

CHAPTER 9

The Theorem of Menelaus

9.1 DUALITY

9.2 THE THEOREM OF MENELAUS

The theorem we will study in this chapter is ancient, dating from about the year AD 100. It was originally discovered by Menelaus of Alexandria (70–130), but it did not become well known until it was rediscovered by Ceva in the seventeenth century. The theorem is powerful and has many interesting applications, some of which will be explored in later chapters.

9.1 DUALITY

In the last chapter we studied the problem of determining when three lines through the vertices of a triangle are concurrent; in this chapter we study the problem of determining when three points on the sidelines of a triangle are collinear. The relationship between these two problems is an example of *duality*. Before studying the main result of the chapter we will pause in this section to consider the principle of duality because discerning larger patterns such as duality can lead to a much deeper understanding of the theorems of geometry than just studying each geometric result in isolation.

Roughly speaking, the principle of duality asserts that any true statement in geometry should remain true when the words point and line are interchanged. Just as two points lie on exactly one line, so two lines intersect in exactly one point; just as three points may be collinear, so three lines may be concurrent. The use of the term *incident* makes these statements sound even more symmetric. For example, two distinct points are incident with exactly one line and two distinct lines are (usually) incident with exactly one point.

Passing to the extended Euclidean plane allows us to eliminate the exceptions to these rules, so the extended plane is the natural setting for duality. Specifically, the statement that two distinct lines are incident with exactly one point is true without exception in the extended plane. This makes it completely dual to the statement that two distinct points are incident with exactly one line (Euclid's first postulate). Girard Desargues (1591–1661) was the first to make systematic use of ideal points and his work lead eventually to the development of a whole new branch of geometry known as *projective geometry*. A later chapter will include a study of the theorem of Desargues, which is a beautiful result that relates two dual properties of triangles.

Desargues's theorem is foundational in projective geometry, but we will not pursue that branch of geometry any further.

The triangle itself is self dual. It can be thought of as determined by three noncollinear points (the vertices) or by three nonconcurrent lines (the sidelines). When this latter view of the triangle is being emphasized, the triangle is referred to by its dual name *trilateral*. Vertices and sidelines (*not* sides) are dual aspects of the triangle, so they should be interchanged when theorems regarding triangles are being dualized. This is the reason it is natural to use the sidelines of a triangle, rather than its sides, in so many of the theorems we have encountered in this course.

The theorems of Ceva and Menelaus will be our primary examples of dual theorems. The theorem of Ceva gives a criterion that tells us when three lines through the vertices of a triangle are concurrent; the theorem of Menelaus tells us that essentially the same criterion can be used to determine when three points on the sidelines of a triangle are collinear. Apparently what happened historically is that Ceva rediscovered the theorem of Menelaus and then discovered his own theorem by applying the principle of duality.

The principle of duality was first formalized by Charles Julien Brianchon (1785–1864) when he applied it to a theorem of Blaise Pascal (1623–1663). The theorems of Pascal and Brianchon serve as a beautiful illustration of a pair of dual theorems. The theorem of Pascal asserts that if a hexagon is inscribed in a circle, then the three points determined by pairs of opposite sidelines are collinear. Brianchon's theorem asserts that if a hexagon is circumscribed about a circle, then the lines determined by pairs of opposite vertices are concurrent. We will study those theorems later in the chapter.

9.2 THE THEOREM OF MENELAUS

In order to simplify the statements in this section, let us make a definition.

Definition. Let $\triangle ABC$ be a triangle. Three points L , M , and N such that L lies on \overleftrightarrow{BC} , M lies on \overleftrightarrow{AC} , and N lies on \overleftrightarrow{AB} are called *Menelaus points* for the triangle. We say that a Menelaus point is *proper* if it is not equal to any of the vertices of the triangle.

It will be assumed that the vertices of the triangle are ordinary points, but one or more of the Menelaus points may be ideal points. Let us begin with some GSP exploration of when three Menelaus points are collinear.

EXERCISES

- *9.2.1.** Construct a triangle $\triangle ABC$ and proper Menelaus points L , M , and N for $\triangle ABC$. Calculate the quantity

$$d = \frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA}.$$

Move the vertices of the triangle and the Menelaus points to verify that if the Menelaus points are collinear, then $d = 1$.

- *9.2.2.** Verify that $d = 1$ if L is an ideal point and M and N lie on a line parallel to \overleftrightarrow{BC} .

