Neighborhoods of S^1-like continua in 4-manifolds

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Abstract. Let X be a compact subset of an orientable 4-manifold. The problem of determining conditions under which such a compactum has neighborhoods that are homeomorphic to $S^1 \times B^3$ is studied in this paper. It is shown that X has arbitrarily close neighborhoods homeomorphic to $S^1 \times B^3$ if and only if X has the shape of some S^1-like continuum and X satisfies the inessential loops condition.

0. Introduction.

In this paper we study the problem of determining which compact subsets of 4-manifolds have close neighborhoods that collapse to 1-dimensional spines. As is explained in [**10**], the study of this problem is motivated by the desire to understand engulfing of 2-dimensional polyhedra in piecewise linear 4-manifolds. The technology of 4-manifold topology does not seem to be well enough developed for us to completely characterize such compacta. We restrict our attention, therefore, to the case in which the neighborhood collapses to a copy of the circle, S^1. In that case the fundamental groups which arise are infinite cyclic and so we can apply the Z-theory of Freedman and Quinn [**2**], [**3**]. Our main theorem characterizes those compact subsets of 4-manifolds that have arbitrarily close neighborhoods with spines homeomorphic to S^1.

Theorem 1. *Suppose X is a compact subset of the orientable 4-manifold M^4. Then X has arbitrarily close neighborhoods homeomorphic to $S^1 \times B^3$ if and only if*

(1) *X has the shape of some S^1-like continuum, and*
(2) *X satisfies the inessential loops condition.*

Let Y be an S^1-like continuum. Then Y is the inverse limit of an inverse sequence in which each space is S^1. Thus there is a standard embedding of Y in S^4 as the intersection of a nested sequence of thin tubes, each tube homeomorphic with $S^1 \times B^3$. We will identify Y with this embedded copy of Y. The following complement theorem is then a corollary to Theorem 1. We use $\mathrm{Fd}(X)$ to denote the fundamental dimension of X.

Corollary. *Suppose X is a compact subset of S^4, $Fd(X) = 1$, X satisfies the inessential loops condition, and Y is an S^1-like continuum standardly embedded in S^4. Then $S^4 - X \cong S^4 - Y$ if and only if $Sh(X) = Sh(Y)$.*

A less general theorem than Theorem 1 is proved in [**10**]. The theorem proved in [**10**] gives sufficient conditions in order for arbitrarily close neighborhoods homeomorphic to

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$S^1 \times B^3$ to exist, but the conditions given there are not necessary. There are some crucial differences between the proof given here and that in [**10**], but the main construction of the present proof is based on that in [**10**] and so we will refer to [**10**] frequently in the course of proving Theorem 1. The reader will therefore find it necessary to consult [**10**] in order to completely understand the proofs in this paper.

1. Definitions and notation.

Suppose X is a compact subset of the interior of the n-manifold M. We say that X satisfies the *inessential loops condition* (ILC) if for every neighborhood U of X in M there exists a neighborhood V of X in U such that each loop in $V - X$ which is homotopically inessential in V is also inessential in $U - X$. We use "$\simeq$" to denote "is homotopic to" and "$\cong$" to denote either "is homeomorphic to" or "is isomorphic to," depending on the context. When we say that X *has arbitrarily close neighborhoods homeomorphic to* $S^1 \times B^3$ we mean that for every neighborhood U of X there exists a neighborhood N of X such that $N \subset U$ and $N \cong S^1 \times B^3$.

The statement $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ means that X and Y have the same shape. The *fundamental dimension* of X is defined by $\mathrm{Fd}(X) = \min\{\dim Y \mid \mathrm{Sh}(X) = \mathrm{Sh}(Y)\}$. Let P be a polyhedron. A space X is said to be P*-like* if X can be written as the inverse limit of an inverse sequence in which each space is homeomorphic to P. This is equivalent to the assertion that for every $\epsilon > 0$ there exists an onto map $f : X \to P$ such that the diameter of $f^{-1}(y)$ is smaller than ϵ for every $y \in P$. An S^1-like continuum is also called a solenoid.

Consult [**4**] for other definitions related to shape theory. It is not necessary to be familiar with very much shape theory in order to read this paper. In fact, the main use of shape theory is in the following characterization of compacta which have the shape of S^1-like continua. The proposition follows easily from the definitions in shape theory (cf. [**4**]).

PROPOSITION 1.1. *Let X be a compact subset of the n-manifold M. Then X has the shape of some S^1-like continuum if and only if for every neighborhood U of X there exists a smaller neighborhood V of X in U and maps $\alpha : V \to S^1$ and $\beta : S^1 \to U$ such that $\beta \circ \alpha \simeq$ inclusion.*

If P is a polyhedron endowed with a triangulation and i is an integer, then $P^{(i)}$ denotes the i-skeleton of P in that triangulation. We use the notation $P \searrow L$ to mean that P collapses to L.

