CHAPTER 6

# Quadrilaterals



The objects we have studied until now have been remarkably simple: just lines, triangles, and circles. In the following chapters we will need to use polygons with more sides; specifically, we will study four-sided and six-sided figures. This chapter contains the necessary information about four-sided figures.

## 6.1 BASIC DEFINITIONS

Let us begin by repeating the definitions from Chapter 0. Four points $A$, $B$, $C$, and $D$ such that no three of the points are collinear determine a *quadrilateral*, which we will denote by $\square ABCD$. Specifically,

$$\square ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}.$$

The four segments are the *sides* of the quadrilateral and the points $A$, $B$, $C$, and $D$ are the *vertices* of the quadrilateral. The sides $\overline{AB}$ and $\overline{CD}$ are called *opposite sides* of the quadrilateral as are the sides $\overline{BC}$ and $\overline{AD}$. Two quadrilaterals are *congruent* if there is a correspondence between their vertices so that all four corresponding sides are congruent and all four corresponding angles are congruent.

In the past, the term *quadrangle* was frequently used for what we now call a quadrilateral. Obviously the difference between the two terms is that one emphasizes the fact that the figure has four sides while the other emphasizes the fact that the figure contains four angles. In the same way either of the terms *triangle* or *trilateral* can be used to name a three-sided or three-angled figure. We will follow current practice in using the terms triangle and quadrilateral even though it could be argued that this is inconsistent terminology.[1] The duality between sides and vertices will be discussed again in Chapter 9.

Each quadrilateral separates the plane into an inside and an outside. A first course on the foundations of geometry will often avoid defining the interior of a

[1] It has been argued that the term *quadrangle* is preferable to *quadrilateral* and that we should resist the use of the term quadrilateral [3, page 52].

general quadrilateral because defining it precisely requires care.[2] In fact, a rigorous definition of interior belongs to the branch of mathematics called *topology* since the definition uses the famous Jordan Curve Theorem, which asserts that a closed curve separates the plane into an inside and an outside. In this course we will not worry about such foundational issues, but will accept the intuitively obvious fact that a quadrilateral in the plane has an interior. GSP also has no qualms about this either and will allow you to define the interior of a general quadrilateral.

## 6.2 CONVEX AND CROSSED QUADRILATERALS

We will distinguish three classes of quadrilaterals: convex, concave, and crossed. There are several equivalent ways to define the most important of these, the convex quadrilaterals.

A quadrilateral is *convex* if each vertex lies in the interior of the opposite angle. Specifically, the quadrilateral $\square ABCD$ is convex if $A$ is in the interior of $\angle BCD$, $B$ is in the interior of $\angle CDA$, $C$ is in the interior of $\angle DAB$, and $D$ is in the interior of $\angle ABC$. This is significant because it allows us to use additivity of angle measure. For example, if $\square ABCD$ is convex, then $\mu(\angle ABC) = \mu(\angle ABD) + \mu(\angle DBC)$. Equivalently, a quadrilateral is convex if the region associated with the quadrilateral is convex in the sense that the line segment joining any two points in the region is completely contained in the region. It can also be shown that a quadrilateral is convex if and only if the two diagonals intersect at a point that is in the interior of both diagonals [16, Theorem 6.7.9].

Our definition of quadrilateral says nothing about how the sides intersect. The requirement that no three vertices are collinear prevents adjacent sides from intersecting at any point other than their common endpoint, but opposite sides can intersect. We will call a quadrilateral whose sides intersect at an interior point a *crossed* quadrilateral. It is easy to see that a convex quadrilateral cannot be crossed, so we have defined two disjoint collections of quadrilaterals. Any quadrilateral that is neither convex nor crossed will be called a *concave* quadrilateral.

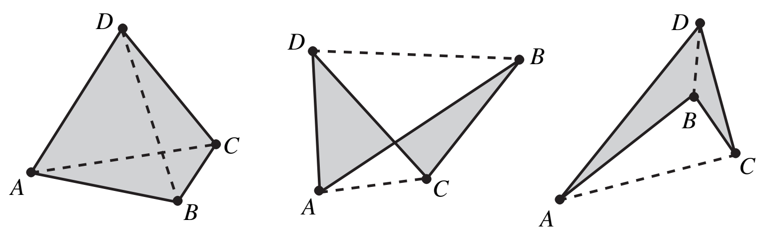


**FIGURE 6.1:** Convex, crossed, and concave

Figure 6.1 shows three quadrilaterals; the first is convex, the second is crossed, and the third is concave. In each case the interior is shaded and the two diagonals

[2]It is customary in such a course to define the interior of a quadrilateral only if the quadrilateral is convex—see §6.2 for the definition of convex.

are shown as dashed lines. Observe that the three kinds of quadrilaterals can be distinguished by their diagonals. The diagonals of a convex quadrilateral are both inside the quadrilateral, the diagonals of a crossed quadrilateral are both outside the quadrilateral, while one diagonal of a concave quadrilateral is inside and the other is outside.

