A 4-DIMENSIONAL 1-LCC SHRINKING THEOREM

M. Bestvina, R. J. Daverman, G. A. Venema, and J. J Walsh

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Abstract. This paper contains several shrinking theorems for decompositions of 4-dimensional manifolds. Let $f : M \to X$ be a closed, cell-like mapping of a 4-manifold M onto a metric space X and let Y be a closed subset of X such that $X - Y$ is a 4-manifold and Y is locally simply co-connected in X. The main result states that f can be approximated by homeomorphisms if Y is a 1-dimensional ANR. The techniques of the proof also show that f can be approximated by homeomorphisms in case Y is an arbitrary 0-dimensional closed subset. Combining the two results gives the same conclusion in case Y contains a closed, 0-dimensional subset C such that $Y - C$ is a 1-dimensional ANR.

The construction in the paper also gives a proof of a taming theorem for 1-dimensional ANR's.

1. Introduction

An important problem in the topology of manifolds is the problem of understanding cell-like images of manifolds. This study began with the work of R. L. Moore in dimension two and continued with the work of R. H. Bing in dimension three. In studying cell-like images of n-manifolds, $n \geq 5$, a fundamental tool is a marvelous recognition criterion for detecting manifolds. It is provided by Edwards's Cell-like Approximation Theorem [E3], which assures that such an image space is a genuine manifold if it is finite dimensional and has a minimal general position feature known as the Disjoint Disks Property.

No comparable recognition criterion is known for 4-manifolds, but we take a step in that direction here. One of the ingredients in the proof of Edwards's result is a 1-LCC Shrinking Theorem, first conjectured by Cannon [C]: if $f : M \to X$ is a cell-like mapping defined on an n-manifold and X contains a closed $(n-3)$-dimensional subset Y such that $X - Y$ is an n-manifold and Y is 1-LCC embedded in X (the term is defined later), then f is a near-homeomorphism (that is, f can be approximated, arbitrarily closely, by homeomorphisms). In particular, the 1-LCC

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Shrinking Theorem implies that X is an n-manifold homeomorphic to M. Our main result is the following special case of the 1-LCC Shrinking Theorem in dimension 4.

1-LCC Shrinking Theorem for ANR's. *Let $f : M \to X$ be a closed, cell-like mapping of a 4-manifold M onto a metric space X and let Y be a closed subset of X. If*

(1) *$X - Y$ is a 4-manifold,*
(2) *Y is 1-LCC in X, and*
(3) *Y is a 1-dimensional ANR,*

then f is a near-homeomorphism (and X is a 4-manifold).

The analogous theorem in high dimensions is true without the hypothesis that Y is an ANR, but we do not know whether the 4-dimensional theorem is valid without that hypothesis. The precise hypothesis that is actually needed in the proof is a version of local simple connectivity. That hypothesis is satisfied by compacta that are not necessarily ANR's. For example, any 0-dimensional set also satisfies the necessary hypothesis. Hence the proof has the following corollary. The corollary can also be proved by other techniques and is well-known to experts in the field. It seems, however, that it has not previously appeared in print.

Corollary 1. *Let $f : M \to X$ be a closed, cell-like mapping of a 4-manifold M onto a metric space X and let Y be a closed subset of X. If*

(1) *$X - Y$ is a 4-manifold,*
(2) *Y is 1-LCC in X, and*
(3) *Y is 0-dimensional,*

then f is a near-homeomorphism (and X is a 4-manifold).

Combining the two results gives the following slightly better corollary.

Corollary 2. *Let $f : M \to X$ be a closed, cell-like mapping of a 4-manifold M onto a metric space X and let Y be a closed subset of X. If*

(1) *$X - Y$ is a 4-manifold,*
(2) *Y is 1-LCC in X, and*
(3) *Y contains a closed, 0-dimensional subset C such that $Y - C$ is a 1-dimensional ANR,*

then f is a near-homeomorphism (and X is a 4-manifold).

As a by-product we obtain a proof for a special case of another result due to Edwards, still unpublished [E4], called the 1-LCC Taming Theorem. This theorem shows that the local homotopy condition (the 1-LCC condition) that characterizes "tameness" in high dimensional manifolds has the same effect in dimension 4. Specifically, for a 1-dimensional compact set Y in a PL 4-manifold M, Y is 1-LCC embedded in M if and only if Y has embedding dimension 1. The latter means that for each $\epsilon > 0$ there exists an ϵ-regular neighborhood N of some 1-complex such that $\operatorname{Int} N \supset Y$. Embedding dimension 1 is a true tameness condition in this setting because, for example, two homotopic embeddings of Y in M are ambient isotopic if both images have embedding dimension 1. Our methods establish this tameness result for 1-dimensional ANR's.

1-LCC Taming Theorem for ANR's. *If for $i \geq 1$, Y_i is a 1-dimensional ANR and is a 1-LCC embedded, closed subset of a PL 4-manifold M, then $Y = \cup_{i=1}^{\infty} Y_i$ has embedding dimension 1.*

For several years we believed the techniques used here would prove the full strength 1-LCC Shrinking Theorem in dimension 4; however, the present manuscript should be taken as an indication of the unsettled nature of that result.

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2. Definitions, Notation, and Preliminary Lemmas

All manifolds are assumed to be separable. A compact subset A of a manifold M is said to be a *cell-like set* if A can be deformed to a point in any neighborhood of itself. It is well-known that cell-likeness is a topological property. A map $f : M \to X$ is said to be a *cell-like mapping* if $f^{-1}(x)$ is a nonempty cell-like subset of M for every $x \in X$.

Suppose $f : M \to X$ is a closed, cell-like mapping of a 4-dimensional manifold M onto a metric space X. We will use d to denote the metric on M and ρ to denote the metric on X. We use $N(f)$ to denote the *nondegeneracy set of f*; i.e.,

$$N(f) = \Big\{ x \in M \mid \{x\} \neq f^{-1}(f(x)) \Big\}.$$

A subset Y of a metric space X is said to be *locally 1-co-connected in X*, abbreviated as 1-LCC in X, if for each $y \in Y$ and neighborhood U of y there exists another neighborhood V of y, $V \subset U$, such that each map $\partial B^2 \to V - Y$ can be extended to a map $B^2 \to U - Y$.

Measuring Closeness. Since we are not assuming that either M or X is compact, closeness must be measured by a majorant function $\epsilon(x) > 0$ rather than by a number $\epsilon > 0$. All majorant functions $\epsilon : X \to (0, \infty)$ and $\delta : M \to (0, \infty)$ are assumed to be continuous (even where this is not explicitly stated). If $f, g : M \to X$, the statement $\rho(f, g) < \epsilon$ means that, for every $x \in M$, both $\rho(f(x), g(x)) < \epsilon(f(x))$ and $\rho(f(x), g(x)) < \epsilon(g(x))$. A subset $Z \subset X$ *has diameter less than* ϵ if $\operatorname{diam} Z < \epsilon(x)$ for every $x \in Z$. (So $\rho(f, g) < \epsilon$ if and only if $\operatorname{diam}\{f(x), g(x)\} < \epsilon$ for every $x \in M$.) A homotopy $\mu_t : M \to X$ is an *ϵ-homotopy* if the track of each point has diameter $< \epsilon$.

Now suppose that $f : M \to X$ is a closed cell-like mapping and that, in addition, X contains a closed subset Y such that $X - Y$ is a 4-manifold. The following lemma allows us to approximate f by another cell-like mapping whose nondegeneracy set is contained in the preimage of Y.

