## C H A P T E R 8

# Ceva's Theorem

8.1 EXPLORING CEVA'S THEOREM
8.2 SENSED RATIOS AND IDEAL POINTS
8.3 THE STANDARD FORM OF CEVA'S THEOREM
8.4 THE TRIGONOMETRIC FORM OF CEVA'S THEOREM
8.5 THE CONCURRENCE THEOREMS
8.6 ISOTOMIC AND ISOGONAL CONJUGATES AND THE SYMMEDIAN POINT

In this chapter we finally complete the proofs of the various concurrency theorems that were investigated in earlier chapters. We will prove a theorem due to Giovanni Ceva (1647–1734) that gives a necessary and sufficient condition for three lines through the vertices of a triangle to be concurrent and then we will derive all of the concurrency theorems as corollaries of that one result. We could have given *ad hoc* proofs of each of the concurrency results separately, but it seems better to expose the unifying principle behind the various results. One nice aspect of doing things this way is that the proof of Ceva's general theorem is not much more difficult than the proof of any one of the special concurrency results.

## 8.1 EXPLORING CEVA'S THEOREM

All of the concurrency theorems studied so far have a common element. In each one there is a triangle and there is one line that passes through each of the vertices of the triangle. They are all special cases of the following general problem.

**Concurrency Problem.** *Let $\triangle ABC$ be a triangle and let $\ell$, $m$, and $n$ be three lines such that $A$ lies on $\ell$, $B$ lies on $m$, and $C$ lies on $n$. Find necessary and sufficient conditions under which $\ell$, $m$, and $n$ are concurrent.*

The lines through the vertices will either be parallel to their opposite sidelines or they will intersect the opposite sidelines. We begin by looking at the two possibilities separately. Those lines that intersect the opposite sideline are more common and have a special name.

**Definition.** Let $\triangle ABC$ be a triangle and let $L$, $M$, and $N$ be points on the sidelines $\overleftrightarrow{BC}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{AB}$, respectively. The lines $\ell = \overleftrightarrow{AL}$, $m = \overleftrightarrow{BM}$, and $n = \overleftrightarrow{CN}$ are called *Cevian lines* (or simply *cevians*) for the triangle $\triangle ABC$. A Cevian line is *proper* if it passes through exactly one vertex of the triangle $\triangle ABC$.

A Cevian line is specified by naming the vertex it passes through along with the point at which it intersects the opposite sideline. Thus the assertion that $\overleftrightarrow{AL}$ is a Cevian line for $\triangle ABC$ is understood to mean that $L$ is a point on $\overleftrightarrow{BC}$.

Exercise 5.3.3 provides a hint about what happens when three proper Cevian lines are concurrent. As you saw there, the following quantity is important:

$$d = \frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA}.$$

This quantity is defined and is positive provided all three Cevian lines are proper.

In the following exercises you will do some GSP exploration that will help to clarify the statement of the theorem we seek. It is assumed in all of the exercises that $\triangle ABC$ and $\ell$, $m$, and $n$ are as in the statement of the Concurrency Problem.

## EXERCISES

***8.1.1.** Construct three noncollinear points $A$, $B$, and $C$, the three sidelines of $\triangle ABC$, and movable points $L$, $M$, and $N$ on the sidelines $\overleftrightarrow{BC}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{AB}$, respectively. Explore various possible positions for the vertices of the triangle and the points $L$, $M$, and $N$ to verify the following result: *If three proper Cevian lines are concurrent, then* $d = 1$. Find examples of concurrent Cevian lines in which all three of the points $L$, $M$, and $N$ lie on the triangle $\triangle ABC$ and other examples in which at least one of the three points is not on $\triangle ABC$. Verify that $d = 1$ in all cases in which the proper Cevians are concurrent.

***8.1.2.** Find an example in which $d = 1$ but the three Cevian lines are not concurrent.

***8.1.3.** If the three Cevians are concurrent, how many of the points $L$, $M$, and $N$ must lie on the triangle $\triangle ABC$? In your example for the preceding exercise, how many of the points were on the triangle?

***8.1.4.** Find points $L$, $M$, and $N$ such that the three Cevian lines are mutually parallel. (Use the Parallel Lines command under the Construct menu to make sure the lines are exactly parallel, not just approximately parallel.) What is $d$ equal to in this case? How many of the points $L$, $M$, and $N$ lie on the triangle $\triangle ABC$?

