Complements of 2-spheres in 4-manifolds

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Abstract. Complements of topologically embedded 2-spheres in compact 4-manifolds are investigated. The main technical result gives a necessary condition for such a complement to be homotopically dominated by a compact polyhedron. Corollaries give conditions under which the end of the complement will have a collar or a weak collar.

Introduction

Let $h : S^2 \to M^4$ be a topological embedding of the 2-sphere into the interior of the compact 4-manifold M^4. We use Σ to denote $h(S^2)$ and W to denote $M - \Sigma$; then W is a non-compact 4-manifold with one end which we call ϵ. We wish to investigate properties of ϵ that are related to the way in which Σ is embedded in M. In particular, we give a condition which implies that neighborhoods of ϵ are homotopically dominated by finite complexes and a second condition which implies that the end is collared (or weakly collared).

Let us begin with a definition. A compact set X in the interior of a manifold M is said to be *globally 1-alg* if for every neighborhood U of X in M there is a neighborhood V of X in U such that the image of $\pi_1(V - X)$ in $\pi_1(U - X)$ is abelian. We are most interested in the case in which $X = \Sigma$, as in the paragraph above. In that case duality shows that the image of $\pi_1(V - \Sigma)$ in $\pi_1(U - \Sigma)$ will be cyclic. The order of this cyclic group is determined by $\Sigma \cdot \Sigma$, the integer-valued self-intersection number of $[\Sigma] \in H_2(M; \mathbb{Z})$. If Σ is globally 1-alg and $\Sigma \cdot \Sigma = 0$, then $\pi_1(\epsilon)$ is infinite cyclic. Otherwise $\pi_1(\epsilon)$ is a finite cyclic group whose order is $\Sigma \cdot \Sigma$. This is explained in Lemma 1, below.

We can now state our main results.

Theorem. *If $h : S^2 \to M^4$ is a topological embedding into the interior of the 4-manifold M and $\Sigma = h(S^2)$ is globally 1-alg, then any 0-neighborhood of the end of $M - \Sigma$ is finitely dominated.*

In the infinite cyclic case we can obtain a collar on the end.

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Corollary 1. *If Σ is a 2-sphere topologically embedded in the interior of the 4-manifold M and π_1 of the end of $M - \Sigma$ is infinite cyclic, then $M - \Sigma \cong M - K$ for some locally flat 2-sphere K in M. Moreover, there exists a neighborhood N of Σ such that $N \cong S^2 \times B^2$ and $N - \Sigma \cong \partial N \times [0, 1)$.*

We do not know whether the end of the complement of every globally 1-alg 2-sphere can be collared. The Theorem does show that the end is always tame. In §3 we prove that the finiteness obstruction $\sigma(\epsilon) \in \widetilde{K}_0(\mathrm{Z}[\pi_1(\epsilon)])$ is trivial and so we are at least able to obtain a weak collar. (See [**2**, §11.9B] for the definition of a weak collar.)

Corollary 2. *If Σ is a 2-sphere topologically embedded in the interior of the 4-manifold M and π_1 of the end of $M - \Sigma$ is a finite cyclic group, then the end of $M - \Sigma$ has a weak collar.*

The Theorem and its Corollaries generalize results in [**3**], where such theorems are proved for the special case in which the ambient manifold is S^4. This generalization is interesting in its own right, but is also motivated by an application to a different problem. That problem is the problem of giving a homotopy characterization of local flatness for surfaces embedded in 4-manifolds. The conjecture is that a local version of the 1-alg property will imply local flatness for a surface N topologically embedded in the interior of the 4-manifold M. The surprising fact is that the global structure of the end of $M - N$ plays a role in this local problem. In [**4**], Corollary 1 above is used to prove the following result: *If Σ is a 2-sphere topologically embedded in the interior of the 4-manifold M^4 so that Σ is locally 1-alg at x for every $x \in \Sigma$ and $\Sigma \cdot \Sigma = 0$, then Σ is locally flat at x for every $x \in \Sigma$.*

It should be noted that the Theorem only asserts that the global 1-alg condition is a sufficient condition to guarantee that the end of the complement of Σ is finitely dominated. In fact it is not a necessary condition, even in case Σ is a 2-sphere in S^4. In [**4**] we show that a necessary condition would involve $\pi_2(M - \Sigma)$. We also show by example that conditions on $\pi_1(M - \Sigma)$ alone will not suffice; in [**4**] we construct an example of a topological 2-sphere $\Sigma \subset S^4$ such that $\pi_1(S^4 - \Sigma) \cong \mathrm{Z}$ but $\pi_2(S^4 - \Sigma)$ is nontrivial. Such an example must necessarily be wildly embedded with $\pi_1(\epsilon)$ unstable.