Lemma 2.1. *Suppose $f : M \to X$ is a closed, cell-like mapping of a 4-manifold M onto a metric space X and that, in addition, X contains a closed subset Y such that $X - Y$ is a 4-manifold. Then for every $\epsilon : X \to (0, \infty)$ there exists a cell-like mapping $g : M \to X$ such that*

(1) *$g|g^{-1}(X - Y)$ is a homeomorphism,*
(2) *$g^{-1}(Y) = f^{-1}(Y)$,*
(3) *$g|g^{-1}(Y) = f|f^{-1}(Y)$, and*
(4) *$\rho(f, g) < \epsilon$.*

Proof. This follows from Corollary 2.5 of [A]. □

As a consequence of Lemma 2.1, there is no loss of generality in assuming, in the statement of the main theorem, that $N(f) \subset f^{-1}(Y)$. If, in addition, Y is 1-dimensional, we can choose a point $x \in M - f^{-1}(Y)$ and we may then replace the manifold M in the statements of the theorems with the manifold $M - \{x\}$. By [FQ, Theorem 8.2], $M - \{x\}$ has a PL manifold structure. Thus we will assume henceforth that the manifold M in the statements of our theorems is a PL 4-manifold.

Definition. Suppose P is a polyhedron which is a closed subset of M and $\delta : M \to (0, \infty)$. A *$\delta$-regular neighborhood of P* is a subpolyhedron V of M such that V is a regular neighborhood of P and the regular neighborhood collapse $V \searrow P$ induces a δ-homotopy of V.

The fact that Y is 1-dimensional means that Y can be approximated by 1-dimensional polyhedra. The approximating polyhedra may be lifted to M via the CE map f. The next lemma spells out how we will make use of that fact.

Lemma 2.2. *Suppose $f : M \to X$ is a closed, cell-like mapping of a 4-manifold M onto a metric space X and that X contains a closed, 1-dimensional subset Y such that $X - Y$ is a 4-manifold. Then for every $\epsilon : X \to (0, \infty)$ there exist an open neighborhood U of $f^{-1}(Y)$, a 1-dimensional polyhedron $P \subset U$, and a homotopy $\mu_t : U \to M$ such that*

(1) *$\mu_0(x) = x$ for every $x \in U$,*
(2) *$\mu_1(x) \in P$ for every $x \in U$,*
(3) *$\mu_t|P$ is the identity for every t, and*
(4) *$f \circ \mu_t$ is an ϵ-homotopy.*

Proof. Since X is a finite dimensionsal cell-like image of a manifold, it is an ANR; hence there exists $\delta : X \to (0, \infty)$ such that for any space S and any two maps $h_0, h_1 : S \to X$ satisfying $\rho(h_0, h_1) < 2\delta$, h_0 and h_1 are ϵ-homotopic; furthermore, the homotopy can be required to be stationary at all $s \in S$ for which $h_0(s) = h_1(s)$. To obtain P, start with a δ-open (relative to X) cover $\mathcal{W}$ of Y; apply 1-dimensionality of Y to refine $\mathcal{W}$ to another open cover $\mathcal{W}'$ of Y by connected sets, no three of which intersect; and identify the 1-complex P' corresponding to the nerve of $\mathcal{W}'$. Set

$$U' = \bigcup_{W' \in \mathcal{W}'} W'$$

and $U = f^{-1}(U')$. In the next paragraph we describe how to produce an embedding $\lambda : P' \to U$ and, simultaneously, a retraction $r : U \to P = \lambda(P')$ satisfying $\rho(f \circ r, f \circ \iota) < 2\delta$ (where ι denotes inclusion $\iota : U \hookrightarrow M$). Hence, $f \circ r$ and $f \circ \iota$ are ϵ-homotopic via a homotopy H_t fixing P pointwise. Lifting properties of cell-like maps (cf. [D, Theorem 16.7] for a proof in the compact case) ensure H_t can be approximately lifted to a homotopy $\mu_t : U \to M$ with the desired properties.

Assume no element of $\mathcal{W}'$ is contained in the union of the others. For each $W' \in \mathcal{W}'$ select $v \in f^{-1}(W')$ not belonging to the preimage of any other $W'' \in \mathcal{W}'$; for each intersecting pair $W', W'' \in \mathcal{W}'$ choose an arc $a(W', W'') \subset f^{-1}(W' \cup W'')$ joining the selected points. After a general position adjustment, the union of all such $a(W', W'')$ will be a copy P of P'. Require that the closures of the various $f^{-1}(W' \cap W'')$ in M be pairwise disjoint. Tietze's Extension Theorem yields a retraction of $\overline{f^{-1}(W' \cap W'')} \cup a(W', W'')$ to $a(W', W'')$. For fixed $W' \in \mathcal{W}'$, the union $P_{W'}$ of all $a(W', W'')$, W'' variable, is a compact absolute retract, so the retraction partially defined on (a closed subset of) $f^{-1}(W')$ extends to a retraction $f^{-1}(W') \to P_{W'}$, and the compilation of these piecewise defined retractions produces the desired $r : U \to P$. □

A second important consequence of the fact that Y is 1-dimensional is that Y has enough codimension so that 1-dimensional polyhedra can be pushed off Y. In fact we need the stronger property that 1-dimensional polyhedra can be pushed off the preimage of a neighborhood of Y via a controlled homotopy.

Lemma 2.3. *Suppose $f : M \to X$ is a closed, cell-like mapping of a 4-manifold M onto a metric space X and that X contains a closed, 1-dimensional subset Y such that $X - Y$ is a 4-manifold. Then for every $\epsilon : X \to (0, \infty)$ and every neighborhood U of $f^{-1}(Y)$ there exists an open neighborhood V of $f^{-1}(Y)$ such that if K is any 1-dimenisonal polyhedron in M then there exists a homotopy $\lambda_t : K \to M$ such that*

1. *$\lambda_0(x) = x$ for every $x \in K$,*
2. *$\lambda_1(x) \in M - V$ for every $x \in K$,*
3. *$\lambda_t | K \cap (M - U)$ is the identity for every t, and*
4. *$f \circ \lambda_t$ is an ϵ-homotopy.*

Proof. The techniques are standard, so we merely sketch the proof. Let K_1 be the 1-skeleton of a triangulation of M whose mesh is small relative to ϵ. It suffices to prove the lemma for the special case $K = K_1$; i.e, to find a neighborhood V such that K_1 can be pushed off V with a controlled homotopy. (In the general case, first use general position to push K into K_1 and then apply the homotopy of K_1 to complete the push of K off V.)

Use the fact that X is a 4-dimensional generalized manifold and Y is 1-dimensional to approximate $f|K_1$ by a map $f_1 : K_1 \to X - Y$. Since $X - Y$ is a 4-manifold, f_1 may be approximated by an embedding f_2. Because X is an ANR, there is a small homotopy from $f|K_1$ to f_2. The fact that f is cell-like allows that homotopy to be lifted to M. Define V to be the preimage under f of a neighborhood of Y that misses $f_2(K_1)$. □

If, in addition, Y is 1-LCC, then 2-dimensional polyhedra may be pushed off the preimage of Y.