2. Proof of Theorem 1.

This section contains the proof of one direction of Theorem 1 and the proof of the Corollary. In addition, a more technical Proposition is stated which implies the other direction of Theorem 1. The remainder of the paper will be devoted to the proof of the Proposition.

Suppose, first, that X has arbitrarily close neighborhoods homeomorphic to $S^1 \times B^3$. Let a neighborhood U of X be given and let $f : (B^2, \partial B^2) \to (U, U - X)$ be a map. There exists a neighborhood N of X such that $N \subset U - f(\partial B^2)$ and $N \cong S^1 \times B^3$. The neighborhood N has a PL structure induced by the homeomorphism $N \cong S^1 \times B^3$. Approximate $f|f^{-1}(\operatorname{Int} N)$ by a map which is in general position with repect to that PL structure. This gives a new continuous function $f' : B^2 \to U$ which agrees with f on ∂B^2 but which also has the property that $f'(B^2)$ misses the core, $S^1 \times \{0\}$, of N. Use the radial structure of $B^3 - \{0\}$ to push the image of f' out of $\operatorname{Int} N$ and hence off X. This shows that X satisfies ILC. Since $X = \cap_{i=1}^{\infty} N_i$, where each $N_i \cong S^1 \times B^3$ and $N_{i+1} \subset \operatorname{Int} N_i$, we see that X is homeomorphic to the inverse limit of an inverse sequence in which each space has the homotopy type of S^1. It follows that X has the shape of an S^1-like continuum. This proves that the conditions (1) and (2) listed in Theorem 1 are necessary.

The converse of Theorem 1 follows from the next Proposition. We may assume that the manifold in which X is embedded is a piecewise linear manifold because every connected 4-manifold has a PL structure in the complement of a point [**6**, Corollary 2.2.3].

Proposition 2.1. *Suppose X is a compact subset of the interior of the PL 4-manifold M^4, X has the shape of some S^1-like continuum, and X satisfies the inessential loops condition. Then for every neighborhood U of X there exist a polyhedron $K \subset U$ with $K \cong S^1$, a PL regular neighborhood N of K in U, and a homeomorphism $h : M \to M$ such that $h|M - U$ is the identity and $h(X) \subset N$.*

In case M is orientable, $N \cong S^1 \times B^3$. Notice, also, that $X \subset h^{-1}(N)$; so $h^{-1}(N)$ is the neighborhood needed to complete the proof of Theorem 1.

We now turn to the proof of the corollary. Suppose X and Y are as in the statement of the corollary and that $S^4 - X \cong S^4 - Y$. Then $\operatorname{Sh}(X) = \operatorname{Sh}(Y)$ by [**11**]. Conversely, if $\operatorname{Sh}(X) = \operatorname{Sh}(Y)$, then $S^4 - X \cong S^4 - Y$ by [**10**, Theorem 4.3].

3. Constructing neighborhoods.

Let X be a compact subset of the interior of the PL 4-manifold M and assume that X satisfies conditions (1) and (2) of Theorem 1. In this section we describe the construction of a special sequence of neighborhoods of X which will be used to prove Proposition 2.1. The construction closely parallels that in [**10**, §2], but we include most of the details here since there are several crucial changes which must be made. The notation established in this section will be used throughout the remainder of the paper. We will assume that X has the shape of a nontrivial S^1-like continuum since the case in which X is cell-like has already been done by Freedman [**1**, Theorem 1.11].

Suppose a positive integer n has been given. Let U_0 be a compact, connected PL manifold neighborhood of X in U. Because X has the shape of an S^1-like continuum, there is a neighborhood U_1 of X and a polyhedron K_1 in U_0, $K_1 \cong S^1$, such that the inclusion map $U_1 \hookrightarrow U_0$ is homotopic in U_0 to a map $\beta_1 : U_1 \to K_1$ (see Proposition 1.1). By [**10**, Lemma 1.1], we can push K_1 off X and so we can arrange that $U_1 \cap K_1 = \emptyset$ by just replacing U_1 with a smaller neighborhood, if necessary. We may also assume that U_1 is a compact, connected PL manifold. By [**9**, Theorem 3.1], X does not separate U_1. Thus we can find a finite collection of PL arcs in $U_1 - X$ which connect all the components of ∂U_1. We remove a small regular neighborhood of the union of these arcs from U_1; the result is a new U_1 with the additional property that $U_0 - U_1$ is connected. This construction is continued inductively to define neighborhoods $U_0, U_1, \ldots, U_{2n}$ and 1-dimensional polyhedra $K_1, K_2, \ldots, K_{2n}$ which satisfy the following properties for each $i = 1, \ldots, 2n$:

(1) U_i is a compact, connected PL manifold neighborhood of X in the interior of U_{i-1}.
(2) U_i does not separate U_{i-1}.
(3) K_i is a compact polyhedron in $U_{i-1} - U_i$ such that $K_i \cong S^1$.
(4) There is a homotopy $f_i : U_i \times [0,1] \to U_{i-1}$ such that $f_i(x,0) = x$ and $f_i(x,1) \in K_i$ for every $x \in U_i$.