The results of this paper apply only in dimension 4. In high dimensions the global 1-alg condition by itself is not strong enough to imply that that the end of the complement of a topologically embedded codimension 2 sphere is finitely dominated. Examples are discussed in [**4**].

Notation: Throughout the remainder of this paper, M will denote a compact 4-manifold, $h : S^2 \to \operatorname{int} M$ will denote a topological embedding and $\Sigma = h(S^2)$.

Then $M - \Sigma$ is a noncompact manifold with one end which we call ϵ. If V is a closed neighborhood of the end ϵ, we write $\overline{V} = V \cup \Sigma$.

REMARK: All the results above hold true if Σ is a compactum having the shape of S^2; i.e., the topological embedding h can be replaced with a shape equivalence from S^2 to a compact subset Σ of M^4.

1. PROOF OF THE THEOREM

There are some technical results needed for the proof and we will state them as lemmas. The first lemma explains the structure of $\pi_1(\epsilon)$.

LEMMA 1. *If Σ is globally 1-alg embedded in* $\operatorname{int} M$*, then the fundamental group of the end ϵ is stable and cyclic. More precisely, $\pi_1(\epsilon) \cong \mathrm{Z}/n\mathrm{Z}$, where $n = \Sigma \cdot \Sigma$, the integer-valued self-intersection number of $[\Sigma] \in H_2(M; \mathrm{Z})$.*

PROOF: Since Σ is an ANR, we can choose a nested sequence $\{V_n\}_{n=1}^{\infty}$ of closed, connected, PL manifold neighborhoods of ϵ such that $\cap V_n = \emptyset$ and each $\overline{V}_{n+1} \hookrightarrow \overline{V}_n$ factors, up to homotopy, through a retraction $r_{n+1} : \overline{V}_{n+1} \to \Sigma$. Consider the following diagram.

$$\begin{array}{ccccccc}
H_2(\overline{V}_{n+1};\mathrm{Z}) & \xrightarrow{\alpha_1} & H_2(\overline{V}_{n+1}, V_{n+1};\mathrm{Z}) & \xrightarrow{\alpha_2} & H_1(V_{n+1};\mathrm{Z}) & \xrightarrow{\alpha_3} & H_1(\overline{V}_{n+1};\mathrm{Z}) \\
\beta_1\downarrow & & \beta_2\downarrow & & \beta_3\downarrow & & \beta_4\downarrow \\
H_2(\overline{V}_n;\mathrm{Z}) & \xrightarrow{\gamma_1} & H_2(\overline{V}_n, V_n;\mathrm{Z}) & \xrightarrow{\gamma_2} & H_1(V_n;\mathrm{Z}) & \xrightarrow{\gamma_3} & H_1(\overline{V}_n;\mathrm{Z})
\end{array}$$

Notice that β_2 is an isomorphism (excision) and $\beta_4 = 0$ (by the choice of V_n), so a simple diagram chasing argument shows that $\operatorname{im}\beta_3 = \operatorname{im}\gamma_2$. Now duality gives $H_2(\overline{V}_n, V_n; \mathrm{Z}) \cong \check{H}^2(\Sigma) \cong \mathrm{Z}$, and the duality isomorphism takes $x \in H_2(\overline{V}_n, V_n; \mathrm{Z})$ to $x \cdot [\Sigma] \in \mathrm{Z}$. Furthermore, the images of $H_2(\overline{V}_n; \mathrm{Z})$ and $H_2(\overline{V}_{n+1}; \mathrm{Z})$ in $H_2(\overline{V}_n, V_n; \mathrm{Z})$ are equal, so $\operatorname{im}\gamma_1$ is generated by $\gamma_1([\Sigma])$, where $[\Sigma] \in H_2(V_n; \mathrm{Z})$. Thus $\operatorname{im}\beta_3 = \operatorname{im}\gamma_2 \cong \mathrm{Z}/n\mathrm{Z}$ where $n = \Sigma \cdot \Sigma$. It follows that the inverse sequence $\{H_1(V_1; \mathrm{Z}) \leftarrow H_1(V_2; \mathrm{Z}) \leftarrow \cdots\}$ is stable and

