HOMOTOPY GROUPS OF COMPLEMENTS OF TOPOLOGICAL KNOTS

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1. Introduction

Throughout this talk we will assume that h is a topological embedding of the $(n-2)$-sphere into the n-sphere, $h : S^{n-2} \to S^n$. We use Σ to denote the image set $h(S^{n-2})$ and refer to Σ as a *topological knot*. We will study the homotopy groups of the knot complement $S^n - \Sigma$ and try to determine the largest possible dimension in which the first nonstandard homotopy group can occur. Specifically, we wish to determine the largest possible value of k such that $\pi_i(S^n - \Sigma) \cong \pi_i(S^1)$ for $i < k$, but $\pi_k(S^n - \Sigma) \neq \pi_k(S^1)$.

Members of the audience are probably thinking that answers to questions like that are well known. But what is known generally applies only to embeddings that are fairly nice. (Here "fairly nice" means that the embedding is smooth, piecewise linear, or locally flat.) The purpose of this talk is to demonstrate that the answer is quite different in the topological setting.

2. Homotopy groups of complements of smooth knots

Let us begin by reviewing the well-known facts in the smooth setting. When combined, the two facts below indicate that the first nontrivial homotopy group of the complement of a smooth knot can occur in any dimension below the middle dimension, but cannot occur in the middle dimension or above. Similar facts hold in the PL and locally flat settings.

Fact 1. *For each k, $1 \leq k < n/2$, there is a smooth embedding $h : S^{n-2} \to S^n$ such that $\pi_i(S^n - \Sigma) \cong \pi_i(S^1)$ for $1 \leq i < k$ but $\pi_k(S^n - \Sigma) \neq \pi_k(S^1)$.*

Examples can be constructed which are boundaries of manifolds obtained by attaching handles to the standard $(n+1, n-1)$-ball pair [3]. Thus the examples are actually slice knots.

Fact 2. *If h is smooth and $\pi_i(S^n - \Sigma) \cong \pi_i(S^1)$ for $1 \leq i < n/2$, then $S^n - \Sigma$ has the homotopy type of S^1.*

Fact 2 is proved by showing that the homology of the universal cover of $S^n - \Sigma$ vanishes. If n is odd, this follows quite easily from Poincaré duality. If n is even, more care is needed to prove that the homology group in the middle dimension vanishes. One approach is to use "Milnor Duality" [2] .

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Milnor Duality. Under certain conditions, an infinite cyclic cover of a compact n-manifold will have the duality properties of an $(n-1)$-manifold. In other words,

$$H_k(\widetilde{M}^n; F) \cong H^{n-k-1}(\widetilde{M}^n, \partial\widetilde{M}^n; F)$$

for any field F.

Milnor Duality can be used to prove Fact 2 in case n is even. In that case the homology in dimension $n/2$ is dual to the cohomology in dimension $(n/2-1)$ and is, therefore, trivial.

3. Homotopy groups of complements of topological knots

In the case of topological knots, the first nontrivial homotopy group can occur above the middle dimension. The surprising fact is that it can occur in any dimension up to $n-2$. The following example will appear in [4].

Example. *For each n and k, $1 \leq k \leq n-2$, there exists a topological embedding $h : S^{n-2} \to S^n$ such that $\pi_i(S^n - \Sigma) \cong \pi_i(S^1)$ for $i < k$ but $\pi_k(S^n - \Sigma) \neq 0$. The embedding is smooth except at one point.*

If $k \geq n/2$, the knot in the example must necessarily be wild (because of Fact 2). There is a close relationship between the global homotopy groups of the knot complement and the homotopy groups of the end of the complement. We explore this relationship in the next three results (to appear in [4]).

Notation. We use W to denote the knot complement $S^n - \Sigma$, $\widetilde{W}$ to denote the infinite cyclic cover of W, and ϵ to denote the end of W.

Theorem 1. *If $\pi_i(\epsilon) \cong \pi_i(S^1)$ for $i \leq n-k-1$, then $H_k(\widetilde{W}; F) \cong H^{n-k-1}(\widetilde{W}; F)$ for every field F.*

Corollary 1. *If $\pi_i(W) \cong \pi_i(\epsilon) \cong \pi_i(S^1)$ for $i \leq k$, then $H_i(\widetilde{W}; \mathbb{Z}) = 0$ for $i \geq n-k-1$.*

Corollary 2. *If $\pi_i(W) \cong \pi_i(\epsilon) \cong \pi_i(S^1)$ for $i < n/2$, then W has the homotopy type of S^1.*

Conversely, the homotopy groups of the end of W can be controlled by controlling the global homotopy groups of W. In particular, if $\pi_i(W) \cong \pi_i(S^1)$ for every i, then any nontrivial homotopy group of the end *must* appear in dimension 1.

A geometric version of the following result was proved by Hollingsworth and Rushing [1]. We give an elementary proof based on duality.

Theorem 2. *If $\pi_i(W) \cong \pi_i(S^1)$ for every i and $\pi_1(\epsilon) \cong \mathbb{Z}$, then $\pi_i(\epsilon) = 0$ for $2 \leq i \leq n-3$ and $\pi_{n-2}(\epsilon) \cong \mathbb{Z}$.*

If we combine Theorem 2 with Corollary 2 we get the following result. It is a topological version of Fact 2.

Corollary 3. *If $\pi_i(W) \cong \pi_i(\epsilon) \cong \pi_i(S^1)$ for $i < \frac{n}{2}$, then W has the proper homotopy type of $S^1 \times \mathbb{R}^{n-1}$.*

4. A noncompact version of Milnor Duality

The proof of Theorem 1 is based on a noncompact version of Milnor Duality. We assume the following notation for the remainder of this section: W is a noncompact PL n-manifold, W has one end ϵ, and $p : \widetilde{W} \to W$ denotes an infinite cyclic cover. As in Milnor's original work, it is convenient to use coefficients in a field F.

Definition.

$$H^k(\widetilde{W}, \tilde{\epsilon}; F) = \varinjlim H^k(\widetilde{W}, p^{-1}(U); F)$$

where the limit is taken over the collection of all neighborhoods U of the end ϵ, ordered by inclusion.

Definition. We will say that $H_k(\widetilde{W}, \tilde{\epsilon}; F)$ *is profinitely generated over* F if for every neighborhood U of ϵ there exists a neighborhood V of ϵ, $V \subset U$, such that the image of $H_k(\widetilde{W}, p^{-1}(V); F)$ in $H_k(\widetilde{W}, p^{-1}(U); F)$ is finitely generated over F.

In [4] we prove the following noncompact version of Milnor Duality and use it to prove Theorem 1, above.

Theorem 3. *If $H_i(\widetilde{W}, \tilde{\epsilon}; F)$ is profinitely generated over F for $n-k-2 \leq i \leq n-k$, then $H_k(\widetilde{W}; F) \cong H^{n-k-1}(\widetilde{W}, \tilde{\epsilon}; F)$.*

References

[1] J. G. Hollingsworth and T. B. Rushing, *Homotopy characterization of weakly flat codimension 2 spheres*, Amer. J. Math. **98** (1976), 385–394.

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