***8.1.5.** Let $\ell$ be the line through $A$ that is parallel to the sideline $\overleftrightarrow{BC}$. Find points $M$ and $N$ so that $\ell$ is concurrent with the Cevian lines $\overleftrightarrow{BM}$ and $\overleftrightarrow{CN}$. When the three lines are concurrent, how many of the points $M$ and $N$ lie on the triangle? There is no point $L$ in this case, but it is still possible to compute the other two factors in $d$. Calculate

$$d' = \frac{AN}{NB} \cdot \frac{CM}{MA}$$

when the three lines are concurrent.

***8.1.6.** Now let $\ell$ be the line through $A$ that is parallel to the sideline $\overleftrightarrow{BC}$ and let $m$ be the line through $B$ that is parallel to the sideline $\overleftrightarrow{AC}$. Is there a Cevian line $n = \overleftrightarrow{CN}$ such that the three lines $\ell$, $m$, and $n$ are concurrent? If so, calculate $AN/NB$. Is $N$ on the triangle?

***8.1.7.** Is it possible for all three of the lines $\ell$, $m$, and $n$ to be parallel to the opposite sidelines and also concurrent?

***8.1.8.** Suppose $\ell = \overleftrightarrow{AB}$. What must be true of $m$ and $n$ if the three lines are concurrent?

In the exercises above you probably discovered that $d = 1$ if three proper Cevian lines are concurrent or if they are all mutually parallel. But the converse does not hold: it is possible that $d = 1$ even if the lines are not concurrent or mutually parallel. It appears that the converse depends on how many of the points $L$, $M$, and $N$ lie on the triangle $\triangle ABC$. You also saw that there are various exceptional cases in which the lines $\ell$, $m$, and $n$ are not Cevian lines but are still concurrent. It would appear that the statement of a theorem that covers all the possibilities and all the special cases we have encountered would have to be quite complicated. The next order of business is to extend the Euclidean plane and distance measurements in that plane in a way that allows us to give one succinct statement that summarizes all of these observations.

## 8.2 SENSED RATIOS AND IDEAL POINTS

This section contains several new ideas and definitions that will allow us to give a concise statement of the most general form of Ceva's Theorem.

The first issue we address is how to include in $d$ information about how many of the points $L$, $M$, and $N$ are on the triangle and how many are not. We will accomplish that by adding a sign to each of the three ratios in the formula for $d$. Start with three distinct collinear points $P$, $Q$, and $R$. Since both distances involved are positive, the ratio $PQ/QR$ is always positive. We will introduce a sensed ratio, which is sometimes positive and sometimes negative.

**Definition.** Assume $P$, $Q$, and $R$ are three distinct collinear points. Define the *sensed ratio* $\mathbf{PQ}/\mathbf{QR}$ by

$$\frac{\mathbf{PQ}}{\mathbf{QR}} = \begin{cases} \dfrac{PQ}{QR} & \text{if } Q \text{ is between } P \text{ and } R\text{, and} \\[2ex] -\dfrac{PQ}{QR} & \text{if } Q \text{ is not between } P \text{ and } R. \end{cases}$$

Note that boldface is used to denote the sensed ratio in order to distinguish it from the unsensed ratio.

Given a line it is possible to assign a direction to it. Once a direction is assigned, it makes sense to speak of *directed distances*. The directed distance **AB** between two points $A$ and $B$ on the line is considered to be positive if the direction from $A$ to $B$ agrees with the direction assigned to the line and is negative otherwise. Directed distances such as this are not well defined because there are two ways to assign a direction to a given line. But reversing the direction of the line changes the sign of all the directed distances between points on the line. Thus the ratio of two directed distances is well defined in the sense that the same answer is obtained regardless of which direction is assigned to the line. The sensed ratio in the previous paragraph should be thought of as the quotient of two such directed distances.

When we state Ceva's Theorem, we will replace the positive quantity $d$ (defined in the last section) by a real number $s$, where $s$ is the product of the sensed ratios.

Specifically, we will define

$$s = \frac{\mathbf{AN}}{\mathbf{NB}} \cdot \frac{\mathbf{BL}}{\mathbf{LC}} \cdot \frac{\mathbf{CM}}{\mathbf{MA}}.$$

Notice that $s = d$ in case all three of the points $L$, $M$, and $N$ lie on the triangle $\triangle ABC$ and $s = d$ in case exactly one of the points lies on the triangle. If either two of the points or none of the points $L$, $M$, and $N$ lie on the triangle $\triangle ABC$, then $s$ is negative and $s = -d$. Using $s$ in the statement of the theorem will allow us to specify the numerical value of $d$ and impose restrictions on the number of points that lie on the triangle at the same time.