$$\varprojlim_n H_1(V_n; \mathrm{Z}) \cong \mathrm{Z}/n\mathrm{Z}.$$

Finally, the fact that Σ is globally 1-alg implies that the inverse sequence $\{\pi_1(V_1) \leftarrow \pi_1(V_2) \leftarrow \cdots\}$ is also stable and $\pi_1(\epsilon) \cong \varprojlim_n \{\pi_1(V_n)\} \cong \varprojlim_n \{H_1(V_n; \mathrm{Z})\}$. ∎

Using Lemma 1, take $V \subset V_0$ to be a pair of connected neighborhoods of ϵ such that

(1) ∂V and ∂V_0 are connected,
(2) the image of $\pi_1(V)$ in $\pi_1(V_0)$ is isomorphic to $\pi_1(\epsilon)$, and
(3) the image of $\pi_1(\partial V)$ in $\pi_1(V)$ contains the image of $\pi_1(\epsilon)$ in $\pi_1(V)$.

Observe that the inclusion induced map $\pi_1(V) \to \pi_1(V_0)$ factors through $H_1(V)$, so $\ker[\pi_1(V) \to \pi_1(V_0)]$ is a normally generated by a finite set. We can therefore do 1-dimensional surgery on the interior of V and arrange that $i_* : \pi_1(\epsilon) \to \pi_1(V)$ is an isomorphism. Consequently, $\pi_1(\partial V) \to \pi_1(V)$ will be surjective by (3) above. We shall prove that V is dominated by a finite complex. Then every 0-neighborhood is dominated by a finite complex by [**6**, Proposition 4.3].

LEMMA 2. *If $\widetilde{V}$ is the universal covering space of V and $\{E_n\}$ is a nested sequence of connected neighborhoods of the end of $\widetilde{V}$ with $\bigcap E_n = \emptyset$, then the compact support cohomology group $H_c^1(\widetilde{V}; \mathrm{Z}) = \varinjlim H^1(\widetilde{V}, E_n) = 0$.*

PROOF: See the proof of [**3**, Lemma 3]. ∎

LEMMA 3. *$H_2(V, \partial V; \mathrm{Z})$ is finitely generated.*

PROOF: From the exact sequence of the pair $(\overline{V}, V)$, it follows that $H_2(V; \mathrm{Z})$ is finitely generated because $H_3(\overline{V}, V; \mathrm{Z}) \cong \check{H}^1(\Sigma) = 0$. Thus the Lemma follows from the exact sequence of the pair $(V, \partial V)$. ∎

We now define a subpolyhedron W of V as follows. Let $\partial V(3)$ denote the collection of all closed 3-simplices of ∂V. For each $\sigma \in \partial V(3)$, let L_σ be a proper PL ray in V starting at the barycenter of σ, let N_σ be a regular neighborhood of L_σ in V and let $\operatorname{int}_V N_\sigma$ be the topological interior of N_σ in V. Without loss of generality we may assume that

(4) each ∂N_σ is PL homeomorphic to R^3 (the union of an increasing sequence of PL 3-balls),
(5) $N_\sigma \cap \partial V = \sigma$, and
(6) the family $\{\operatorname{int}_V N_\sigma | \sigma \in \partial V(3)\}$ is pairwise disjoint.

Define $W = V - \cup\{\operatorname{int}_V N_\sigma | \sigma \in \partial V(3)\}$. Then

(7) $W \cap \partial V = \partial V^{(2)}$, the 2-skeleton of ∂V, and
(8) $\pi_1(\epsilon) \cong \pi_1(V) \cong \pi_1(W)$.