Let $\hat{f}_i : K_{i+1} \to K_i$ be the map defined by $\hat{f}_i(x) = f_i(x,1)$. Since we have assumed that X has the shape of a nontrivial S^1-like continuum, we may add that

(5) $\hat{f}_i : K_{i+1} \to K_i$ is essential for every i.

So far we have used only the fact that X has the shape of an S^1-like continuum. The ILC hypothesis allows us to gain some control over π_1 as well. By [**8**], $\pi_1(U_i, U_i - X) = 0 = \pi_2(U_i, U_i - X)$, so we can push $f_i(K_{i+1} \times [0,1])$ off X. We do so and then inductively choose U_{i+1} small enough so that $U_{i+1} \cap f_i(K_{i+1} \times [0,1]) = \emptyset$.

Put each of the maps $f_i|K_{i+1} \times [0,1]$ in general position keeping $f_i|K_{i+1} \times \{0,1\}$ fixed. Then $f_i(K_{i+1} \times [0,1)) \cap f_{i+1}(K_{i+2} \times [0,1])$ will consist of a finite number of points. If i is even and ≥ 2, then take a small neighborhood in $f_{i+1}(K_{i+2} \times [0,1])$ of each such intersection point and push the neighborhood along $f_i(K_{i+1} \times [0,1])$ until it is pushed off the $f_i(K_{i+1} \times \{0\})$-end of $f_i(K_{i+1} \times [0,1])$. This removes the points of intersection between $f_i(K_{i+1} \times [0,1))$ and $f_{i+1}(K_{i+2} \times [0,1])$. The price we must pay is that there are new self-intersections introduced in $f_{i+1}(K_{i+2} \times [0,1])$, and $f_{i+1}(K_{i+2} \times [0,1])$ is stretched out so that it no longer stays in U_i, but now maps into U_{i-1}. We can therefore add the following four additional conditions to the list of properties satisfied by the sequences constructed thus far.

(6) $f_i(K_{i+1} \times [0,1]) \cap f_{i+1}(K_{i+2} \times [0,1]) = K_{i+1}$ if $i \geq 2$ is even.
(7) $f_i(K_{i+1} \times [0,1]) \cap f_j(K_{j+1} \times [0,1]) = \emptyset$ for $|i - j| > 1$.
(8) $f_i(K_{i+1} \times [0,1]) \subset U_{i-1} - U_{i+1}$ if i is even.
(9) $f_i(K_{i+1} \times [0,1]) \subset U_{i-2} - U_{i+1}$ if i is odd.

LEMMA 3.1. *If X satisfies the inessential loops condition, then the neighborhoods U_1,*

$U_2, \dots, U_{2n}$ can be chosen so that the inclusion induced homomorphism $\pi_2(U_i, U_i - U_j) \to \pi_2(U_{i-1}, U_{i-1} - U_{j+1})$ is the trivial homomorphism whenever $1 \le i < j \le 2n-1$.

PROOF: The proof is the same as that of [**10**, Lemma 2.1]. ∎

We now construct a new sequence, $V_0, V_1, \dots, V_n$, of neighborhoods of X. These new neighborhoods will improve on the U_i's in the following sense: V_i will contain a copy, K_i', of S^1 such that for $2 \le i \le n$ there is a homotopy of V_i in V_{i-2} to K_i' which keeps K_i' fixed. Thus V_i will homotopically mimic a regular neighborhood of K_i'. Begin by letting $V_0 = U_0$, $K_1' = K_1$, and N_1 be a regular neighborhood of K_1 in U_0.

For each $i \le n$, we use g_i to denote the homotopy $g_i : K_{2i+1} \times [0,1] \to U_{2i-3}$ which consists of $f_{2i}|K_{2i+1}$ followed by $f_{2i-1}|f_{2i}(K_{2i+1} \times \{1\})$. Approximate g_1 with a general position map of $K_3 \times [0,1]$. Now $g_1(K_3 \times \{1\})$ is no longer a subset of K_1, but is still contained in N_1 and is homotopic to the original there. Since $g_1|K_3 \times [0,1]$ is in general position, the only singularities will be a finite number of double points. Let g_1' be the embedding of $K_3 \times [0,1]$ obtained by piping each of these double points off the $(K_3 \times \{1\})$-end of $g_1(K_3 \times [0,1])$. We define L_1 to be $g_1'(K_3 \times [0,1])$ and K_2' to be $g_1'(K_3 \times \{1\})$. Now choose two relative regular neighborhoods, P_1 and P_1', of L_1 modulo K_3 in such a way that P_1' is much thinner than P_1 is. We want these two neighborhoods to fit together correctly near K_3. The simplest way to accomplish this is to be specific about their construction: start with a triangulation of U_0 which includes L_1 as a subcomplex and then define P_1 to be the union of all simplices in the second barycentric subdivision which meet $L_1 - K_3$ and define P_1' to be the union of all simplices in the fourth barycentric subdivision which lie in P_1 and meet L_1. We then define U_1' to be $\overline{U_1 - P_1}$ and define V_1 to be $U_1' \cup P_1'$. The construction of V_1 is illustrated schematically in Figure 1. It is important to notice that condition (6) implies that $g_2(K_5 \times [0,1]) \subset V_1$.