Lemma 2.4. *Suppose $f : M \to X$ is a closed, cell-like mapping of a 4-manifold M onto a metric space X and that X contains a closed, 1-dimensional subset Y such that $X - Y$ is a 4-manifold. Then for every $\epsilon : X \to (0, \infty)$ and every neighborhood U of $f^{-1}(Y)$ there exists an open neighborhood V of $f^{-1}(Y)$ such that if K is any 2-dimensional polyhedron in M then there exists a homotopy $\lambda_t : K \to M$ such that*

1. *$\lambda_0(x) = x$ for every $x \in K$,*
2. *$\lambda_1(x) \in M - V$ for every $x \in K$,*
3. *$\lambda_t | K \cap (M - U)$ is the identity for every t, and*
4. *$f \circ \lambda_t$ is an ϵ-homotopy.*

Proof. The proof is essentially the same as that of the previous lemma. The one difference is that the 1-LCC property must be used to approximate a map from a 2-dimenisonal polyhedron into X by one that maps into $X - Y$. □

Finally, there is a point in the proof at which we need Y to be an ANR. The last lemma in this section spells out how that property will be used. The key point in the lemma is the fact that the same ϵ' works for all i. We will also make use of the fact that any 0-dimensional closed set satisfies the conclusion of the lemma.

Lemma 2.5. *Suppose X is a metric ANR and Y is a closed subset of X such that Y is locally simply connected. For every $\epsilon : X \to (0, \infty)$ there exist $\epsilon' : X \to (0, \infty)$ and a nested sequence $\{U'_i\}$ of neighborhoods of Y in X such that $\cap U'_i = Y$ and any loop in U'_i of diameter less than ϵ' bounds a disk in U'_{i-1} of diameter less than ϵ.*

Proof. Given $\epsilon : X \to (0, \infty)$, use the fact that Y is an ANR to choose $\epsilon_1 : X \to (0, \infty)$ such that any loop in Y of diameter less than ϵ_1 bounds a singular disk of diameter less than ϵ. Define $\epsilon' = \epsilon_1/3$. Then choose a nested sequence of neighborhoods $\{U'_i\}$ of Y in X such that there is an ϵ'-deformation retraction of U'_i to Y in U'_{i-1}. It is clear that ϵ' and $\{U'_i\}$ satisfy the conclusions of the lemma. □

3. Proof of the Main Lemma

The objective of this section is to prove the following theorem. A sequence of applications of Theorem 3.1 will be used to establish the 1-LCC Shrinking Theorem.

Theorem 3.1. *Suppose $f : M \to X$ is a closed, cell-like mapping of a PL 4-manifold M onto a metric space X and that X contains a closed, 1-dimensional ANR Y such that $N(f) \subset f^{-1}(Y)$ and Y is 1-LCC in X. Then for every $\epsilon : X \to (0, \infty)$ and for every $\delta : M \to (0, \infty)$ there exist a cell-like map $g : M \to X$, a 1-dimensional polyhedron P in M, and a δ-regular neighborhood V of P such that $N(g) \subset g^{-1}(Y) \subset V$ and $\rho(f, g) < \epsilon$.*

We use I to denote the closed unit interval $[0, 1]$ and $\pi : M \times I \to M$ to denote the projection map.

Main Lemma. *Suppose $f : M \to X$ and $Y \subset X$ are as in Theorem 3.1. Then for every $\epsilon : X \to (0, \infty)$ and for every $\delta : M \to (0, \infty)$ there exist an open neighborhood U of $\overline{N(f)}$, a closed 1-dimensional polyhedron $P \subset M$, a δ-regular neighborhood V of P, and a homeomorphism $h : M \times I \to M \times I$ such that*

(1) *$h(x, 0) = (x, 0)$ for every $x \in M$,*
(2) *$h(x, 1) \in V$ for every $x \in U$, and*
(3) *$f \circ h_t$ is an ϵ-homotopy, where $h_t : M \to M$ is defined by $h_t(x) = \pi(h(x, t))$.*

Proof of Theorem 3.1, assuming the Main Lemma. Let U, P, V, and h be as in the conclusion of the Main Lemma. Notice that $h_1 : M \to M$ is a homeomorphism. Thus we can define g by $g = f \circ h_1^{-1}$. Now $N(g) = h_1(N(f))$, so $\overline{N(g)} \subset h_1(U) \subset V$. Since $\rho(f, f \circ h_1) < \epsilon$, we also have $\rho(f \circ h_1^{-1}, f) < \epsilon$. Hence $\rho(f, g) < \epsilon$. $\square$

The idea of the proof of the Main Lemma. The remainder of this section is devoted to the proof of the Main Lemma. The idea is to use a handle cancelling argument similar to that in the proof of the Controlled h-cobordism Theorem [FQ, Theorem 7.2A] to construct a special product structure on $M \times I$. We will find a neighborhood U of $\overline{N(f)}$ and a regular neighborhood V of a 1-dimensional polyhedron P and then construct the product structure to have two properties: first, any fiber that starts out in $U \times \{0\}$ must end in $V \times \{1\}$ and, second, the projection of each fiber into X must be small. Thus there are two forms of control that must be maintained at all times during the argument: the first ensures that fibers move towards P and the second ensures that each fiber has small image in X.

Although it is possible to apply a 4-dimensional Controlled h-cobordism Theorem, we prefer to work out the proof by hand, explaining how to cancel handles of various indices. The reason for doing this is that it is just as difficult to explain how to construct the controlled deformations needed in the hypotheses of the Controlled h-cobordism Theorem as it is to explain how to cancel the handles. In addition, we think the proof is geometrically clearer if we explain how to construct the product structure directly.

The logical structure of the proof. Rather than spell out all the ϵ's and δ's before hand, we will start the construction at the beginning and work through it. As we go, we will highlight the conditions that must be met in order to achieve the necessary control. This is not the strictly logical way in which to present the proof, but we believe it is the best way to present the geometric ideas that support the proof. In order to produce the strict logical version of the proof, one would have to make a first pass through the proof noting all the conditions that must be satisfied and then go back to the beginning of the proof and construct the regions R_i, below, in such a way that all these conditions are satisfied.

Construction of the regions R_i. Fix n (a large positive integer to be specified later). We will construct a finite sequence $R_0, R_1, \ldots, R_n$ of regions in $M \times I$ with

$R_0 \supset R_1 \supset R_2 \supset \cdots \supset R_n$. Each R_i, $i > 0$, will consist of two parts:

(1) $U_{i-1} \times [0, 1/(i+1))$ for some open neighborhood U_{i-1} of $\overline{N(f)}$, and
(2) $\{ (x,t) \in M \times I \mid 1/(i+1) \le t \le 1 \text{ and } x \in \Phi^i_t(V_i) \}$ where Φ^i_t is a PL isotopy of M such that Φ^i_t is the identity for $0 \le t \le 1/(i+1)$ and V_i is the interior of a thin regular neighborhood of a 1-dimensional polyhedron $P_i \subset U_i$.

The first part is called the *thick part* of R_i while the second part is called the *thin part* of R_i. The 2-dimensional polyhedron

$$C_i = \{ (x,t) \in M \times I \mid 1/(i+1) \le t \le 1 \text{ and } x \in \Phi^i_t(P_i) \}$$

is called the *core* of the thin part of R_i. Notice that the core of the thin part is 2-dimensional and $M \times I$ is 5-dimensional, so 2-dimensional polyhedra in $M \times I$ can be general positioned off C_i. Note too that $R_i \cap (M \times \{1\}) = \Phi^i_1(V_i) \times \{1\}$, which is a thin regular neighborhood of the 1-dimensional polyhedron $\Phi^i_1(P_i) \times \{1\}$. The various isotopies Φ^i_t will all move points approximately the same amount, limited by a specified function of the initially given ϵ and δ.