Two other reasons Ceva's Theorem is complicated to state in complete generality are the fact that mutually parallel lines behave like concurrent lines and the fact that a line through a vertex might be parallel to the opposite sideline and hence not a Cevian line. The correct setting for Ceva's Theorem is actually the extended Euclidean plane, in which any two lines intersect. We will define the extended Euclidean plane to consist of all points in the Euclidean plane together with some additional ideal points, called "points at infinity." These ideal points are added to the plane in such a way that two parallel lines intersect at one of these ideal points.

**Definition.** The *extended Euclidean plane* consists of all the points in the ordinary Euclidean plane together with one additional point for each collection of mutually parallel lines in the plane. The points in the Euclidean plane will be called *ordinary points* and the new points that are added are called *ideal points* or *points at infinity*. The set of all ideal points is called the *line at infinity*.

Intuitively what we have done is we have added one point to each line in the plane. This point is "at infinity" in the sense that it is infinitely far away from any of the ordinary points. We can think of it as being a point out at the end of the line, although there is only one ideal point on each line so we approach the same ideal point on a line by traveling towards either of the two ends of the line. We will not attempt to picture the ideal points in our diagrams since they are infinitely far away and thus beyond what we can see. Their role is to complete the ordinary plane in such a way that it is not necessary to mention exceptional special cases in the statements of theorems. For example, every pair of distinct lines in the extended Euclidean plane intersect in exactly one point (with no exceptions). If the two lines are parallel in the ordinary plane, they share an ideal point and that is their point of intersection. If the two lines are not parallel in the ordinary plane, then they have different ideal points but intersect at an ordinary point.

Here is a summary of the important properties of the ideal points.

- Each ordinary line contains exactly one ideal point.
- Parallel ordinary lines share a common ideal point.
- Ordinary lines that are distinct and nonparallel have distinct ideal points.

In the extended plane every line through a vertex of an ordinary triangle is a Cevian line. To see this, consider an ordinary triangle $\triangle ABC$ and a line $\ell$ that passes through vertex $A$. Either $\ell$ intersects the sideline $\overleftrightarrow{BC}$ at a point $L$ or $\ell \parallel \overleftrightarrow{BC}$. In the first case $\ell$ is obviously a Cevian line, but in the second case $\ell$ is also a Cevian line with $L$ equal to the ideal point on $\overleftrightarrow{BC}$.

Now that we have added additional points to the plane, we must extend our definition of sensed ratio to include them. In order to understand the next definition, think of two fixed points $A$ and $B$ and a movable point $C$ on the line $\overleftrightarrow{AB}$. As $C$ approaches either end of the line, $C$ is not between $A$ and $B$ so the sensed ratio $\mathbf{AC}/\mathbf{CB}$ is negative. Furthermore, the distances $AC$ and $CB$ both approach infinity. But the difference between $AC$ and $CB$ is constant, so they approach infinity at the same rate and their ratio has a limit of 1. In other words, if $I$ is the ideal point on $\overleftrightarrow{AB}$, then

$$\lim_{C \to I} \frac{\mathbf{AC}}{\mathbf{CB}} = -1.$$

This reasoning justifies the following definition.

**Definition.** Let $A$ and $B$ be distinct ordinary points and let $I$ be the ideal point on $\overleftrightarrow{AB}$. Define

$$\frac{\mathbf{AI}}{\mathbf{IB}} = -1.$$

Finally, we must extend ordinary arithmetic of fractions to cover the possibility that a Cevian line might not be proper.[1] We will adopt the convention that $a/b = c/d$ provided $ad = bc$. Thus it is possible for $b$ to be zero in the fraction $a/b$, but if $b = 0$ and $a/b = c/d$, then either $a = 0$ or $d = 0$. In particular, if we write

$$\frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA} = 1,$$

this does not necessarily imply that $LC \neq 0$. Instead it means that if $LC = 0$, then at least one of $AN$, $BL$, or $CM$ must also be zero.

## EXERCISES

***8.2.1.** Construct two lines $\ell$ and $m$. Mark the point of intersection of the two lines. Now rotate one line and keep the other fixed. Watch what happens to the point of intersection as the lines pass the position at which they are parallel. Explain how what you observe justifies the choice of just one ideal point on each line even though the line has two ends.
[Hint: There is a special version of the selection tool that can be used to rotate objects.]

***8.2.2.** Construct a line and points $A$, $B$, and $X$ on the line. Calculate the ratio $AX/XB$. (GSP will only calculate an unsensed ratio so you will have to supply the sign.)

---

[1] This can happen, for example, in the altitude concurrence theorem.

