

CHAPTER 7

The Nine-Point Circle

7.1 THE NINE-POINT CIRCLE

7.2 THE NINE-POINT CENTER

7.3 FEUERBACH'S THEOREM

One of the most remarkable discoveries of nineteenth century Euclidean geometry is the fact that there is one circle that contains nine important points associated with a triangle. In 1765 Euler proved that the medians of the sides and the feet of the altitudes of a triangle lie on a single circle. In other words, the medial and orthic triangles always share the same circumcircle. Furthermore, the center of this common circumcircle lies on the Euler line of the original triangle.

It was not until 1820 that Charles-Julien Brianchon (1783–1864) and Jean-Victor Poncelet (1788–1867) proved that the midpoints of the segments joining the orthocenter to the vertices lie on the same circle. As a result this circle became known as the nine-point circle. Later, Karl Wilhelm Feuerbach (1800–1834) proved that the nine-point circle has the additional property that it is tangent to all four of the equicircles; for this reason Feuerbach's name is often associated with the nine-point circle.

7.1 THE NINE-POINT CIRCLE

Let us begin with a statement of the theorem.

Nine-point Circle Theorem. *If $\triangle ABC$ is any triangle, then the midpoints of the sides of $\triangle ABC$, the feet of the altitudes of $\triangle ABC$, and the midpoints of the segments joining the orthocenter of $\triangle ABC$ to the three vertices of $\triangle ABC$ all lie on a single circle.*

EXERCISES

- *7.1.1. Construct a triangle and the nine points indicated in the theorem. Verify that they all lie on a circle, regardless of the shape of the triangle.
- *7.1.2. Make a tool that constructs the nine-point circle for a triangle. The tool should accept the vertices of the triangle as givens and should return both the circle and the nine points as results. Label all the points as in Figure 7.1. In particular, the vertices of the triangle are A , B , and C , the midpoints of the sides are D , E , and F as before, the feet of the altitudes are A' , B' , and C' , the orthocenter is H , and the midpoints of the segments from H to the vertices of the triangle are K , L , and M . Experiment with triangles of different shapes to get a feel for the nine-point circle. Is it possible for the nine-point circle to be completely contained inside $\triangle ABC$? What is true of $\triangle ABC$ in that case?

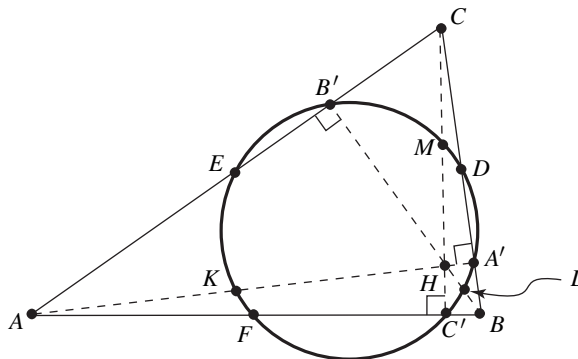


FIGURE 7.1: The nine-point circle

Three points determine a circle, so the strategy for proving the nine-point circle theorem is to start with the circle determined by three of the points and then prove that the other six points also lie on that circle. Specifically, let $\triangle ABC$ be a triangle and let D , E , and F be the midpoints of the sides of $\triangle ABC$. Define γ to be the circumcircle for the medial triangle $\triangle DEF$. Then the midpoints of the sides of the triangle obviously lie on γ . We will first show that the feet of the altitudes lie on γ and then we will show that the midpoints of the segments joining the orthocenter to the three vertices lie on γ .

EXERCISES

- *7.1.3.** Go back to your GSP sketch from Exercise 7.1.2. Move the vertices A , B and C so that your diagram looks as much like Figure 7.1 as possible. Use angle measurement and calculation to verify that each of the quadrilaterals $\square DEFA'$, $\square EFD B'$, and $\square DEFC'$ satisfies the following conditions:
- one pair of opposite sides is parallel,
 - the other pair of opposite sides is congruent, and
 - both pairs of opposite angles are supplements.
- *7.1.4.** When the vertices of $\triangle ABC$ are moved so that the diagram changes, there continue to be three trapezoids that satisfy the conditions of the previous exercise. But the order of the vertices must be adjusted. Try to understand the pattern before proceeding.
- 7.1.5.** Assume $\triangle ABC$ is a triangle such that C' lies between F and B (see Figure 7.1). Prove that $\square DEFB$ is a parallelogram. Use the converse to Thales's theorem to prove that $DC' = DB$. Prove that $\square DEFC'$ is a cyclic quadrilateral. Conclude that C' always lies on the circle determined by D , E , and F .
- *7.1.6.** Go back to your GSP sketch of the nine-point circle and move the vertices of the triangle $\triangle ABC$ to determine the other possible locations for C' on the line \overleftrightarrow{AB} . Modify the argument in the preceding exercise as necessary in order to conclude that in every case C' lies on the circle determined by D , E , and F .