Begin with $R_0 = M \times I$. To get started, let U_0 denote the preimage under f of the ϵ-neighborhood of Y in X. Then apply Lemma 2.2 to obtain an open neighborhood U_1 of $\overline{N(f)}$, a 1-dimensional polyhedron $P_1 \subset U_1$, and a homotopy $\mu^1_t : U_1 \to U_0$ which pushes U_1 into P_1 in a controlled way. Let V_1 be the interior of a thin regular neighborhood of P_1 in U_1 and set

$$R_1 = U_0 \times [0, \tfrac{1}{2}) \cup V_1 \times [\tfrac{1}{2}, 1].$$

The isotopy Φ^1_t is the identity.

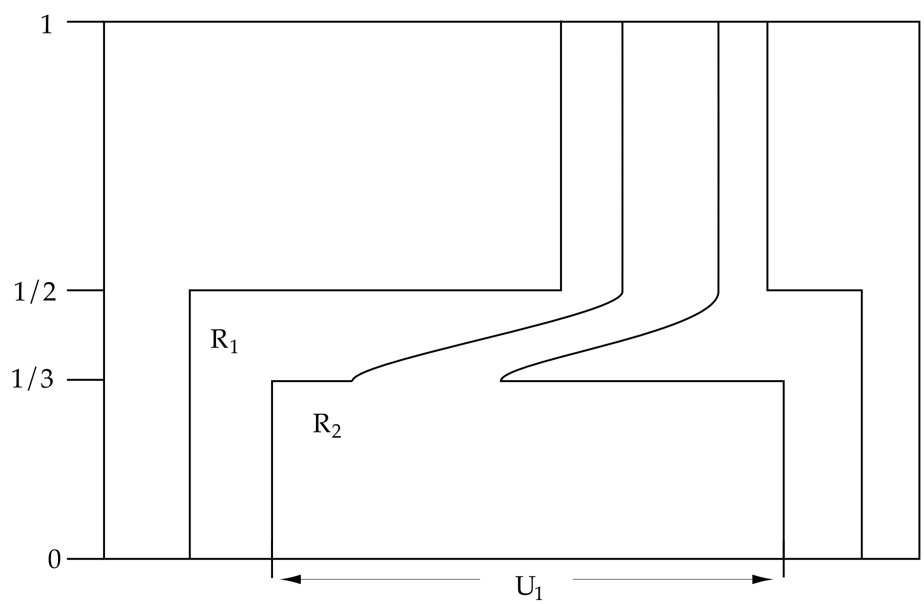

Figure 1

We next explain how to construct R_2. Apply Lemma 2.2 again to obtain a smaller neighborhood U_2 of $\overline{N(f)}$, a 1-dimensional polyhedron $P_2 \subset U_2$, and a homotopy $\mu_t^2 : U_2 \to U_1$ that pushes U_2 into P_2 in a controlled way. Since P_2 is 1-dimensional and M is 4-dimensional, there is a PL isotopy $\phi_t^2 : M \to M$ such that $\phi_t^2 | P_2$ is close to $\mu_t^1 | P_2$. We may assume that ϕ_t^2 is the identity for $0 \le t \le \frac{1}{3}$ and that $\phi_t^2 = \phi_1^2$ for $\frac{1}{2} \le t \le 1$. The track of μ_t^1 is contained in U_0, so we may also assume that $\phi_t^2 | M - U_0$ is the identity. Set $\Phi_t^2 = \Phi_t^1 \circ \phi_t^2$. (Since Φ_t^1 is the identity, $\Phi_t^2 = \phi_t^2$.) Taking V_2 to be the interior of a thin regular neighborhood of P_2, we can define

$$R_2 = U_1 \times [0, \tfrac{1}{3}) \cup \{\, (x,t) \in M \times I \mid \tfrac{1}{3} \le t \le 1 \text{ and } x \in \Phi_t^2(V_2) \,\}.$$

The regions R_1 and R_2 are pictured in Figure 1.

The construction of R_3 is similar. By Lemma 2.2 there exist a neighborhood U_3 of $\overline{N(f)}$, a 1-dimensional polyhedron $P_3 \subset U_3$, and a homotopy $\mu_t^3 : U_3 \to U_2$ that pushes U_3 into P_3 in a controlled way. Again, $\mu_t^2 | P_3$ can be approximately covered by a PL isotopy $\phi_t^3 : M \to M$ such that ϕ_t^3 is the identity for $0 \le t \le 1/4$, is constant for $t \ge 1/3$, and is the identity outside U_1; with appropriate controls on μ_t^2 and ϕ_t^3, the composite $\Phi_t^3 = \Phi_t^2 \circ \phi_t^3$ will not move points of M much more than Φ_t^2 does. Take V_3 to be the interior of a thin regular neighborhood of P_3. Then we can define R_3 as follows.

$$R_3 = U_2 \times [0, \tfrac{1}{4}) \cup \{\, (x,t) \in M \times I \mid \tfrac{1}{4} \le t \le 1 \text{ and } x \in \Phi_t^3(V_3) \,\}.$$

The construction is continued inductively. It results in $n+1$ regions $R_0, R_1, \ldots, R_n$. In later statements it will be convenient to have R_j defined for every integer j. Hence we define $R_j = R_0$ for $j < 0$ and $R_j = \emptyset$ for $j > n$.

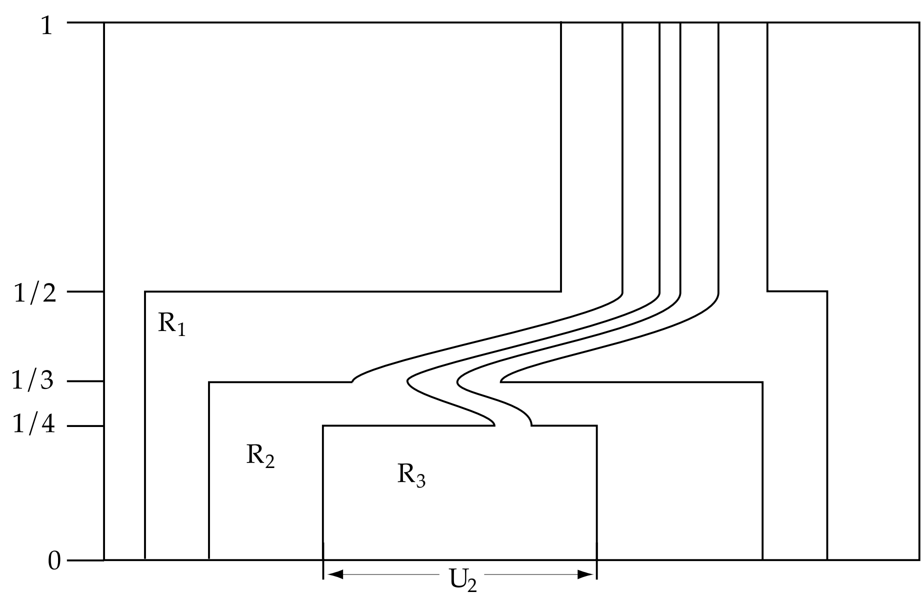

Figure 2

Constructing the homeomorphism h. Start with thin (closed) collars C_0 and C_1 of $M \times \{0\}$ and $M \times \{1\}$, respectively, in $M \times I$. Then take a handle decomposition of the remainder, $M \times I - (C_0 \cup C_1)$. The handle decomposition contains handles of indices 0, 1, 2, 3, 4, and 5. We use $\mathcal{H}$ to denote the collection of handles, W_i to denote the union of C_0 and all handles of index $\leq i$, and W^j to denote the union of C_1 and all handles of index $\geq j$. Let $\partial_+ W_i = \partial W_i - (M \times \{0\})$ and $\partial_+ W^j = \partial W^j - (M \times \{1\})$. Note that $\partial_+ W_i = \partial_+ W^{i+1}$. The diameter of the handles in $\mathcal{H}$ should be small relative to the distance between the frontiers of the regions R_i. In particular, make the handles so small that if H is a handle and $H \cap R_i \neq \emptyset$ for some i, then $H \subset R_{i-1}$. Observe that for each handle $H \in \mathcal{H}$ there exists an i such that $H \subset R_{i-1} - R_{i+1}$.