For which points $X$ on $\overleftrightarrow{AB}$ is this ratio defined? Watch what happens to the ratio as $X$ moves along the line. Use your observations to draw a graph of the function

$$f(X) = \frac{\mathbf{AX}}{\mathbf{XB}}.$$

[Hint: $f$ is a function whose domain consists of points on $\overleftrightarrow{AB}$ and whose range consists of real numbers, so your graph should have the line $\overleftrightarrow{AB}$ as its horizontal axis and the real line as the vertical axis.]

**8.2.3.** Let $A$ and $B$ be two distinct points. Prove that for each real number $r$ there is exactly one point on the extended line $\overleftrightarrow{AB}$ such that $\mathbf{AX}/\mathbf{XB} = r$.

**8.2.4.** Draw an example of a triangle in the extended Euclidean plane that has one ideal vertex. Is there a triangle in the extended plane that has two ideal vertices? Could there be a triangle with three ideal vertices?

## 8.3 THE STANDARD FORM OF CEVA'S THEOREM

We are finally ready for the statement of Ceva's theorem.

**Ceva's Theorem.** *Let $\triangle ABC$ be an ordinary triangle. The Cevian lines $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are concurrent if and only if*

$$\frac{\mathbf{AN}}{\mathbf{NB}} \cdot \frac{\mathbf{BL}}{\mathbf{LC}} \cdot \frac{\mathbf{CM}}{\mathbf{MA}} = +1.$$

The statement of the theorem assumes that the triangle is an ordinary triangle, so $A$, $B$, and $C$ are all ordinary points. There is no assumption that the points $L$, $M$, and $N$ are ordinary; nor is there there an assumption that the point of concurrency is ordinary. In addition, $L$, $M$, and $N$ may be equal to vertices of the triangle. Thus the theorem covers all possible lines through the vertices of the triangle and gives a complete solution to the Concurrency Problem stated at the beginning of the chapter.

Because we have stated Ceva's theorem in its ultimate generality, a complete proof will involve looking at a number of special cases and this might make the proof appear to be complicated. In order to make certain that the simplicity of the basic theorem is not lost, we will begin with a proof of the special case in which the point of concurrency is inside the triangle; this case covers most of the applications. It is the special case in which all three of the points $L$, $M$, and $N$ are in the interior of the segments that form the sides of $\triangle ABC$.[2]

---

[2] It follows from the crossbar theorem and related foundational results that the point of concurrency of the lines $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ is inside the triangle $\triangle ABC$ if and only if all three of the points $L$, $M$, and $N$ are in the interiors of the segments that form the sides of $\triangle ABC$. That fact will be assumed in this section.

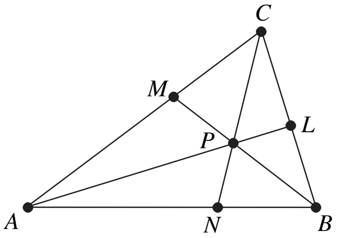

**FIGURE 8.1:** A special case of Ceva's theorem

## EXERCISES

**8.3.1.** Let $\triangle ABC$ be a triangle and let $L$, $M$, and $N$ be points in the interiors of the sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Prove that if the three Cevian lines $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are concurrent, then

$$\frac{\mathbf{AN}}{\mathbf{NB}} \cdot \frac{\mathbf{BL}}{\mathbf{LC}} \cdot \frac{\mathbf{CM}}{\mathbf{MA}} = 1.$$

[Hint: Let $k$ be the line through $A$ that is parallel to $\overleftrightarrow{BC}$, let $R$ be the point at which $\overleftrightarrow{BM}$ intersects $k$, and let $S$ be the point at which $\overleftrightarrow{CN}$ intersects $k$. Use similar triangles to show that $AN/NB = SA/CB$, $CM/MA = CB/AR$, and $BL/LC = AR/SA$. The result follows by algebra.]

**8.3.2.** Let $\triangle ABC$ be a triangle and let $L$, $M$, and $N$ be points in the interiors of the sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Prove that if

$$\frac{\mathbf{AN}}{\mathbf{NB}} \cdot \frac{\mathbf{BL}}{\mathbf{LC}} \cdot \frac{\mathbf{CM}}{\mathbf{MA}} = 1,$$

then the three Cevian lines $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are concurrent.
[Hint: By the Crossbar Theorem there is a point $P$ at which $\overline{AL}$ intersects $\overline{BM}$. Choose $N'$ to be the point at which $\overrightarrow{CP}$ intersects $\overline{AB}$. Note that $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN'}$ are concurrent. Use the part of the theorem you have already proved and Exercise 8.2.3 to prove that $N = N'$.]