Figure 1

The construction of V_2 is similar to that of V_1. Begin by choosing a regular neighborhood N_2 of K_2' in $N_1 \cap P_1'$. Now L_1 is homeomorphic to $K_3 \times [0,1]$ and shrinking out the fibers defines a map $L_1 \to g_1'(K_3 \times \{1\})$ which can be approximated by a homeomorphism h_1 of M such that h_1 is the identity off a close neighborhood of L_1. Put $g_2 : K_5 \times [0,1] \to U_1'$ in general position and pipe the singularities over the $K_5 \times \{1\}$ end to get an embedding $g_2'|K_5 \times [0,1]$ with the property that $g_2'(K_5 \times \{1\}) \subset h_1^{-1}(N_2)$. We let $L_2' = g_2'(K_5 \times [0,1])$

and define L_2 to be $h_1(L_2')$. Notice that L_2 is just L_2' stretched out so that it stretches all the way from K_5 into N_2. We define K_3' to be the end of L_2 which is in N_2; i.e., $K_3' = h_1(g_2'(K_5 \times \{1\}))$. Let P_2 and P_2' be a pair of relative regular neighborhoods of L_2 modulo K_5 (defined in a way which is analogous to the definition of P_1 and P_1', above) and define $U_3' = U_3 - P_2$ and $V_2 = U_3' \cup P_2'$.

This construction is continued inductively and results in sequences which satisfy the following conditions for each i.

(10) V_i is a compact PL manifold and $V_i \subset \operatorname{Int} V_{i-1}$.
(11) N_i is a regular neighborhood in V_{i-1} of a PL circle K_i'.
(12) $N_i \subset \operatorname{Int} N_{i-1}$.
(13) L_i is a 2-dimensional polyhedron which is a product stretching from K_{2i+1} to K_{i+1}'.
(14) $V_i \searrow U_{2i-1}' \cup L_i$ and $U_{2i-1}' \cap L_i = K_{2i+1}$.

4. Homotopy properties of neighborhoods.

The most important properties of the neighborhoods which were constructed in the previous section are spelled out in the lemmas of this section.

Lemma 4.1. *Whenever $1 \leq i \leq n$, the inclusion map $V_i \hookrightarrow V_{i-2}$ is homotopic in V_{i-2} to a map $\rho_i : V_i \to N_i$ via a homotopy which keeps N_{i+1} fixed.*

Proof: Because of property (14), it is enough to explain how to define the homotopy on $U_{2i-1}' \cup L_i$. Notice, first, that the restiction of f_{2i-1} to U_{2i-1}' defines a homotopy of U_{2i-1}' to K_{2i-1}. If we follow this with the homotopy which shrinks out fibers in L_{i-1}, we get a homotopy of U_{2i-1}' to K_i'. In addition, there is a homotopy of L_i to K_{i+1}' which consists of shrinking out the fibers of L_i. We want to define our homotopy of $U_{2i-1}' \cup L_i$ by pasting these two homotopies together. The problem with doing so is that the two homotopies do not agree on the overlap: $U_{2i-1}' \cap L_i = K_{2i+1}$

Let us use $\alpha_0 : K_{2i+1} \to V_i$ to denote the restriction of f_{2i-1}, $\alpha_1 : K_{2i+1} \to V_i$ to denote the homotopy which consists of the restriction of f_{2i} followed by f_{2i-1}, and use α_2 to denote the homotopy defined by shrinking out the fibers of L_i. We show that these three are homotopic homotopies. Once that fact is established, we construct the homotopy needed to complete the proof by filling in a small collar of K_{2i+1} in L_i with those two homotopies. Since α_0 extends to U_{2i-1} and α_2 extends to all of L_i, we are finished.

The homotopy, α_t, between α_0 and α_1 is described as follows: for a fixed value of t, α_t consists of the homotopy obtained by doing $f_{2i}|K_{2i+1} \times [0, t]$ during the time interval $[0, \frac{t}{2}]$ and then $f_{2i-1}|f_{2i}(K_{2i+1} \times \{t\})$ during the remaining interval $[\frac{t}{2}, 1]$.

The difference between α_1 and α_2 is a result of the pipings done in removing the intersection points between $f_{2i-1}(K_{2i} \times [0, 1])$ and $f_{2i-2}(K_{2i-1} \times [0, 1])$. Since this is done with a regular homotopy, we have that $\alpha_1 \simeq \alpha_2$.