We will use the handle decomposition to construct a new product structure on $M \times I$. This product structure serves as the image of $h : M \times I \to M \times I$ and thus implicitly defines h. In the proof below, the handles will be absorbed, one at a time, into the collars C_0 and C_1. At the end of this procedure there will be no handles left and so the collars C_0 and C_1 will exactly cover $M \times I$ and their union will define a product structure on $M \times I$. This product structure will be the one we seek provided we maintain size control during the construction. As mentioned earlier, there are two kinds of control to be considered.

The first kind of control is meant to ensure that fibers that start near $\overline{N(f)}$ end in V_1. This will be accomplished by requiring that at each step of the proof there is an integer k such that

(a) if $H \in \mathcal{H}$ and $H \cap R_i \neq \emptyset$, then $H \subset R_{i-k}$, and
(b) if α is a fiber arc in the product structure of either C_0 or C_1 and if $\alpha \cap R_i \neq \emptyset$, then $\alpha \subset R_{i-k}$.

Note that the two conditions above make sense even if $k > i$.

The collars and the handle decomposition have been chosen so that $k = 1$ satisfies these conditions at the beginning of the proof. As we work through the proof, we will see that the value of k must increase. This will happen only a finite number of times, so at the end of the proof there will still be a finite number k with the two properties listed above. Thus, at the end of the proof, the union of the two collars will define a product structure with the property that if α is a fiber arc in the product structure and if $\alpha \cap R_i \neq \emptyset$, then $\alpha \subset R_{i-2k}$. In particular, if $\alpha \cap (U_i \times \{0\}) \neq \emptyset$, then $\alpha \subset R_{i-2k}$ and so $\alpha \cap (M \times \{1\})$ is a point in $\Phi_1^{i-2k}(V_{i-2k})$. We choose $n \gg 2k$ so that any fiber that begins in $U_n \times \{0\}$ will end in $V_1 \times \{1\}$.

The second kind of control is control in X. We require that, at each stage of the proof, $\operatorname{diam} f(\pi(H))$ and $\operatorname{diam} f(\pi(\alpha))$ are small in X for every handle H and for every fiber arc α in the product structure of either C_0 or C_1. This is true at the beginning of the proof simply because each handle and each collar arc is small in $M \times I$. During the proof both the handles and the collar arcs will grow in size (as measured in $M \times I$), so it will be necessary to impose additional conditions on the construction of the regions R_i in order to control the size of the projections in X. After each step of the proof the necessary conditions will be highlighted.

Cancelling 5-handles. Let H^5 be a 5-handle in $\mathcal{H}$. There exists an i such that $H^5 \subset R_i$ but $H^5 \not\subset R_{i+1}$. It follows that $H^5 \cap R_{i+k+1} = \emptyset$. Let β_1 be a vertical arc from H^5 straight up to a level $U_{i-1} \times \{s\}$ that is higher than the thick part of R_{i+k+1}. Then use the homotopy μ_t^{i-1} to find an arc β_2 in $U_{i-2} \times \{s\}$ that joins the endpoint of β_1 to a point directly below the thin part of R_{i-1}. By general position, we may assume that β_2 misses the thin part of R_{i+k+1}. Finally, let β_3 be an arc from the endpoint of β_2 that follows the thin part of R_{i-1} up to $M \times \{1\}$. In this way we construct an arc $\beta = \beta_1 * \beta_2 * \beta_3$ such that $\beta \subset R_{i-1} - R_{i+k+1}$ and β joins a point of H^5 to a point of $M \times \{1\}$. Use general position to homotope β off the cores of the handles of index ≤ 3 and off the cocores of the handles of index ≥ 4 so that $\beta \subset \partial_+ W^4 = \partial_+ W_3$. This will increase the number of regions β can intersect, but by at most k regions in each direction; thus $\beta \subset R_{i-1-k} - R_{i+2k+1}$. From the point of view of the top of the cobordism, H^5 looks like a 0-handle. Use β to introduce a small cancelling handle pair (H^4, H^3). From the point of view of the top of the cobordism the new handles have indices 1 and 2, but from the point of view of the bottom of the cobordism they have indices 4 and 3. The new 4-handle H^4 geometrically cancels H^5 in the sense that their union is a 5-cell attached to C_1 along a face. We absorb this 5-cell into C_1. This removes H^5 and H^4 from $\mathcal{H}$. The net effect is to trade the 5-handle H^5 for the new 3-handle H^3.

Size control. The new handle H^3 introduced in the handle trade spills across more regions than did the original handles and the same is true of the fiber arcs in the new collar C_1. Specifically, $H^3 \subset R_{i-1-k} - R_{i+2k+1}$, so we must replace the old k by a new k which is $3 \cdot (\text{old}\, k) + 2$. We now have $k = 3 \cdot 1 + 2 = 5$. All the objects in the previous paragraph are small in $M \times I$ except for the arc β. But β_1 projects to a point in X, while β_2 follows the track of a point under μ_t^{i-1} and β_3 follows the track of some point under Φ_t^{i-1}. Hence each of the three projects to a small arc in X.

Cancelling 0-handles. Let H^0 be a 0-handle in $\mathcal{H}$. There exists an i such that $H^0 \subset R_i$ but $H^0 \not\subset R_{i+1}$. It follows that $H^0 \cap R_{i+k+1} = \emptyset$. Choose $x \in \partial H^0$. Construct an arc β_3 in $R_i - R_{i+k+1}$ joining x to a point $x' \in U_{i-1} \times \{s\}$. (Here β_3 can either be vertical, if x is below $U_{i-1} \times \{s\}$, or lie in the track of some point under Φ_t^i.) Choose $y \in U_{i-1} - U_i$ so that $f(\pi(x'))$ and $f(y)$ are close. Specify a vertical arc β_1 from $(y, 0)$ to a level $U_{i-1} \times \{s\} \subset R_i$ higher than the thick part of R_{i+k+1}. Finally, let β_2 be an arc in $U_{i-1} \times \{s\}$ joining (y, s) to x'. By general position we may assume that β_2 misses R_{i+k+1}. The path $\beta = \beta_1 * \beta_2 * \beta_3$ joins x to $M \times \{0\}$, and it can be chosen so that $\beta \subset R_{i-1} - R_{i+k+1}$ and $f(\pi(\beta))$ is small in X. Use the collar structure to push β out of $C_0 \cup C_1$ and then use general position to push β into $\partial_+ W_1$. This leaves $\beta \subset R_{i-2k-1} - R_{i+3k+1}$. Use β to introduce a new (1,2)-handle pair and then absorb H^0 and the new 1-handle into C_0. In this way H^0 is traded for a new 2-handle.