- *9.2.3. Find an example of a triangle and Menelaus points such that $d = 1$ even though the Menelaus points are not collinear.
- *9.2.4. Construct a triangle $\triangle ABC$ and a line ℓ that does not pass through any of the vertices of the triangle. Determine the answer to the following question: *What are the possible numbers of points of intersection of ℓ with the triangle?*

The exercises above indicate that if the Menelaus points are collinear, then $d = 1$. But they also show that the converse does not hold. Again the number of Menelaus points that lie on the triangle itself is important, so we must use

$$s = \frac{\mathbf{AN}}{\mathbf{NB}} \cdot \frac{\mathbf{BL}}{\mathbf{LC}} \cdot \frac{\mathbf{CM}}{\mathbf{MA}},$$

the product of the sensed ratios, rather than d . All of this is probably expected since it is exactly analogous to what happened in the case of Ceva's theorem.

One difference between this situation and that of Ceva's theorem is that the number of Menelaus points that are expected to lie on the triangle is even, so the number of factors in s that are negative is even rather than odd. This is because any line in the plane that does not contain any of the vertices of the triangle $\triangle ABC$ will either miss the triangle entirely or will intersect it in exactly two points. You should have convinced yourself of that fact when you did your GSP exploration in Exercise 9.2.4. Pasch's Axiom, which was assumed in Chapter 0 makes this precise. We can conclude that if the three Menelaus points are all proper, then either zero or two of them lie on the triangle. In that case either three or one of the sensed ratios in s will be negative, so s itself is negative.

We can now state Menelaus's theorem in full generality. Again we assume that the vertices of the triangle are ordinary points in the Euclidean plane, but we allow the possibility that one or more of the Menelaus points is ideal.

Theorem of Menelaus. *Let $\triangle ABC$ be an ordinary triangle. The Menelaus points L , M , and N for $\triangle ABC$ are collinear if and only if*

$$\frac{\mathbf{AN}}{\mathbf{NB}} \cdot \frac{\mathbf{BL}}{\mathbf{LC}} \cdot \frac{\mathbf{CM}}{\mathbf{MA}} = -1.$$

The next few exercises outline a proof of Menelaus's theorem. It would be possible to prove the theorem as a corollary of Ceva's theorem, but that proof is no simpler than a proof based on similar triangles. Since it is no more difficult, we give a proof based only on elementary Euclidean geometry and do not use Ceva's theorem.

EXERCISES

- 9.2.5. Prove the theorem in case one or more of the Menelaus points is improper. Specifically, prove each of the following statements and then explain why this suffices to prove the theorem in every case in which at least one of the Menelaus points is improper.
- (a) If $L = C$ and the three Menelaus points are collinear, then either $N = A$ or $M = C$ and (in either case) $s = -1$.

(b) If $L = C$ and $s = -1$, then either $N = A$ or $M = C$.

9.2.6. Assume that all three Menelaus points are proper and ordinary. Prove that if the three Menelaus points are collinear, then $s = -1$.

[Hint: Assume, first, that L and M lie on the triangle and N does not (see Figure 9.1). Drop perpendiculars from A , B , and C to the line \overleftrightarrow{LM} and call the feet R , S , and T , respectively. Let $r = AR$, $s = BS$, and $t = CT$. Use similar triangles to express each of the sensed ratios AN/NB , BL/LC , and CM/MA in terms of r , s , and t and then use algebra to derive the Menelaus formula. Use GSP to determine what the other possible figures look like and modify the argument to fit them as well.]

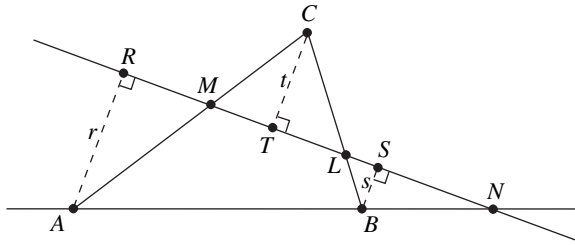


FIGURE 9.1: Proof of Menelaus's theorem in case two points lie on the triangle

9.2.7. Assume that all three Menelaus points are proper and that exactly one of them is ideal. Prove that $s = -1$ if the three Menelaus points are collinear.

9.2.8. Prove that it is impossible for the Menelaus points to be collinear if exactly two of them are ideal points. Prove that $s = -1$ in case all three Menelaus points are ideal. [In that case they are collinear because they all lie on the line at infinity.]

9.2.9. Prove the converse of Menelaus's theorem; i.e., prove that if $s = -1$, then the three Menelaus points are collinear.

There is also a trigonometric form of Menelaus's theorem. The proof is much like the proof of the trigonometric form of Ceva's theorem.

Trigonometric Form of the Theorem of Menelaus. Let $\triangle ABC$ be an ordinary triangle. The Menelaus points L , M , and N for $\triangle ABC$ are collinear if and only if

$$\frac{\sin(\angle BAL)}{\sin(\angle LAC)} \cdot \frac{\sin(\angle CBM)}{\sin(\angle MBA)} \cdot \frac{\sin(\angle ACN)}{\sin(\angle NCB)} = -1.$$

EXERCISES

9.2.10. Prove the trigonometric form of the theorem of Menelaus.