Now we proceed to prove Ceva's Theorem in general. We will begin by proving that if the three lines are concurrent, then $s = 1$. Once that has been established, the proof used in the last exercise (above) can easily be modified to prove the converse.

Assume in the following exercises that $\triangle ABC$ is triangle, the lines $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are Cevian lines for $\triangle ABC$, and

$$s = \frac{\mathbf{AN}}{\mathbf{NB}} \cdot \frac{\mathbf{BL}}{\mathbf{LC}} \cdot \frac{\mathbf{CM}}{\mathbf{MA}}.$$

It is also assumed that $A$, $B$, and $C$ are ordinary points in the Euclidean plane, but $L$, $M$, and $N$ are points in the extended plane. Thus one or more of the points $L$,

$M$, and $N$ may be ideal, in which case the corresponding factors in $s$ are $-1$. It is also possible that one or more of $L$, $M$, and $N$ may be a vertex of $\triangle ABC$. In that case there might be a 0 in the denominator of $s$. When that happens the factors of $s$ should not be considered separately, but the entire numerator and the entire denominator should be considered. If one of the factors in the denominator of $s$ is 0, then $s = -1$ simply means that there is also a factor in the numerator that is equal to 0.

## EXERCISES

**8.3.3.** Prove that if $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are proper Cevian lines that are concurrent at an ordinary point $P$, then either all three of the points $L$, $M$, and $N$ lie on $\triangle ABC$ or exactly one of them does.
[Hint: The three sidelines divide the exterior of $\triangle ABC$ into six regions. Consider the possibility that $P$ lies in each of them separately. Don't forget that one or more of the points $L$, $M$, and $N$ might be ideal.]

**8.3.4.** Prove that if $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are proper Cevian lines that are concurrent at an ideal point, then exactly one of the three of the points $L$, $M$, and $N$ lies on $\triangle ABC$.

**8.3.5.** Prove that if $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are proper Cevian lines that are concurrent at an ordinary point $P$ and all three of $L$, $M$, and $N$ are ordinary, then $s = 1$.
[Hint: The case in which all three of the points $L$, $M$, and $N$ lie on $\triangle ABC$ was covered in Exercise 8.3.1, so you may assume that only $M$ lies on $\triangle ABC$. Proceed as in the proof of Exercise 8.3.1; there are two possible diagrams, both different from that in Exercise 8.3.1, but you should still be able to find the similar triangles you need.]

**8.3.6.** Prove that if $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are proper Cevian lines that are concurrent at an ideal point and all three of $L$, $M$, and $N$ are ordinary, then $s = 1$.
[Hint: Concurrent at an ideal point means parallel. Essentially the same proof works again.]

**8.3.7.** Prove that if $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are proper Cevian lines that are concurrent at an ordinary point $P$ and $L$ is ideal but $M$ and $N$ are ordinary, then $s = 1$.
[Hint: Take another look at your sketch and your conclusions in Exercise 8.1.5.]

**8.3.8.** Prove that if $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are proper Cevian lines that are concurrent at an ordinary point $P$ and $L$ and $M$ are ideal but $N$ is ordinary, then $s = 1$.
[Hint: Take another look at your sketch and your conclusions in Exercise 8.1.6.]

**8.3.9.** Prove that at least one of $L$, $M$, and $N$ must be ordinary if the proper Cevian lines $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are concurrent.

**8.3.10.** Prove that if $L = B$ (so that $\overleftrightarrow{AL}$ is the sideline $\overleftrightarrow{AB}$ of the triangle) and the three Cevians $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are concurrent, then either $\overleftrightarrow{BM} = \overleftrightarrow{AB}$ or $\overleftrightarrow{CN} = \overleftrightarrow{BC}$. Prove that $s = 1$ in either case.

**8.3.11.** Prove that if $s = 1$ then the Cevian lines $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are concurrent.
[Hint: Proceed as in Exercise 8.3.2. Be sure to check that this proof works in every case.]

## 8.4 THE TRIGONOMETRIC FORM OF CEVA'S THEOREM

There is second form of Ceva's Theorem that is sometimes more convenient to apply than the standard form. This second version of the theorem expresses the concurrence criterion in terms of sines of angles rather than distances, so it will be preferred for applications in which it is simpler to measure the angles than the lengths.