Size control. The new k is $5 \cdot (\text{old}\, k) + 2$. Thus $k = 27$. The arc β has a small projection into X. There is one new form of size control needed in order that the arc β_2 exist and have small projection in X.

Additional requirement on the construction of regions. The neighborhoods U_i must be chosen to satisfy the following additional requirement.

(AR1) Any point $x' \in U_{i-1}$ may be joined by a path β in U_{i-1} to a point $y \in U_{i-1} - U_i$ in such a way that $f(\beta)$ is small in X.

Remark. One convenient way to achieve (AR1) is to incorporate it into the proof of Lemma 2.2. Using the facts that Y is 1-dimensional and f is cell-like, we can build the 1-dimensional polyhedron P so that $P \subset U - f^{-1}(Y)$. This refinement in Lemma 2.2 would allow us to choose U_i so that $U_i \cap P_{i-1} = \emptyset$. Then the arc β is simply an initial segment of the track of x' under the homotopy μ^{i-1}. Lemma 2.3 can also be used.

Cancelling 4-handles. Let H^4 be a 4-handle in $\mathcal{H}$. As before, there exists an i such that $H^4 \subset R_i$ but $H^4 \not\subset R_{i+1}$; hence $H^4 \cap R_{i+k+1} = \emptyset$. Let α be the cocore of H^4. Then α is an arc beginning and ending on $\partial_+ W_4$. But $\mathcal{H}$ no longer contains any 5-handles, so $W_4 = W_5$ and α begins and ends on ∂C_1. For each endpoint of α, add the corresponding collar arc in C_1. The result is a new arc $\hat\alpha$ which begins and ends on $M \times \{1\}$ and satisfies $\hat\alpha \subset R_{i-k} - R_{i+2k+1}$. We claim that there is a controlled homotopy that pushes $\hat\alpha$ up to the top of $M \times I$, keeping the endpoints of $\hat\alpha$ fixed. To accomplish this, first push $\hat\alpha$ vertically so that it lies entirely in one level $M \times \{s\}$, a little above the thick part of R_{i+2k+1}, together with the thin part of R_{i-k}. Then use the homotopy μ_t^{i-k-1} in the level $M \times \{s\}$ to pull it into the thin part of R_{i-k}. Next push the arc up through the thin part of R_{i-k}, moving parallel to the core of R_{i-k}. By general position we may assume that the tracks of the last two homotopies miss the thin part of R_{i+2k+1}. The track of the homotopy forms a singular disk $D \subset R_{i-k-1} - R_{i+2k+1}$. Push D out of the two collars; this leaves $D \subset R_{i-2k-1} - R_{i+3k+1}$. Finally, use general position to push D off the cores of the 1- and 2-handles and off the cocores of the 4- and 3-handles. After performing all these homotopies we have $D \subset \partial_+ W_2 = \partial_+ W^3$ and $D \subset R_{i-3k-1} - R_{i+4k+1}$. Now use D (desingularized) to introduce a new (2,3)-handle pair and cancel H^4 together with the new 3-handle by absorbing them into C_1. This has the effect of trading H^4 for a new 2-handle.

Size control. The disk D has small size in X because it is formed using the homotopies μ_t^{i-k} and Φ_t^{i-k}. The new k is $7 \cdot (\text{old}\, k) + 2$. Thus $k = 191$.

Cancelling 1-handles. Let H^1 be a 1-handle and choose i such that $H^1 \subset R_i - R_{i+k+1}$. Let α be the core of H^1; then α is an arc joining two points of ∂C_0. Add to α the two collar arcs in C_0 corresponding to the endpoints of α. The result is an arc $\hat\alpha \subset R_{i-k} - R_{i+2k+1}$ joining two points in $(U_{i-k-1} - U_{i+2k}) \times \{0\}$. Push the arc $\hat\alpha$ parallel to the thin part of R_{i-k} into a level $(U_{i-k-1} - V_{i+2k+1}) \times \{s\}$ a little above the thick part of R_{i+2k+2}. Then use the fact that Y is 1-dimensional to find a homotopy of $\hat\alpha$ in that level that pushes $\hat\alpha$ off $U_{i+2k+1} \times \{s\}$ and keeps the endpoints fixed. Finally, push the arc straight down into $M \times \{0\}$. The track of the juxtaposition of these three homotopies forms a disk D. By general position, we may assume that this disk misses the thin part of R_{i+2k+2}. Thus $D \subset R_{i-k-1} - R_{i+2k+2}$. Pushing D off the collars leaves $D \subset R_{i-2k-1} - R_{i+3k+2}$. Pushing D into $\partial_+ W_2$ leaves $D \subset R_{i-3k-1} - R_{i+4k+2}$. Use D to introduce a new $(2,3)$-handle pair. The new 2-handle cancels H^1, so the two can be absorbed into the collar C_0. This entire procedure has the net effect of trading H^1 for a 3-handle.

Size control. The new k is $7 \cdot (\text{old}\, k) + 4$. Thus $k = 1341$. The disk D has small size in X as long as we impose the following additional requirements on the construction of the regions R_i.

Additional requirement on the construction of regions. In order for the disk D, above, to satisfy $f(\pi(D))$ is small in X, we must add another requirement on the construction of the regions R_i.

(AR2) If α is an arc in U_{i-k-1} such that the endpoints of α are in $U_{i-k-1} - U_{i+2k}$, then there is a homotopy of α, rel endpoints, to an arc $\beta \subset U_{i-k-2} - U_{i+2k+1}$ such that the homotopy is small in X.

This condition can be achieved by use of Lemma 2.3.

Cancelling 2- and 3-handles. At this point, our handle decomposition $\mathcal{H}$ contains only handles of indices 2 and 3, attached to C_0. For the remainder of the proof it will be convenient to work with the dual handle decomposition, $\mathcal{H}^*$, which also consists only of 2- and 3-handles, but attached to C_1. We will use W_2^* to denote the union of C_1 and all the 2-handles in $\mathcal{H}^*$ and W_3^* to denote W_2^* union the 3-handles of $\mathcal{H}^*$. Let $\partial_+ W_2^* = \partial W_2^* - (M \times \{1\})$. In the 4-manifold $\partial_+ W_2^*$ there are two collections of 2-spheres: the belt spheres for the 2-handles (the B-spheres) and the attaching spheres for the 3-handles (the A-spheres). We would like to change the handle decomposition so that each A-sphere intersects exactly one B-sphere and the two intersect transversely in one point. Of course we must do this while maintaining size control.

In order to complete the proof, we must analyze the boundary homomorphism

$$\partial : H_3(W_3^*, W_2^*) \to H_2(W_2^*, C_1).$$

The group $H_3(W_3^*, W_2^*)$ is free abelian with the 3-handles as generators and the group $H_2(W_2^*, C_1)$ is free abelian with the 2-handles as generators. Since $M \times I$ is a product, ∂ must be an isomorphism. We need to prove that ∂ is an isomorphism with geometric control in the sense of [Q1]. The control space is $X \times I$. In order to define geometric module structures on $H_3(W_3^*, W_2^*)$ and $H_2(W_2^*, C_1)$, we must define a control map $c : M \times I \to X \times I$. First define $r : M \times I \to I$ by defining r to be equal to i/n on the frontier of R_i and then using the Tietze Extension Theorem to extend r to a continuous map of all of $M \times I$ into I such that $r(R_i - R_{i+1}) \subset [i/n, (i+1)/n]$ for each i. Then define $c : M \times I \to X \times I$ by $c(x) = (f(\pi(x)), r(x))$. Notice that this one control map captures both kinds of control that we need: if

$c(x)$ and $c(y)$ are close in $X \times I$, then $f(\pi(x))$ is close to $f(\pi(y))$ in X and there must be i and j with $|i-j|$ small relative to n such that $x, y \in R_i - R_j$.

