{i,j}$ to denote the composition $\alpha_{j+1} \circ \cdots \circ \alpha_i : S_i^1 \to S_j^1$. We represent each point in S^1 by an angle $\theta \in [0, 2\pi)$ in the natural way. We want to define an embedding $f_0 : S^0 \to \Sigma^1$ in such a way that $f_0(-1)$ and $f_0(+1)$ are in different path components of Σ^1. Specifically, let us choose

$$\begin{aligned} f_0(-1) &= (0, 0, 0, \dots), \quad \text{and} \\ f_0(+1) &= (0, \pi, \frac{3\pi}{2}, \frac{3\pi}{4}, \frac{11\pi}{8}, \frac{11\pi}{16}, \frac{43\pi}{32}, \dots). \end{aligned}$$

The important thing to notice is that any path joining $f_0(-1)$ to $f_0(+1)$ in S_k^1, $k \geq 4$, will wind around S_0^1 more than k times. We state this formally as follows.

OBSERVATION. *Let A_i be an arc in S^1_i which joins the i-th coordinate of $f_0(-1)$ to the i-th coordinate of $f_0(+1)$. For every i there exists n_i such that if $j \geq n_i$ then the loop $B_{i,j}$ in S^1_0 formed by $\alpha_{i,1}(A_i)$ and $[\alpha_{j,1}(A_j)]^{-1}$ will be homologically nontrivial in S^1_0.*

Now define Σ^{k+1} to be the k-fold suspension of Σ^1. Then

$$\Sigma^{k+1} = \varprojlim \left\{ S^{k+1}_i, \alpha^{k+1}_i \right\}$$

with each $S^{k+1}_i \cong S^{k+1}$ and $\alpha^{k+1}_i : S^{k+1}_i \to S^{k+1}_{i-1}$ the k-fold suspension of α^1_i. Let $f_k : S^k \to \Sigma^{k+1}$ be the k-fold suspension of f_0, let $p_i : \Sigma^{k+1} \to S^{k+1}_i$ be the natural projection map and let $f^i_k = p_i \circ f_k$.

The proof that Σ^{k+1} has the properties listed in the introduction is most naturally stated in terms of approaching maps. The definition of an approaching map may be found in [**Mr**$_1$]. We need the following concept.

DEFINITION: Suppose A and B are compacta with $A \subset B$, X is a compact subset of the ANR M and $f : A \to X$ is a map. We say that f *extends to an approaching map of B into M* if there exists a map $F : B \times [0, \infty) \to M$ such that F approaches X and $F(x,t) = f(x)$ for every $x \in A$.

LEMMA 3. *If M is an ANR containing Σ^{k+1}, then there is no extension of f_k to an approaching map $F : B \times [0, \infty) \to M$ for any CE compactum B containing S^k.*

PROOF: Suppose, on the contrary, that B and F exist. Then for each i the map f^i_k extends to a map $g_i : B^{k+1} \to S^{k+1}_i$ such that g_i is homotopic to $\alpha^{k+1}_{i+1} \circ g_{i+1}$ keeping S^k fixed. [Here B^{k+1} denotes the standard $(k+1)$-ball whose boundary is S^k.] Put g_i in general position keeping S^k fixed. Then there is an arc A_i in $g_i(B^{k+1} \cap S^1_i)$ which joins the i-th component of $f_0(-1)$ to the i-th component of $f_0(1)$. These arcs determine loops $B_{i,j}$ in S^1_0 just as in the Observation, above. If the homotopy from g_i to $\alpha^{k+1}_{i+1} \circ g_{i+1}$ is put in general position in S^{k+1}_0 and then intersected with S^1_0, it gives a homology from A_i to $\alpha_{i+1}(A_{i+1})$. But this means that the loops $B_{i,j}$ are all null-homologous in S^1_0 and the Observation is contradicted. ∎

PROPOSITION 2. *There do not exist a UV^{k+1} space Z and maps $g : Z \to \Sigma^{k+1}$ and $f' : S^k \to Z$ such that $g \circ f' = f_k$.*

PROOF: Suppose, on the contrary, that Z, g, and f' exist. Let M_1 and M_2 be ANR's containing Z and Σ^{k+1}, respectively. Now g can be extended to $g : N \to M_2$, where N is a neighborhood of Z in M_1. Define $F' : S^k \times [0, \infty) \to N$ by $F'(x,t) = f'(x)$. Since Z is k-UV, we can extend F' to $B^{k+1} \times \{n\}$ for each nonnegative integer n in such a way that $F'(B^{k+1} \times \{n\})$ approaches Z as n approaches ∞. The fact that Z is $(k+1)$-UV allows us to extend F' to an approaching map $F' : B^{k+1} \times [0, \infty) \to M_1$. Define $F = g \circ F'$. Then F is an approaching map whose existence contradicts Lemma 3. ∎

Proposition 3. $\pi_k^{(k)}(\Sigma^{k+1}) = 0$.