So we have the homotopy we need pushing $U'_{2i-1}\cup L_i$ to N_i. Observe that the homotopy keeps K'_{i+1} fixed and that it takes place in $U_{2i-2}\cup P'_{i-1}$. Hence the homotopy can be chosen to be fixed on the regular neighborhood N_{i+1} of K'_{i+1}. Now $U_{2i-2}\cup P'_{i-1}\subset U'_{2i-5}$, so the track of the homotopy is contained in V_{i-2}. ∎

LEMMA 4.2. *The inclusion induced homomorphism* $\pi_2(V_i, V_i-V_j)\to\pi_2(V_{i-2}, V_{i-2}-V_{j+2})$ *is trivial whenever* $1\le i<j\le n-2$.

PROOF: Let $g:(B^2,\partial B^2)\to(V_i,V_i-V_j)=(U'_{2i-1}\cup P'_i, U'_{2i-1}\cup P'_i-(U'_{2j-1}\cup P'_{j-1}))\subset(U'_{2i-1}\cup P'_i, U'_{2i-1}\cup P'_i-(U_{2j+1}\cup P'_{j+1}))$ represent an element of $\pi_2(V_i,V_i-V_j)$. By [**10**, Lemma 1.1] and general position it is possible to push the image of the 1-skeleton off $U'_{2j+1}\cup P'_{j+1}$. We can then use Lemma 3.1 to push the image of g off U'_{2j+2} in $U_{2i-2}\subset V_{i-2}$. Finally, put $g(B^2)$ in general position with repect to L_{j+2} and pipe any points of $g(B^2)\cap L_{j+1}$ off the K'_{j+2} end of L_{j+2}, thus making $g(B^2)$ disjoint from V_{j+2}. ∎

LEMMA 4.3. *For each* $k\ge 0$ *there exists an* $\ell\ge 0$ *such that the inclusion induced homomorphism*

$$\pi_k(V_i-N_{j+1}, V_i-V_j)\to\pi_k(V_{i-\ell}-N_{j+\ell+1}, V_{i-\ell}-V_{j+\ell})$$

is trivial whenever $\ell+1\le i<j\le n-\ell-1$.

PROOF: The case $k=0$ is easy: $\ell=0$ will work because V_i is connected and neither N_{j+1} nor V_j separates V_i. By [**10**, Lemma 1.1], $\pi_1(V_i,V_i-V_j)=0$ and the core of N_{j+1} has codimension 3, so $\ell=0$ works in case $k=1$ as well.

We next consider the case $k=2$. Let us use the notation of [**7**]. Recall from [**7**] that if $B\subset A$, then $\pi'_2(A,B)$ denotes the quotient of $\pi_2(A,B)$ by the normal subgroup generated by elements of the form

$$(h_{[\omega]}[\alpha])[\alpha]^{-1} \qquad (\star)$$

where $[\alpha]\in\pi_2(A,B)$, ω is a loop in B and $h_{[\omega]}[\alpha]$ denotes $[\alpha]$ acted on by $[\omega]$ under the usual action of the fundamental group on $\pi_2(A,B)$.

Let $p:\widetilde{V}_i\to V_i$ denote the universal cover and let $\bar{Z}$ denote $p^{-1}(Z)$ whenever Z is a subset of V_i. Consider the commutative diagram

$$\begin{array}{ccc}
\pi'_2(\widetilde{V}_i-\bar{N}_{j+1}, \widetilde{V}_i-\bar{V}_j) & \xrightarrow{i_*} & \pi'_2(\widetilde{V}_i-\bar{N}_{j+3}, \widetilde{V}_i-\bar{V}_{j+2}) \\
\downarrow{\scriptstyle\phi_1} & & \downarrow{\scriptstyle\phi_2} \\
H_2(\widetilde{V}_i-\bar{N}_{j+1}, \widetilde{V}_i-\bar{V}_j) & \longrightarrow & H_2(\widetilde{V}_i-\bar{N}_{j+3}, \widetilde{V}_i-\bar{V}_{j+2}) \\
\downarrow{\scriptstyle\psi_1} & & \downarrow{\scriptstyle\psi_2} \\
\bar{H}^2_c(\bar{V}_j, \bar{N}_{j+1}) & \xrightarrow{i_\#} & \bar{H}^2_c(\bar{V}_{j+2}, \bar{N}_{j+3}).
\end{array}$$

The maps ϕ_1 and ϕ_2 are the Hurewicz homomorphisms, which in this case are isomorphisms [**7**, Proposition 7.5.1]. The homomorphisms ψ_1 and ψ_2 are isomorphisms by Alexander Duality [**7**, Theorem 6.9.10]. The restriction homomorphism $i_\#$ is the 0 homomorphism because the homotopy of Lemma 4.1 can be lifted to a proper homotopy in $\bar{V}_j$. We conclude that i_* is the trivial homomorphism.