The remainder of the proof consists of two parts. First we will show that for every $\delta : X \times I \to (0, \infty)$, we can construct the regions R_i and the handle decomposition $\mathcal{H}^*$ in such a way that ∂ is a δ-isomorphism in the sense of [Q1]. Once this is accomplished, we apply [Q1, Theorem 8.4] to show that ∂ can be deformed to a geometric isomorphism. This means that the handle decomposition can be adjusted so that ∂ is represented algebraically by the identity matrix, and the A-spheres and the B-spheres have good algebraic intersections. In particular, each A-sphere has algebraic intersection number 1 with one of the B-spheres and algebraic intersection number 0 with all the others. Furthermore, the intersections are controlled in the sense that the excess geometric intersection points can be paired off so that each pair has a singular Whitney disk whose projection into $X \times I$ is small. The final step is to apply the Controlled Disk Embedding Theorem [FQ, Theorem 5.4] to get embedded Whitney disks. The proof is then complete because the Whitney trick can be used to remove excess points of intersection between the A-spheres and the B-spheres and then the handles can be cancelled in pairs.

Diagonalizing the boundary homomorphism. We must show that for any $\delta : X \times I \to (0, \infty)$, the construction can be done in such a way that ∂ is a δ-isomorphism. Since ∂ is obviously an isomorphism, this means that we must prove that both ∂ and ∂^{-1} are δ-homomorphisms.

If H^3 is a 3-handle and $\partial(H^3) = n_1 H_1^2 + \cdots + n_j H_j^2$, then H^3 must intersect each H_i^2. The first coordinates of $c(H^3)$ and $c(H_i^2)$ will be close because the diameter of $f(\pi(H))$ is small for every handle $H \in \mathcal{H}^*$. The I-coordinates of H^3 and H_i^2 will differ by at most $3k/n$. Thus we can make ∂ a δ-homomorphism for any δ by simply choosing n to be large relative to k.

Let H^2 be a 2-handle. There exists an i such that $H^2 \subset R_i - R_{i+k}$. We define a homotopy ψ_t which pushes H^2 up to the top level without pushing it into R_{i+k}. The homotopy $\psi_t : M \times I \to M \times I$ is defined by

$$\psi_t(x, s) = \begin{cases} (x, s) & \text{if } s \geq t \\ (\Phi_t^{i+k}(x), t) & \text{if } s < t. \end{cases}$$

Notice that ψ_t deforms $M \times I$ to $M \times \{1\}$ and that it moves points parallel to the thin part of R_{i+k}. In particular, if z is any point in the complement of R_{i+k}, then the entire track of z under ψ_t misses R_{i+k}. Consider the track of H^2 under ψ_t. We can adjust $\psi_t | H^2$ so that it consists of a finite sequence of handle slides, each slide being a slide of H^2 over one of the 3-handles. This allows us to write $\psi(H^2 \times I) = m_1 H_1^3 + \cdots + m_j H_j^3$, where each H_ℓ^3 is a 3-handle which misses R_{i+2k}. Thus

$$\partial(m_1 H_1^3 + \cdots + m_j H_j^3) = H^2$$

or

$$\partial^{-1}(H^2) = m_1 H_1^3 + \cdots + m_j H_j^3.$$

This almost gives us what we need. It does show that ∂^{-1} is small in the X coordinate of the control space $X \times I$ since the homotopy ψ projects to a small homotopy in X. The vertical push also satisfies $\psi((R_i - R_{i+1}) \times I) \subset (M \times I) - R_{i+1}$, so the homotopy ψ only decreases coordinates in the I direction. Thus the equations above show that ∂^{-1} does not increase I coordinates by more than k/n. In order to show that ∂^{-1} does not decrease I coordinates by much, we construct a second deformation retraction of $M \times I$ to $M \times \{1\}$ which has that property. Fix an i such that $H^2 \subset R_i$ but $H^2 \not\subset R_{i+1}$. In the construction of R_i there was a controlled homotopy μ^i_t which pushes U_i into P_i, keeping P_i fixed. Define $\xi_t : R_i \to R_{i-1}$ to be the homotopy which does μ^{i-1}_t on each level of the thick part of R_i during the first half of the time interval and then deformation retracts the thin part of R_{i-1} up to $M \times \{1\}$ during the second half of the interval. Notice that $f \circ \pi \circ \xi_t$ is small in X and that $\xi_t(R_i) \subset R_{i-1}$ for every t. Just as above, this allows us to write $\partial^{-1}(H^2)$ as a linear combination of 3-handles such that for each 3-handle H^3_j in the sum, the second coordinate of $c(H^3_j)$ is greater than or equal to $r(H^2) - (k+1)/n$. Thus we conclude that ∂^{-1} does not decrease I coordinates by much either and hence ∂^{-1} can be made a δ-homomorphism for any δ.

Controlled disk embedding. All that remains in order to complete the proof is to use the Whitney trick in the middle level $\partial_+ W^*_2$ to separate the A-spheres and the B-spheres. In the preceeding step of the proof we saw that for each A-sphere there is a B-sphere such that the two spheres have algebraic intersection number 1 and all other algebraic intersection numbers between A- and B-spheres are zero. Furthermore, any excess intersection points can be paired off so that each pair has a Whitney loop whose image in $X \times I$ is small. Each of these Whitney loops must bound a small singular Whitney disk. This imposes an additional requirement on the construction of the R_i. Each loop can be pushed vertically into a level in the thick part of an R_i, so the following condition will give what we need.

(AR3) If $\alpha : S^1 \to U_i$ is a map such that $f(\alpha(S^1))$ is small in X, then α extends to $\bar{\alpha} : B^2 \to U_{i-1}$ such that the diameter of $f(\bar{\alpha}(B^2))$ is small in X.

Remark. It is at this point in the proof that the hypothesis that Y is an ANR is crucial. In condition (AR3), "small" means small relative to the original ϵ in the statement of the Main Lemma. Since all the isotopies Φ^i_t move points approximately the same amount, it is not possible to make the A- and B-spheres that lie in R_i get progressively smaller as i increases. Instead their sizes are all controlled by the same ϵ which must be chosen and fixed at the beginning of the proof when the first region R_1 is constructed. Since Y is an ANR, it satisfies the hypotheses of Lemma 2.5. Hence that Lemma can be used to acheive (AR3).

We now want to use the Whitney trick to make the geometric intersections match the algebraic intersection numbers. In order to do that we must find controlled, framed, embedded Whitney disks for the excess intersection points of the A-spheres and the B-spheres. This part of the proof is exactly the same as the corresponding part of the usual proof of the Controlled h-cobordism Theorem which can be found on pages 110 and 111 of [FQ]. We have completed the portion of the proof corresponding to the first two paragraphs starting in the middle of page 110. The remainder of the proof consists of four parts: First we must construct small immersed transverse (unframed) spheres for the A-spheres and the B-spheres separately. (See the last full sentence on the bottom of page 110.) Second, as noted on the bottom of page 110 and the top of page 111, the unframed transverse spheres can be used to construct small framed transverse spheres and then immersed Whitney disks for the extra points of intersection. (The details of the uncontrolled version of this argument are given on pages 104–106 of [FQ].) Third, the embedded disks must be constructed by an application of the Controlled Disk Embedding Theorem [FQ, Theorem 5.4]. Finally, the Whitney trick is used to remove all the excess intersection points.