As in the standard form, sense must to taken into account in the concurrence condition. For the trigonometric form this is simply a matter of using directed measures for the angles. The measure of an angle $\angle BAC$ is considered to be positive if it is measured in the counterclockwise direction from $\overrightarrow{AB}$ to $\overrightarrow{AC}$ and it is negative if the rotation from $\overrightarrow{AB}$ to $\overrightarrow{AC}$ is clockwise. (This is the same way angles are measured in calculus.) We will use boldface for the angle when it is to be considered a directed angle. Thus $\angle\mathbf{BAC}$ denotes the directed angle from $\overrightarrow{AB}$ to $\overrightarrow{AC}$.

**Trigonometric Form of Ceva's Theorem.** *Let $\triangle ABC$ be an ordinary triangle. The Cevian lines $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$ are concurrent if and only if*

$$\frac{\sin(\angle\mathbf{BAL})}{\sin(\angle\mathbf{LAC})} \cdot \frac{\sin(\angle\mathbf{CBM})}{\sin(\angle\mathbf{MBA})} \cdot \frac{\sin(\angle\mathbf{ACN})}{\sin(\angle\mathbf{NCB})} = +1.$$

The proof is based on the following simple lemma.

**Lemma.** *If $\triangle ABC$ is any ordinary triangle and $L$ is a point on $\overleftrightarrow{BC}$, then*

$$\frac{\mathbf{BL}}{\mathbf{LC}} = \frac{AB \cdot \sin(\angle\mathbf{BAL})}{AC \cdot \sin(\angle\mathbf{LAC})}.$$

***Proof.*** We will prove the theorem in case $L$ is between $B$ and $C$ (see Figure 8.2) and leave the general case as an exercise. Let $h$ denote the height of triangle $\triangle ABC$. Then

$$\frac{\mathbf{BL}}{\mathbf{LC}} = \frac{(1/2)\,h \cdot BL}{(1/2)\,h \cdot LC} = \frac{(1/2)\,AB \cdot AL \cdot \sin(\angle BAL)}{(1/2)\,AL \cdot AC \cdot \sin(\angle LAC)} = \frac{AB \cdot \sin(\angle\mathbf{BAL})}{AC \cdot \sin(\angle\mathbf{LAC})}.$$

The first equation is just algebra, the second equation is based on two applications of Exercise 0.11.1, and the third equation uses the fact that $L$ is between $B$ and $C$ so the two angles have the same direction. □

## EXERCISES

**8.4.1.** Modify the proof above to cover the remaining cases of the lemma.
[Hint: $L$ might equal $B$ or $C$, or $L$ might be outside $\overline{BC}$.]

**8.4.2.** Use the lemma to prove the trigonometric form of Ceva's theorem.

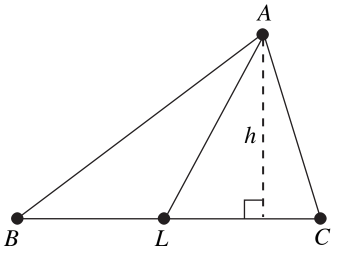

**FIGURE 8.2:** Proof of the lemma

## 8.5 THE CONCURRENCE THEOREMS

In this section we finally apply Ceva's Theorem to prove the concurrence results that we have encountered thus far in the course. Before getting to those proofs, however, we should make one observation about applying Ceva's theorem.

When we want to apply Ceva's theorem we will usually not find the triangle and the Cevian lines conveniently labeled with exactly the letters that were used in the statement of Ceva's Theorem. The pattern that you should see in the expression

$$\frac{\mathbf{AN}}{\mathbf{NB}} \cdot \frac{\mathbf{BL}}{\mathbf{LC}} \cdot \frac{\mathbf{CM}}{\mathbf{MA}}$$

is that the vertices of the triangle are listed in cyclic order around the triangle. The first fraction involves the two endpoints of the first side of the triangle and the point on the corresponding sideline. The second fraction involves the second side of the triangle, oriented so that the second side begins at the point where the first segment ended, and so on. When you apply the theorem you may begin at any vertex of the triangle and you may go around the triangle in either direction. But once you select a starting vertex and a direction around the triangle, you must follow the rest of the pattern.

## EXERCISES

**8.5.1. Existence of the centroid.** Prove that the medians of a triangle are concurrent.

**8.5.2. Existence of the orthocenter.** Prove that the altitudes of any triangle are concurrent.
[Hint: Use the angle sum theorem and the trigonometric form of Ceva's theorem. Be careful with the special case of a right triangle.]

**8.5.3. Existence of the circumcenter.** Prove that the perpendicular bisectors of the sides of any triangle are concurrent.
[Hint: Be sure to notice that the circumcenter breaks the pattern: it is not the point of concurrence of Cevian lines. Use Exercise 5.1.4.]