Now consider an element $[g] \in \pi_2(V_i - N_{j+1}, V_i - V_j)$ where $g : (B^2, \partial B^2) \to (V_i - N_{j+1}, V_i - V_j)$. We can lift g to a map $\tilde{g} : (B^2, \partial B^2) \to (\widetilde{V}_i - \bar{N}_{j+1}, \widetilde{V}_i - \bar{V}_j)$. Because $i_* = 0$, $[\tilde{g}]$ is in the normal subgroup of $\pi_2(\widetilde{V}_i - \bar{N}_{j+3}, \widetilde{V}_i - \bar{V}_{j+2})$ generated by elements of the form $(\star)$ where ω is a loop in $\widetilde{V}_i - \bar{V}_{j+2}$. This means that the image of $[g]$ in $\pi_2(V_i - N_{j+3}, V_i - V_{j+2})$ is in the normal subgroup generated by elements of the form $(\star)$ where ω now is a loop in $V_i - V_{j+2}$ which is null homotopic in V_i. But, by Lemma 4.2, $\omega \simeq *$ in $V_{i-2} - V_{j+4}$ and so each element of the form $(\star)$ dies in $\pi_2(V_{i-2} - N_{j+5}, V_{i-2} - V_{j+4})$. Thus $[g] = 0$ in $\pi_2(V_{i-2} - N_{j+5}, V_{i-2} - V_{j+4})$ and the proof of the case $k = 2$ is complete.

Finally, we prove the case $k > 2$ by induction. This inductive proof is similar to the proof of the "eventual Hurewicz theorem" given in [**5**]. Consider an element $[g] \in \pi_k(V_i - N_{j+1}, V_i - V_j)$. Then $g : (B^k, \partial B^k) \to (V_i - N_{j+1}, V_i - V_j)$ and we can lift g to a map $\tilde{g} : (B^k, \partial B^k) \to (\widetilde{V}_i - \bar{N}_{j+1}, \widetilde{V}_i - \bar{V}_j)$. Since the diagram

$$\begin{array}{ccc} H_k(\widetilde{V}_i - \bar{N}_{j+1}, \widetilde{V}_i - \bar{V}_j) & \longrightarrow & H_k(\widetilde{V}_i - \bar{N}_{j+3}, \widetilde{V}_i - \bar{V}_{j+2}) \\ \psi_1 \downarrow & & \downarrow \psi_2 \\ \bar{H}_c^{4-k}(\bar{V}_j, \bar{N}_{j+1}) & \xrightarrow{i_\#} & \bar{H}_c^{4-k}(\bar{V}_{j+2}, \bar{N}_{j+3}). \end{array}$$

commutes and $i_\# = 0$, we have that $\tilde{g}$ is null-homologous in $(\widetilde{V}_i - \bar{N}_{j+3}, \widetilde{V}_i - \bar{V}_{j+2})$. By induction, there is an ℓ' such that each map of a relative $(k-1)$-simplex into $(V_i - N_{j+3}, V_i - V_{j+2})$ is homotopic in $(V_{i-\ell'} - N_{j+\ell'+1}, V_{i-\ell'} - V_{j+\ell'})$ to a map into $V_{i-\ell'} - V_{j+\ell'}$. In order to simplify the notation in the rest of the proof, let us use (A, B) to denote $(\widetilde{V}_i - \bar{N}_{j+3}, \widetilde{V}_i - \widetilde{V}_{j+2})$ and (C, D) to denote $(\widetilde{V}_{i-\ell'} - \bar{N}_{j+\ell'+1}, \widetilde{V}_{i-\ell'} - \widetilde{V}_{j+\ell'})$. The choice of ℓ' implies that the inclusion map $(A, B) \hookrightarrow (C, D)$ is homotopic to a map $f : (A, B) \to (C, D)$ such that $f(A^{(k-1)}) \subset D$. We have

$$(B^k, \partial B^k) \xrightarrow{\tilde{g}} (A, B) \hookrightarrow (A, A^{(k-1)} \cup B) \xrightarrow{f} (C, D).$$

Now the pair $(A, A^{(k-1)} \cup B)$ is $(k-1)$-connected and $\tilde{g}$ is null-homologous in $(A, A^{(k-1)} \cup B)$, so [**7**, Proposition 7.5.1] shows that $[\tilde{g}] = 0$ in $\pi'_k(A, A^{(k-1)} \cup B)$. The fact that f is homotopic to the inclusion then gives $[g] = 0$ in $\pi'_k(V_{i-\ell'} - N_{j+\ell'+1}, V_{i-\ell'} - V_{j+\ell'})$. The proof given above for the case $k = 2$ then shows that our conclusion follows with $\ell = \ell' + 2$. ∎

Remark: Notice that the number ℓ given by Lemma 4.3, above, is independent of n. The same is true of the ℓ in the Lemma 4.5, below, and that point is important in the proof of Proposition 2.1.