Let Λ denote the group ring $\mathrm{Z}[\pi_1(\epsilon)]$.

LEMMA 4. *If H is the homology or cohomology functor and $\mathcal{B}$ is any coefficient bundle of Λ-modules, then $H(V, \partial V; \mathcal{B}) \cong H(W, \partial V^{(2)}; \mathcal{B})$.*

PROOF: Let $A = \cup\{N_\sigma | \sigma \in \partial V(3)\}$. By (4), each N_σ strong deformation retracts onto ∂N_σ; we abbreviate this $N_\sigma \searrow \partial N_\sigma$. It follows that $A \searrow \partial A$ and consequently $V \searrow W \cup \partial V$. Hence,

$$\begin{aligned} H(V, \partial V; \mathcal{B}) &\cong H(W \cup \partial V, \partial V; \mathcal{B}) \\ &\cong H(W, \partial V^{(2)}; \mathcal{B}), \end{aligned}$$

the last isomorphism being an excision isomorphism. ∎

Since V is homotopy equivalent to $W \cup \partial V = W \cup (\cup\{\sigma | \sigma \in \partial V(3)\})$ and $\partial V(3)$ is a finite collection, we need only show that W is dominated by a finite complex. Since Λ is Noetherian, Theorems B and F in [**9**] imply that it is sufficient to show that W satisfies conditions D_2 and NF_2 in [**9**]. Refer to [**10**, Theorem 5] for an equivalent statement of D_2. We need only consider the case of countably generated free Λ-modules since we will only be involved with countably generated free chain complexes.

LEMMA 5. *W satisfies condition D_2; i.e.,*

(*i*) $H_i(W;\Lambda) = 0$ *for all* $i > 2$, *and*
(*ii*) $H^3(W;\mathcal{B}) = 0$ *for every coefficient bundle* $\mathcal{B}$ *of countably generated free* Λ*-modules.*

PROOF: First, for $j = 3, 4$ we have

$$\begin{aligned} H_j(W, \partial V^{(2)}; \Lambda) &\cong H_j(V, \partial V; \Lambda) \qquad \text{(Lemma 4)} \\ &\cong H_j(\widetilde{V}, \partial\widetilde{V}; \mathrm{Z}) \\ &\cong H_c^{4-j}(\widetilde{V}; \mathrm{Z}) \qquad \text{(duality)} \\ &= 0 \qquad\qquad \text{(Lemma 2).} \end{aligned}$$

From the homology sequence of the pair $(W, \partial V^{(2)})$, we can infer that $H_j(W;\Lambda) = 0$. So (*i*) is proved.

Next, from the cohomology exact sequence of the pair $(W, \partial V^{(2)})$ we can infer that $H^3(W;\mathcal{B}) = 0$ since $H^3(\partial V^{(2)};\mathcal{B}) = 0$ and since $H^3(W, \partial V^{(2)};\mathcal{B}) \cong H^3(V, \partial V;\mathcal{B}) \cong H_1^{\infty}(V;\mathcal{B}) = 0$ (by Lemma 4 and the relative version of [**3**, Lemma 5], respectively). Therefore (*ii*) is proved. ∎

LEMMA 6. *W satisfies condition NF_2; i.e., $H_2(W;\Lambda)$ is a finitely generated Λ-module.*

PROOF: Since the sequence

$$\ldots \to H_2(\partial V^{(2)};\Lambda) \to H_2(W;\Lambda) \to H_2(W, \partial V^{(2)};\Lambda) \to \cdots$$

is exact and Λ is Noetherian, we need only prove that $H_2(W, \partial V^{(2)};\Lambda)$ is finitely generated [**9**, Lemma 1.5]. From the exact sequence

$$H_2(W, \partial V^{(1)};\Lambda) \to H_2(W, \partial V^{(2)};\Lambda) \to H_1(\partial V^{(2)}, \partial V^{(1)};\Lambda) = 0$$

of the triple $(W, \partial V^{(2)}, \partial V^{(1)})$, we see that we will only need to prove that the Λ-module $H_2(W, \partial V^{(1)}; \Lambda)$ is finitely generated.