Proof: Let $[C, \alpha, \beta]$ be an element of $\pi_k^{(k)}(\Sigma^{k+1})$. The fact that Σ^{k+1} is UV^k means that $[\Sigma^{k+1}, \beta \circ \alpha, id]$ also represents an element of $\pi_k^{(k)}(\Sigma^{k+1})$ and so the diagram

$$\begin{tikzcd}
& C \arrow[d, "\beta"'] & \\
S^{k-1} \arrow[r, "\beta\circ\alpha"] & \Sigma^{k+1} \arrow[r, "id"] & \Sigma^{k+1} \\
& \{*\} \arrow[u] &
\end{tikzcd}$$

shows that $[C, \alpha, \beta]$ represents the trivial element of $\pi_k^{(k)}(\Sigma^{k+1})$. ∎

Remark: The proof above shows that $\pi_k^{(m)}(X) = 0$ for every UV^m space X and for every k.

Proposition 4. $\pi_k^{\mathrm{CE}}(\Sigma^{k+1}) \neq 0$.

Proof: Let $\alpha : S^{k-1} \hookrightarrow B^k$ be the inclusion and define $\beta : B^k \to \Sigma^{k+1}$ to be the projection $B^k \to B^k/S^{k-1} = S^k$ followed by f_k. We will show that $[B^k, \alpha, \beta]$ is nontrivial in $\pi_k^{(k+1)}(X)$. If $[B^k, \alpha, \beta] = 0$, then there is a sequence

$$(C_0, \alpha_0, \beta_0), (C_1, \alpha_1, \beta_1), \ldots, (C_n, \alpha_n, \beta_n)$$

such that $(C_0, \alpha_0, \beta_0) = (B^k, \alpha, \beta)$, β_n is constant, and, for each i, C_i is a CE compactum with either $(C_i, \alpha_i, \beta_i) \geq (C_{i+1}, \alpha_{i+1}, \beta_{i+1})$ or $(C_i, \alpha_i, \beta_i) \leq (C_{i+1}, \alpha_{i+1}, \beta_{i+1})$ (see [**Mr**$_2$] for definitions). By replacing each C_i with the mapping cylinder of α_i, if necessary, we can arrange that each α_i is an embedding. Define $D_i = C_i/\alpha_i(S^{k-1})$. Then each D_i has the shape of S^k and β_i determines a map $\beta_i' : D_i \to \Sigma^{k+1}$. We will prove the following claim: If β_i' has the property that β_i' extends to an approaching map $B \times [0, \infty) \to \Sigma^{k+1}$ for some CE compactum B, then so does β_{i-1}'. Since β_n' clearly has this property and β_0' does not (by Lemma 3), this will give us the contradiction we need to complete the proof. The claim is formalized in Lemma 4, below. ∎

Lemma 4. *For $i = 1, 2$, let D_i be a compactum having the shape of S^k, and let $\beta_i : D_i \to \Sigma^{k+1}$ be a map. Suppose $g : D_1 \to D_2$ is a map such that g is a shape equivalence and $\beta_1 = \beta_2 \circ g$. If M is an ANR containing Σ^{k+1}, then β_1 extends to an approaching map of some CE compactum into M if and only if β_2 does.*

Proof: First, suppose that β_2 extends to such an approaching map. Then there exists a cell-like compactum B_2 containing D_2 and an approaching map $F_2 : B_2 \times [0, \infty) \to M$ which extends β_2. Define $g' : D_1 \to B_2$ to be g composed with the inclusion $D_2 \hookrightarrow B_2$ and define B_1 to be the mapping cylinder of g'. Then there is a strong deformation retraction of B_1 to B_2 so B_1 is cell-like. We identify D_1 with $D_1 \times \{0\} \subset B_1$. Define $F_1 : B_1 \times [0, \infty) \to M$ by $F_1(x,t) = F_2(x,t)$ if $x \in B_2$ and $F_1((y,s),t) = \beta_1(y)$ if $(y,s) \in B_1 \times [0,1]$. It is easy to check that F_1 is a well-defined approaching map of B_1 into M which extends β_1.

Second, suppose there exists a cell-like compactum B_1 containing D_1 and an approaching map $F_1 : B_1 \times [0, \infty) \to M$ which extends β_1. In this case we define B_2 to be $B_1 \cup M(g)$ (where D_1 is identified with $D_1 \times \{0\} \subset M(g)$ — the top of $M(g)$) and define F_2 in a way that is analogous to the definition of F_1 in the paragraph above. The only thing that is different in this case is the fact that B_2 does not necessarily strong deformation retract to B_1, so it is less obvious that B_2 is cell-like. But this is where the fact that g is a shape equivalence comes in. Just as the mapping cylinder of a homotopy equivalence of finite polyhedra strong deformation retracts to the top, so the mapping cylinder of a shape equivalence strong deformation retracts to the top in any neighborhood of itself. This is enough to make B_2 cell-like. ∎

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