Observation: Because we have assumed that X has the shape of a nontrivial S^1-like continuum, we can arrange that $\pi_1(K_1)$ injects into $\pi_1(U_0)$. Thus the image of $\pi_1(K_1) \to \pi_1(U_0)$ is isomorphic to Z. If $i < j$ there is a natural map $K'_j \to K'_i$ which is induced by $K'_j \hookrightarrow N_j \hookrightarrow N_i \searrow K'_i$. Property (5) implies that the homomorphism on π_1 induced by this natural map also has image isomorphic to Z for every $i < j$.

Lemma 4.4. *For each $i \geq 4$ the image of the inclusion induced homomorphism $\pi_1(V_i) \to \pi_1(V_{i-2})$ is equal to the image of $\pi_1(N_i) \to \pi_1(V_{i-2})$ and both images are isomorphic to* Z.

Proof: By Lemma 4.1, the inclusion $V_i \hookrightarrow V_{i-2}$ factors, up to homotopy, through a map $V_i \to K'_i$. Thus the image is cyclic. To complete the proof we need to show that the inclusion induced homomorphism $\pi_1(K'_i) \to \pi_1(V_{i-2})$ is an injection. If not, then some multiple of the generator of $\pi_1(K'_i)$ is null homotopic in V_{i-2}. Applying Lemma 4.1 again, we see that this multiple of the generator would then be inessential in N_{i-3}. But this contradicts the observation made above. ∎

Lemma 4.5. *For each $k \geq 0$ there exists an $\ell \geq 0$ such that the inclusion induced homomorphism*

$$\pi_k(V_i - N_{j+1}, N_{i+1} - N_{j+1}) \to \pi_k(V_{i-\ell} - N_{j+\ell+1}, N_{i-\ell+1} - N_{j+\ell+1})$$

is trivial whenever $\ell + 1 \leq i < j \leq n - \ell - 1$.

Proof: The proof is similar to the proof of Lemma 4.3, so we include only a sketch. By Lemma 4.1 we have that $\pi_k(V_i, N_{i+1}) \to \pi_k(V_{i-2}, N_i)$ is trivial for every i, k. The problem is to "excise" N_{j+1}. This can be done on the homology level: by the excision theorem we have $H_k(V_i, N_{i+1}) \cong H_k(V_i - N_{j+1}, N_{i+1} - N_{j+1})$ for every i, j, k. In addition, Lemma 4.4 shows that $\pi_1(N_{i+1}) \to \pi_1(V_{i-1})$ is one-to-one and so $\pi_1(N_{i+1} - N_{j+1}) \to \pi_1(V_{i-1} - N_{j+1})$ is one-to-one for every i, k. Thus a Relative Hurewicz Theorem argument similar to that in the proof of Lemma 4.3 can be used to complete the proof. ∎

5. Proper h-cobordism.

In this section we present the proof of Proposition 2.1. The proof is based on an application of the Controlled h-cobordism Theorem [**3**, Corollary 7.2B]. The idea is to find a product structure on $M \times [0, 1]$ that is controlled by the neighborhoods we have constructed in such a way that each fiber which begins in V_n ends in N_1 and each fiber over $M - U_0$ is straight. That product structure then is used to define a homeomorphism $h : M \to M$ such that $h(X) \subset N_1$ and $h|M - U_0 = id$.

NOTATION: Throughout this section ℓ will denote the largest of the numbers given by either Lemma 4.3 or Lemma 4.5 with $0 \le k \le 4$. Let Y be the compact subset of $M \times [0,1]$ defined by $Y = X \times \{0\} \cup K'_{n+1} \times [0,1]$. Define a sequence of neighborhoods of Y by $W_i = V_i \times [0,1]$ for $0 \le i \le 4\ell$ and $W_i = V_i \times [0, 1/(i-4\ell)] \cup N_{i+1} \times [0,1]$ for $4\ell < i \le n$. Finally, for $0 \le i \le n-1$, let R_i be the region $R_i = \overline{W_i - W_{i+1}}$ and let $R_n = W_n$.

We next specify the various ingredients we need in order to set up an application of [**3**, Corollary 7.2B]. The cobordism we will use is $(M \times [0,1]; M \times \{0\}, M \times \{1\})$, the space E is $M \times [0,1]$, and the parameter space X of [**3**] is $[0,1]$. We must define a control function $f : M \times [0,1] \to [0,1]$. Begin to define f by letting $f((M - \operatorname{Int} V_0) \times [0,1]) = 0$, $f(R_n) = 1$, and $f(R_{i-1} \cap R_i) = i/n$ for each $i \ge 1$. Then use the Tietze Extension Theorem to extend f to all of $M \times [0,1]$ in such a way that $f(R_i) = [i/n, (i+1)/n]$. We may assume that f is simplicial with respect to some triangulations and thus $f : M \times [0,1] \to [0,1]$ is a simplicial NDR.