After the excess intersection points have been removed the A-spheres and the B-spheres will intersect in pairs but will have no other points of intersection in $\partial_+ W_2^*$. This means that the 2-handles and the 3-handles in the handle decomposition will cancel in pairs. Hence we can absorb all the handles into the collars and arrive at the desired controlled product structure on $M \times I$.

Thus the proof of the Main Lemma will be complete once we verify two things: the A-spheres and the B-spheres separately have controlled transverse spheres, and the hypotheses of the controlled disk embedding theorem are satisfied.

The existence of transverse spheres. Let a be one of the A-spheres. Then a is the attaching sphere of a 3-handle $H_*^3 \in \mathcal{H}^*$. Dually, we can view H_*^3 as a 2-handle $H^2 \in \mathcal{H}$. Let D be a 2-disk in ∂H^2 parallel to the core of H^2. Then D intersects a in exactly one point and $\partial D \subset \partial C_0$. Form a larger disk D' by adding to D the product annulus that ∂D spans in C_0. Then D' still intersects a in exactly one point and $\partial D' \subset (U_{i-k-1} - U_{i+2k}) \times \{0\}$ for some i. Form a transverse sphere for a by taking the union of D' and a singular disk in $(U_{i-k-2} - U_{i+2k+3}) \times \{0\}$ spanned by $\partial D'$. In order to make this transverse sphere a subset of $\partial_+ W_2^*$, we must push it into $\partial C_0 - M \times \{0\}$ and then out of the attaching regions of the 2-handles. Note that these operations force us to increase the size of k. Specifically, the new k is $7 \cdot (\text{old}\, k) + 3$.

In a similar way we can construct a transverse sphere for each of the B-spheres. If b is a B-sphere, then b is the belt sphere of a 2-handle $H_*^2 \in \mathcal{H}^*$. Thus a disk in ∂H_*^2 parallel to the core of H_*^2 intersects b in exactly one point and has its boundary in ∂C_1. We can add an annulus in C_1 to form a disk whose boundary is a loop in $(V_{i-k} - V_{i+2k+1}) \times \{1\}$. This loop bounds a singular disk in V_{i-k-1} by the argument given above under "diagonalizing the boundary homomorphism" (where it was shown that ∂^{-1} does not increase the I coordinate much). The fact that V_{i+2k+1} has a 1-dimensional spine allows us to use a general position adjustment to make the disk disjoint from V_{i+2k+1}. The union of the two disks is the transverse sphere we need.

In order to control the sizes of these transverse spheres for the A-spheres we need the R_i to satisfy the following additional requirement.

Additional requirement on the construction of regions. In order for the transverse spheres constructed above to be small in $X \times$I, we must impose one additional requirement on the construction of the regions R_i.

(AR4) If $\alpha : S^1 \to U_{i-k-1} - U_{i+2k+2}$ is a map such that α extends to $\alpha' : B^2 \to U_{i-k-1}$ with the diameter of $\alpha'(B^2)$ small, then α extends to $\bar{\alpha} : B^2 \to U_{i-k-2} - U_{i+2k+3}$ such that the diameter of $f(\bar{\alpha}(B^2))$ is small in X,

The existence of such an extension $\bar{\alpha}$ follows from Lemma 2.4.

The hypotheses of the Controlled Disk Embedding Theorem are satisfied. There are two hypotheses: the control map must have a kind of $(\delta, 1)$-connectedness property and the immersed Whitney disks must have δ-algebraically transverse spheres with δ-algebraically trivial intersections. The fact that the immersed Whitney disks satisfy the algebraic hypothesis is automatic in our situation. The uncontrolled proof of this is found on page 105 of [FQ]. As is noted on page 111 of [FQ], this construction is really a controlled construction. Thus the immersed Whitney disks have δ-algebraically transverse spheres with δ-algebraically trivial intersections.

The control map $c : M \times I \to X \times I$ fails to be $(\delta, 1)$-connected over $X \times I$, since it is not surjective. However, for any loop α in $M \times I$ whose image under c is small, α bounds a singular disk D whose image under c is also small. This follows from (AR3). This property is close enough to $(\delta, 1)$-connectedness to allow the proof of the Controlled Disk Embedding Theorem in [FQ] to go through. □

4. Proofs of the shrinking and taming theorems

Proof of the 1-LCC Shrinking Theorem for ANR's. Apply Theorem 3.1 recursively to obtain a sequence $\{g_i\}$ of cell-like, surjective mappings $M \to X$ as well as sequences $\{P_i\}$ of 1-dimensional polyhedra in M and $\{V_i\}$ of regular neighborhoods such that V_i is a $(1/i)$-regular neighborhood of P_i and $N(g_i) \subset g_i^{-1}(Y) \subset V_i$. Impose controls to insure $\{g_i\}$ converges to a cell-like map g which is within ϵ of f and is 1-1 over $X - Y$, with motion at later stages restricted so severely that $N(g) \subset g^{-1}(Y) \subset V_i$ for each i. Then $g^{-1}(Y)$ has embedding dimension 1, by definition. Edwards's 1-demensional Shrinking Theorem [E2] (cf. [D, Theorem 23.2]) implies that g (and, therefore, f) can be approximated within preassigned ϵ by a homemorphism $M \to X$, as required. □

Proof of Corollary 1. The proof of the corollary is the same as the proof of the 1-LCC Shrinking Theorem for ANR's. The only point in the proof at which the ANR hypothesis was needed was in the application of the Controlled Disk Embedding Theorem. The fact that Y is 0-dimensional is sufficient to achieve Additional Requirement (AR3), so the proof can be completed. □

Proof of Corollary 2. Application of the 1-LCC Shrinking Theorem for ANR's over $X - C$ yields that $X - C$ is a 4-manifold. Due to the hereditary nature of the 1-LCC condition, C itself is 1-LCC in X. An application of Corollary 1 gives the desired result. □

Proof of the 1-LCC Taming Theorem for ANR's. It suffices to check that each Y_i has embedding dimension 1 [E1, Proposition 1.1(4)]. Fix $\epsilon > 0$ and apply Theorem 3.1 to $id : M \to M$, with $\epsilon/2$ and $Y = Y_i$, to obtain a cell-like mapping $g : M \to M$ and a close regular neighborhood V of a 1-dimensional polyhedron P with $\operatorname{Int} V$ containing $g^{-1}(Y_i)$. By Lemma 2.1 we may assume that g is a (small) homeomorphism. Thus we see that $g(V)$ is a small regular neighborhood of $g(P)$ with $\operatorname{Int} g(V) \supset Y_i$. $\square$

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University of Utah, Salt Lake City, Utah 84112
E-mail address: bestvina@math.utah.edu

University of Tennessee, Knoxville, Tennessee 37996-1300
E-mail address: daverman@novell.math.utk.edu

Calvin College, Grand Rapids, Michigan 49546
Current address: National Science Foundation, Arlington, VA 22230
E-mail address: venema@calvin.edu

Cleveland State University, Cleveland, Ohio 44115-2440
E-mail address: walsh@math.csuohio.edu