One fact that will be assumed is that every trapezoid is convex [16, Exercise 6.28]. In particular, every parallelogram is convex.

### EXERCISES

***6.2.1.** Construct three quadrilaterals, one of each kind. Construct the interior of each. Observe that GSP determines the interior of a quadrilateral by the order in which you select the vertices and that the interior may not correspond with the segments you have constructed between those vertices.

**6.2.2.** Review Exercise 1.4.8. Prove the following theorem.
**Varignon's Theorem.** *The midpoint quadrilateral of any quadrilateral is a parallelogram.*
[Hint: Use Euclid's Proposition VI.2 (page 7).]

***6.2.3.** Construct a convex quadrilateral and its associated midpoint quadrilateral. Calculate the area of each. What is the relationship between the areas? Find analogous relationships for concave and crossed quadrilaterals.

The theorem above is named for the French mathematician Pierre Varignon (1654–1722). The area relationship in the last exercise is often stated as part of Varignon's theorem.

## 6.3 CYCLIC QUADRILATERALS

We have seen that every triangle can be circumscribed. By contrast, the vertices of a quadrilateral do not necessarily lie on a circle. To see this, simply observe that the first three vertices determine a unique circle and that the fourth vertex may or may not lie on that circle. A quadrilateral whose vertices lie on a circle is quite special and such quadrilaterals will be useful in the next chapter.

**Definition.** A *cyclic quadrilateral* is a quadrilateral that is convex and whose vertices lie on a circle.

A quadrilateral $\square ABCD$ is *inscribed* in the circle $\gamma$ if all the vertices of $\square ABCD$ lie on $\gamma$.

### EXERCISES

***6.3.1.** Construct a circle $\gamma$ and construct four points $A$, $B$, $C$, and $D$ (in cyclic order) on $\gamma$. Measure angles $\angle ABC$ and $\angle CDA$ and calculate the sum of the measures. Do the same with angles $\angle BCD$ and $\angle DAB$. What relationship do you observe?

**6.3.2.** Prove the following theorem.
**Euclid's Proposition III.22.** *If $\square ABCD$ is a convex quadrilateral inscribed in the circle $\gamma$, then the opposite angles are supplements; i.e.,*

$$\mu(\angle ABC) + \mu(\angle CDA) = 180^\circ = \mu(\angle BCD) + \mu(\angle DAB).$$

[Hint: Divide the angles using diagonals of the quadrilateral and apply the Inscribed Angle Theorem.]

**6.3.3.** Prove the following theorem.

**Converse to Euclid's Proposition III.22.** *If $\square ABCD$ is a convex quadrilateral such that the opposite angles are supplements, then $\square ABCD$ is a cyclic quadrilateral.*

[Hint: The idea is very much like that in the proof of Exercise 0.10.4. Let $\gamma$ be the circumscribed circle for $\triangle ABC$. Use the fact that $\square ABCD$ is convex to prove that there is a point $D'$ such that $D'$ lies on $\overrightarrow{AD}$ and $\gamma$. Show that the assumption $D \neq D'$ leads to a contradiction.]

***6.3.4.** Construct a circle $\gamma$ and construct four points $A$, $B$, $C$, and $D$ (in cyclic order) on $\gamma$. Let $a, b, c$, and $d$ denote the lengths of the sides of $\square ABCD$. Define $s = (1/2)(a + b + c + d)$. Verify that

$$\alpha(\square ABCD) = \sqrt{(s - a)(s - b)(s - c)(s - d)}.$$

The formula in the last exercise is known as *Brahmagupta's formula*. It is named for the Indian mathematician Brahmagupta who discovered the formula in the seventh century. Heron's formula can be viewed as the special case in which $d = 0$.

## 6.4 DIAGONALS

Before you work the following exercises, you should review Exercise 1.4.9.

### EXERCISES

**6.4.1.** Prove that a quadrilateral is a parallelogram if and only if the diagonals bisect each other.

**6.4.2.** Prove that a quadrilateral is a rhombus if and only if the diagonals are perpendicular and bisect each other.

**6.4.3.** Prove that a quadrilateral is a rectangle if and only if the diagonals are congruent and bisect each other.

**6.4.4.** Complete the following sentence: A quadrilateral is a square if and only if the diagonals ....