The exercises above show that C' lies on the circumcircle of $\triangle DEF$. We could use the same argument (with the labels of points changed appropriately) to conclude

that A' and B' lie on this circle as well. We can therefore conclude that the feet of all three of the altitudes lie on the circle γ .

The next set of exercises will complete the proof of the nine-point circle theorem by showing that the remaining three points lie on γ as well. We continue to use the notation of Figure 7.1.

EXERCISES

7.1.7. Prove that $\overleftrightarrow{EM} \parallel \overleftrightarrow{AH}$ and conclude that $\overleftrightarrow{EM} \perp \overleftrightarrow{BC}$. Prove that $\overleftrightarrow{EF} \parallel \overleftrightarrow{BC}$ and conclude that $\overleftrightarrow{EM} \perp \overleftrightarrow{EF}$. Prove that E lies on the circle with diameter \overline{MF} .

[Hint: Use Exercise 0.10.4.]

7.1.8. Prove that C' also lies on the circle with diameter \overline{MF} .

7.1.9. Use the two preceding exercises to prove that M lies on the circle determined by E , F , and C' .

By uniqueness of the circumcircle, there is only one circle that contains the three points E , F , and C' . Thus the exercises above show that M lies on γ . Similar arguments show that K and L lie on γ , so the proof of the nine-point circle theorem is complete.

The last few exercises actually prove more than is stated in the theorem—not only do the points K , L , and M lie on the nine-point circle, but \overline{KD} , \overline{LE} , and \overline{MF} are diameters of the circle.

7.2 THE NINE-POINT CENTER

It should be clear from Figure 7.1 that the orthocenter H is not the center of the nine-point circle. In fact the center of the nine-point circle is a new triangle center that we have not encountered before.

Definition. The center of the nine-point circle is the *nine-point center* of $\triangle ABC$. It is denoted by N .

EXERCISES

***7.2.1.** Make a tool that constructs the nine-point center of a triangle.

[Hint: the nine-point center is the circumcenter of the medial triangle, so you should be able to make this tool by combining two others.]

***7.2.2.** Construct a triangle, its circumcenter, its orthocenter, and its nine-point center. Verify that the nine-point center is the midpoint of the segment joining the circumcenter to the orthocenter.

Here is a statement of the theorem you verified in the last exercise.

Nine-Point Center Theorem. *The nine-point center is the midpoint of the line segment from the circumcenter to the orthocenter.*

In particular, the nine-point center lies on the Euler line. We now know a total of five points on the Euler line: the centroid, the circumcenter, the orthocenter, the de Longchamps point, and the nine-point center.

EXERCISES

7.2.3. Prove the Nine-Point Center Theorem.

[Hint: Use Figure 7.2 and the Secant Line Theorem.]

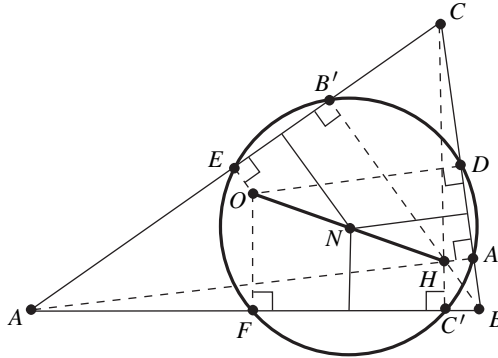


FIGURE 7.2: The orthocenter, the circumcenter, and the nine-point center

7.3 FEUERBACH'S THEOREM

Well after the nine-point circle had been discovered, Feuerbach proved the amazing result that the nine-point circle is tangent to each of the four equicircles.

Feuerbach's Theorem. *The nine-point circle is tangent to each of the four equicircles.*

EXERCISES

- *7.3.1. Construct a triangle $\triangle ABC$ and use tools from your toolbox to construct the incircle, the three excircles, and the nine-point circle. Verify that the nine-point circle is tangent to each of the others. Explore different shapes for $\triangle ABC$ to determine possible configurations for the five circles.
- *7.3.2. The *Feuerbach point* of the triangle is the point of tangency of the nine-point circle and the incircle. Construct the Feuerbach point for your triangle.
- *7.3.3. The *Feuerbach triangle* is the triangle whose vertices are the three points of tangency of the nine-point circle with the three excircles. Construct the Feuerbach triangle for your triangle.

We will not attempt a proof of Feuerbach's Theorem, but a proof may be found in §5.6 of [3].