**8.5.4. Existence of the incenter.** Prove that the interior angle bisectors of any triangle are concurrent.

**8.5.5. Existence of the Gergonne point.** Let $L$, $M$, and $N$ be the points at which the incircle touches the sides of $\triangle ABC$. Prove that $\overline{AL}$, $\overline{BM}$, and $\overline{CN}$ are concurrent.
[Hint: Use the external tangents theorem.]

**8.5.6. Existence of excenters.** Prove that the bisector of an interior angle of a triangle and the bisectors of the remote exterior angles are concurrent.

**8.5.7. Existence of the Nagel point.** Let $\triangle ABC$ be a triangle, let $T_A$ be the point at which the $A$-excircle is tangent to $\overline{BC}$, let $T_B$ be the point at which the $B$-excircle is tangent to $\overline{AC}$, and let $T_C$ be the point at which the $C$-excircle is tangent to $\overline{AB}$. Prove that the segments $\overline{AT_A}$, $\overline{BT_B}$, and $\overline{CT_C}$ are concurrent.
[Hint: Use Exercise 4.5.2 to prove that the ratios in this problem are the reciprocals of the ratios in Exercise 8.5.5.]

## 8.6 ISOTOMIC AND ISOGONAL CONJUGATES AND THE SYMMEDIAN POINT

Ceva's Theorem allows us to define two very interesting transformations of the plane associated with a triangle. Both these transformations are called *conjugates* because applying them twice results in the identity transformation. This is the same sense in which the word "conjugate" is used in connection with complex numbers. In other contexts a transformation with this property would be called an *involution*.

Let $\triangle ABC$ be an ordinary triangle and let $P$ be a point in the extended plane. For simplicity we will assume that $P$ does not lie on any of the sidelines of the triangle. In that case $P$ is the point of concurrence of three proper Cevian lines $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$.

**Definition.** The *isotomic conjugate* of $P$ is the point of concurrence of the lines $\overleftrightarrow{AL'}$, $\overleftrightarrow{BM'}$, and $\overleftrightarrow{CN'}$, where $L'$ is the reflection of $L$ across the perpendicular bisector of $\overline{BC}$, $M'$ is the reflection of $M$ across the perpendicular bisector of $\overline{AC}$, and $N'$ is the reflection of $N$ across the perpendicular bisector of $\overline{AB}$.

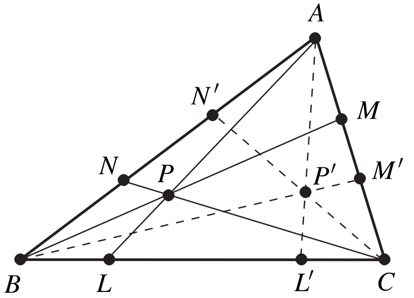

**FIGURE 8.3:** $P'$ is the isotomic conjugate of $P$

**Definition.** The *isogonal* of a Cevian line $\overleftrightarrow{AL}$ is the reflection of $\overleftrightarrow{AL}$ through the angle bisector of $\angle CAB$. Similarly the isogonal of a Cevian line $\overleftrightarrow{BM}$ is the reflection of $\overleftrightarrow{BM}$ through the bisector of $\angle ABC$ and the isogonal of a Cevian line $\overleftrightarrow{CN}$ is the reflection of $\overleftrightarrow{CN}$ through the bisector of $\angle BCA$. The *isogonal conjugate* of $P$ is the point of concurrence of the isogonals of $\overleftrightarrow{AL}$, $\overleftrightarrow{BM}$, and $\overleftrightarrow{CN}$.

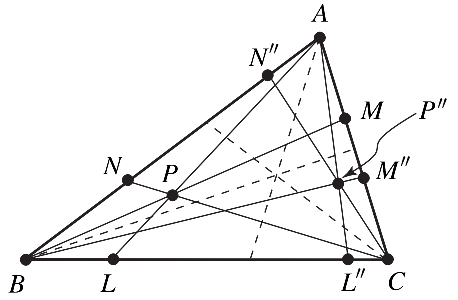

**FIGURE 8.4:** $P''$ is the isogonal conjugate of $P$

Both definitions presume that the new Cevian lines intersect; that is where Ceva's theorem comes in. The next two exercises show that the isotomic conjugate and the isogonal conjugate exist.

## EXERCISES

**8.6.1.** Use Ceva's theorem to prove that the three Cevian lines in the definition of isotomic conjugate are concurrent.

**8.6.2.** Use the trigonometric form of Ceva's theorem to prove that the three Cevian lines in the definition of isogonal conjugate are concurrent.