It is obvious that the identity map $M \times [0,1] \to E$ is δ-1-connected for every $\delta > 0$. We also note that f has local π_1 isomorphic to Z. In order to see this, we show that, whenever $2 \le i < j$, the image of $\pi_1(W_i - W_j)$ in $\pi_1(W_{i-2} - W_j)$ is isomorphic to Z. That is equivalent to showing that the image of $\pi_1(V_i - N_{j+1})$ in $\pi_1(V_{i-2} - N_{j+1})$ is isomorphic to Z. Since N_{j+1} has a spine of codimension 3, this follows from Lemma 4.4.

The only ingredient now missing for an application of the controlled h-cobordism theorem is the fact that $M \times [0,1]$ is a δ-h-cobordism over [0,1]. That follows from the next two lemmas.

LEMMA 5.1. *There is a homotopy $g_t : M \times [0,1] \to M \times [0,1]$ such that*

(1) $g_0 = id$,
(2) $g_t | M \times \{0\} \cup (M - U_0) \times [0,1] = id$ *for every* $t \in [0,1]$,
(3) $g_1(M \times [0,1]) \subset M \times \{0\}$, *and*
(4) *for every* $x \in M \times [0,1]$ *there exists an* i *such that* $g(\{x\} \times [0,1]) \subset R_i \cup \cdots \cup R_{i+8\ell}$.

PROOF: We verify that, if ℓ is as in Lemma 4.3, then the inclusion induced map

$$\begin{aligned} &\pi_k(R_i \cup \cdots \cup R_j, (R_i \cup \cdots \cup R_j) \cap (M \times \{0\})) \\ &\qquad \to \pi_k(R_{i-\ell} \cup \cdots \cup R_{j+\ell}, (R_{i-\ell} \cup \cdots \cup R_{j+\ell}) \cap (M \times \{0\})) \end{aligned}$$

is trivial for every k and for every $i < j$. The homotopy g_t is then constructed by inductively pushing the skeleta of R_i down to $M \times \{0\}$.

Let $\alpha : (B^k, \partial B^k) \to (R_i \cup \cdots \cup R_j, (R_i \cup \cdots \cup R_j) \cap (M \times \{0\}))$ represent an element of $\pi_k(R_i \cup \cdots \cup R_j, (R_i \cup \cdots \cup R_j) \cap (M \times \{0\}))$. Choose a number η between $1/i - 4\ell$ and $1/j - 4\ell$. A vertical push gives a homotopy of α to a map α' with the property that $\alpha' | \partial B^k = \alpha | \partial B^k$ and $\alpha'(B^k) \subset (\alpha(\partial B^k) \times [0, \eta]) \cup (V_i - N_{j+1} \times \{\eta\}$. An application of Lemma 4.3 in the level $M\times\{\eta\}$ gives a homotopy of α' to a map α'' such that $\alpha''(B^k)\subset(V_{i-\ell}-V_{j+\ell})\times[0,\eta]$. A push straight down completes the construction of the homotopy we need to show that α dies in $\pi_k(R_{i-\ell}\cup\cdots\cup R_{j+\ell},(R_{i-\ell}\cup\cdots\cup R_{j+\ell})\cap(M\times\{0\}))$. ∎

Lemma 5.2. *There is a homotopy $g_t : M\times[0,1]\to M\times[0,1]$ such that*

(1) $g_0 = id$,
(2) $g_t|M\times\{1\}\cup(M-U_0)\times[0,1] = id$ *for every* $t\in[0,1]$,
(3) $g_1(M\times[0,1])\subset M\times\{1\}$, *and*
(4) *for every* $x\in M\times[0,1]$ *there exists an* i *such that* $g(\{x\}\times[0,1])\subset R_i\cup\cdots\cup R_{i+8\ell}$.

Proof: Lemma 5.2 follows from Lemma 4.5 in the same way that Lemma 5.1 follows from Lemma 4.3. ∎

Proof of Proposition 2.1: Apply [**3**, Corollary 7.2B] with $\epsilon=\frac{1}{10}$ and the other data of [**3**, Corollary 7.2B] as described in the paragraphs just before the statement of Lemma 5.1. Let $\delta>0$ be the number whose existence is given by the [**3**, Corollary 7.2B]. Choose n large enough so that $8\ell/n<\delta$. Now do the construction of §3 using that value for n. Then Lemmas 5.1 and 5.2 imply that $M\times[0,1]$ is a δ-h-cobordism over [0,1]. The Controlled h-cobordism Theorem gives a product structure on $M\times[0,1]$ which has diameter less than ϵ in [0,1]. Use this product structure to define $h : M\to M$. ∎

Remark: It is only in the very last proof, above, that the hypothesis that X have the shape of an S^1-like continuum is really essential. It is possible to find versions of all the constructions through Lemma 5.2 which work for any X having fundamental dimension 1. In the that case, the neighborhoods could have arbitrary compact 1-dimensional polyhedra as spines and thus the fundamental groups would be finitely generated free groups. But the only such group for which the Disk Theorem [**3**] is known to hold is the free group on one generator, Z.

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