Now let $i : \partial V^{(1)} \to W$ be the inclusion map. Since $i_* : \pi_1(\partial V^{(1)}) \to \pi_1(W) \cong \pi_1(V)$ is onto (by (3)) and W satisfies the condition D_2 (by Lemma 5), it follows from the proof of Lemma 2.1 of [**9**] that $H_2(W, \partial V^{(1)}; \Lambda)$ is a projective Λ-module. Suppose, on the contrary, that $H_2(W, \partial V^{(1)}; \Lambda)$ is not a finitely generated Λ-module. Then it will be a free Λ-module by [**7**, Theorem 2.3] since $\pi_1(\epsilon)$ is cyclic. Consequently, the abelianization $\pi_2(i)_{\mathrm{ab}} \cong H_2(W, \partial V^{(1)}; \Lambda)$ of the homotopy group $\pi_2(i)$ is a free Λ-module. (Refer to [**5**, p. 222] for notation and terminology.) It follows from Theorem 4.3 of [**5**] that W has the homotopy type of an almost finite 2-complex X; i.e., X is the wedge of a finite complex Y (containing $\partial V^{(1)}$) and infinitely many copies of the 2-sphere. Consequently, $H_2(X, Y; \mathrm{Z})$ is free and of infinite rank and it follows from the exact sequence of the triple $(X, Y, \partial V^{(1)})$ that $H_2(W, \partial V^{(1)}; \mathrm{Z}) \cong H_2(X, \partial V^{(1)}; \mathrm{Z})$ is of infinite rank. Finally, the exact sequence of the triple $(W, \partial V^{(2)}, \partial V^{(1)})$

$$H_2(\partial V^{(2)}, \partial V^{(1)}; \mathrm{Z}) \to H_2(W, \partial V^{(1)}; \mathrm{Z}) \to H_2(W, \partial V^{(2)}; \mathrm{Z}) \to H_1(\partial V^{(2)}, \partial V^{(1)}; \mathrm{Z}) = 0$$

will imply that $H_2(W, \partial V^{(2)}; \mathrm{Z})$ is of infinite rank. Thus Lemma 4 implies that $H_2(V, \partial V; \mathrm{Z}) \cong H_2(W, \partial V^{(2)}; \mathrm{Z})$ is of infinite rank. This contradicts Lemma 3. Hence $H_2(W, \partial V^{(1)}; \Lambda)$ is a finitely generated Λ-module and so is $H_2(W; \Lambda)$. ∎

Remark: In case $\pi_1(\epsilon) \cong \mathrm{Z}$ we can use Lemma 5 (*ii*) and the proof of [**2**, Lemma 5] to show that $H_2(W; \mathrm{Z})$ is a projective Λ-module.

2. Infinite cyclic $\pi_1(\epsilon)$

The proof of Corollary 1 is essentially the same as the proof of [**3**, Theorem 1], but for the sake of completeness we provide an outline. We first note that, since $\widetilde{K}_0(\mathrm{Z}[\pi_1(\epsilon)]) = 0$, the end of $M - \Sigma$ has a weak collar U by [**2**, Theorem 11.9B] (see [**3**, Lemma 8]). A *weak collar* is a closed manifold neighborhood U of the end which homotopically looks like a collar. In particular, $\pi_1(U) \cong \mathrm{Z}$ and $\ker[\pi_1(\partial U) \to \pi_1(U)]$ is perfect. Second, it follows from 4-dimensional surgery theory [**2**, §11.6] that there is a compact 4-manifold V such that $\partial V \cong \partial U$ and V has the homotopy type of S^1 (see [**3**, Lemma 9]). As before, we use $\overline{U}$ to denote $U \cup \Sigma$. Next, $\overline{U} \cup_\partial V$ is homeomorphic to S^4 by the Poincaré conjecture (see [**3**, Lemma 10]). We have thus succeeded in embedding a neighborhood of Σ in S^4, and we can therefore use the theorems of [**3**] which apply in case the ambient manifold is S^4. Let $W' = U \cup V \subset S^4$. Then $\pi_1(W') \cong \mathrm{Z}$ by Van Kampen's Theorem. By

[**3**, Theorem 5′], there is a compact manifold $N \subset S^4$ such that

$$\phi : S^2 \times B^2 \xrightarrow{\cong} N, \Sigma = S^4 - W' \subset \operatorname{int} N, \text{ and } N \cap W' = N - \Sigma \cong \partial N \times [0,1).$$

Pushing along the radial structure of $\partial N \times [0,1)$, we can assume that $N \subset U \subset M$ and define $K = \phi(S^2 \times \{0\}) \subset N \subset M$. The proof of the Corollary is complete.