***8.6.3.** Let $\triangle ABC$ be a triangle. Make a tool that constructs the isotomic conjugate of a point $P$ that is not on any of the sidelines of the triangle.

***8.6.4.** Let $\triangle ABC$ be a triangle. Make a tool that constructs the isogonal conjugate of a point $P$ that is not on any of the sidelines of the triangle.

***8.6.5.** Verify that the Gergonne point and the Nagel point are isotomic conjugates.

**8.6.6.** Prove that the Gergonne point and the Nagel point are isotomic conjugates. [Hint: Use Exercise 4.5.2.]

***8.6.7.** Verify that the incenter is, in general, the only point in the interior of a triangle that is its own isogonal conjugate. Find all points in the exterior of the triangle that are isogonal conjugates of themselves.

**8.6.8.** Prove that the incenter is its own isogonal conjugate.

**8.6.9.** Prove that the orthocenter and the circumcenter are isogonal conjugates. [Hint: Let $\gamma$ be the circumcircle of $\triangle ABC$. Extend $\overline{AO}$ and $\overline{AH}$ until they intersect $\gamma$ at points $R$ and $S$ as indicated in Figure 8.5. Let $\overline{PQ}$ be a diameter of $\gamma$ that is parallel to $\overline{BC}$ and let $T$ be a point on $\gamma$ such that $\overline{ST}$ is a diameter (see Figure 8.5).

Prove that $\overleftrightarrow{RT} \perp \overleftrightarrow{PQ}$. Prove that $\angle ROP \cong \angle POT \cong \angle QOS$ and prove that $\angle BOP \cong \angle COQ$. Conclude that $\angle BOR \cong \angle COS$. Apply Exercise 0.10.5 twice to conclude that $\angle BAR \cong \angle CTS$. Finally, use the inscribed angle theorem to conclude that $\angle BAO \cong \angle HAC$.]

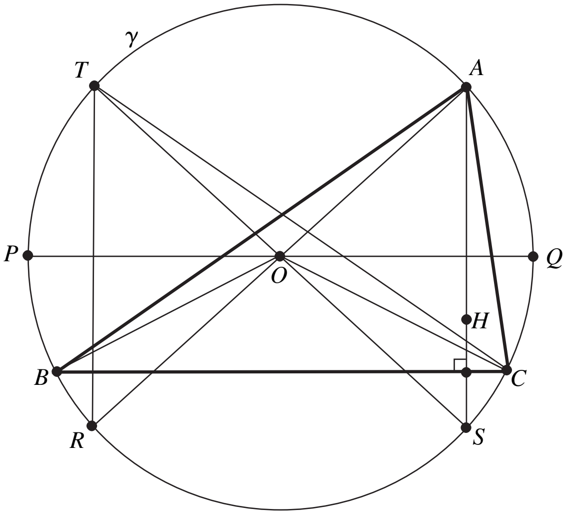

**FIGURE 8.5:** Proof that the circumcenter and the orthocenter are isogonal conjugates

***8.6.10.** What happens to the isogonals of the Cevians through $P$ as $P$ approaches the circumcircle of $\triangle ABC$? What is the isogonal conjugate of a point on the circumcircle?

**8.6.11.** Prove that the isogonal complement of a point on the circumcircle is an ideal point.
[Hint: Observe that the isogonal complement of $P$ is ideal if and only if the isogonal Cevians are parallel. Since the isogonal Cevians have already been proved concurrent, it is enough to show that two of them are parallel. Figure 8.6 shows one possible diagram. Apply Euclid's Proposition III.22 and the angle sum theorem.]

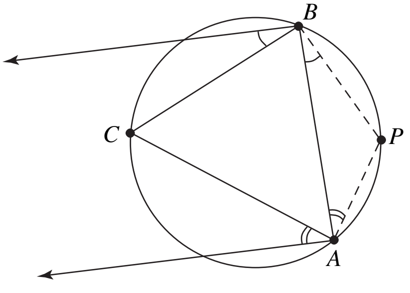

**FIGURE 8.6:** Proof that the isogonal complement of a point on the circumcircle is ideal

**Definition.** A *symmedian* for $\triangle ABC$ is the isogonal of a median of the triangle. The *symmedian point* of $\triangle ABC$ is the point of concurrence of the three symmedians; i.e., the symmedian is the isogonal conjugate of the centroid.

The symmedian point is another triangle center. It is usually denoted by $K$.

## EXERCISES

***8.6.12.** Verify that the isotomic conjugate of the orthocenter is the symmedian point of the anticomplementary triangle.