3. Finite cyclic $\pi_1(\epsilon)$

A *Swan space* for a group G is a space with the homotopy type of a CW-complex, fundamental group G, and universal cover of the homotopy type of some sphere S^k (refer to [**2**, p. 228]). Let Z_n denote the cyclic group of order $n, n < \infty$.

Lemma 7. *Let Σ be a 2-sphere embedded in the interior of the compact 4-manifold M and let ϵ denote the end of $M - \Sigma$. If $\pi_1(\epsilon) \cong \mathrm{Z}_n$, then ϵ has neighborhoods that are Swan spaces for $\pi_1(\epsilon)$.*

Proof: Let E denote the homotopy collar of the end of $M - \Sigma$ (as defined on p. 214 of [**2**]). By [**2**, Theorem 11.9E] there is an open neighborhood U of the end ϵ such that U is homeomorphic to the infinite cyclic cover of a compact manifold N with N homotopy equivalent to $E \times S^1$. Thus $\pi_1(U) \cong \pi_1(E) \cong \mathrm{Z}_n$ and U has two ends. One of the two ends is ϵ, which is finitely dominated by the Theorem. But both ends are proper homotopy equivalent to $E \times [0, \infty)$, so the other end of U must also be finitely dominated and have fundamental group Z_n. Moreover, since $\pi_1(U)$ is finite, the universal cover $\widetilde{U}$ of U also has two finitely dominated ends, both simply connected. Therefore each end of $\widetilde{U}$ has a simply connected weak collar whose boundary is a homology 3-sphere [**2**, Theorem 11.9C]. Thus $\widetilde{U} \cong S^3 \times \mathrm{R}$ [**1**, Corollary 1.3]. In other words, U is a Swan space for $\pi_1(\epsilon) \cong \mathrm{Z}_n$. ∎

Proof of Corollary 2: Let U be an open neighborhood of the end ϵ. By the lemma above we may assume that U is a Swan space for $\pi_1(\epsilon)$. The finiteness obstruction

$$\sigma(U) \in \widetilde{K}_0(\mathrm{Z}[\pi_1(\epsilon)])$$

is trivial by [**8**, Corollary 2.4(i)] and so ϵ has a weak collar [**2**, Theorem 11.9B]. ∎

References

1. M. H. Freedman, *The topology of four-dimensional manifolds*, J. Diff. Geom. **17** (1982), 357–453.
2. M. H. Freedman and F. Quinn, "Topology of 4-manifolds," Princeton University Press, Princeton, 1990.
3. Vo Thanh Liem and G. A. Venema, *Characterization of knot complements in the 4-sphere*, Topology and its Appl. (to appear).

4. ________________, *On the asphericity of knot complements*, preprint.
5. J. G. Ratcliffe, *On complexes dominated by a 2-complex*, in "Combinatorial group theory and topology," Annals of Math. Studies, vol. 111, 1987, pp. 221–259.
6. L. C. Siebenmann, "The obstruction to finding a boundary for an open manifold of dimension greater than five," PhD dissertation, Princeton University, Princeton, New Jersey, 1965.
7. P. F. Smith, *A note on idempotent ideals in group rings*, Arch. Math. **27** (1976), 22–27.
8. C. B. Thomas and C. T. C. Wall, *The topological spherical space form problem I*, Compositio Math. **23** (1971), 101–114.
9. C. T. C. Wall, *Finiteness conditions for CW-complexes*, Annals of Math. **81** (1965), 56–69.
10. ____________, *Finiteness conditions for CW complexes, II*, Proc. Roy. Soc., Ser. A **295** (1966), 129–139.
11. ____________, "Surgery on Compact Manifolds," Academic Press, London, 1970.

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