CHAPTER 5

The medial and orthic triangles

In this chapter we investigate constructions of new triangles from old. We begin by studying two specific examples of such triangles, the medial triangle and the orthic triangle, and then generalize the construction in two different ways.

Note on terminology. A median of a triangle is defined to be the segment from a vertex of the triangle to the midpoint of the opposite side. While this is usually the appropriate definition, there are occasions in this chapter when it is more convenient to define the median to be the line determined by the vertex and midpoint rather than the segment joining them. The reader should use whichever definition fits the context. In the same way an altitude of a triangle is usually to be interpreted as a line, but may occasionally be thought of as the segment joining a vertex to a point on the opposite sideline.

5.1 THE MEDIAL TRIANGLE

Throughout this section $\triangle ABC$ is a triangle, D is the midpoint of $\overline{BC}$, E is the midpoint of the segment $\overline{AC}$, and F is the midpoint of $\overline{AB}$.

Definition. The triangle $\triangle DEF$ is the *medial triangle* of $\triangle ABC$.

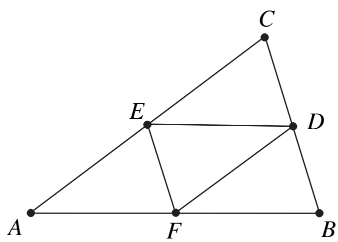

FIGURE 5.1: The medial triangle

EXERCISES

***5.1.1.** Make a tool that constructs the medial triangle of a given triangle.

5.1.2. Prove that the medial triangle subdivides the original triangle into four congruent triangles.

5.1.3. Prove that the original triangle $\triangle ABC$ and medial triangle $\triangle DEF$ have the same medians.

5.1.4. Prove that the perpendicular bisectors of the sides of the original triangle $\triangle ABC$ are the same as the altitudes of the medial triangle $\triangle DEF$.

***5.1.5.** Start with a triangle $\triangle ABC$ and construct a line through each vertex that is parallel to the opposite side of the triangle. Let A'', B'', and C'' be the points at which these three lines intersect. (See Figure 5.2.) The triangle $\triangle A''B''C''$ is called the *anticomplementary triangle* of $\triangle ABC$.[1]

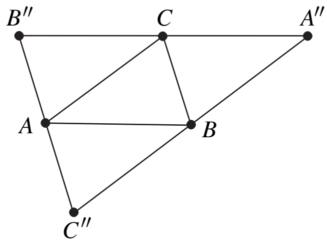

FIGURE 5.2: $\triangle A''B''C''$ is the anticomplementary triangle of $\triangle ABC$

5.1.6. Prove that $\triangle ABC$ is the medial triangle of $\triangle A''B''C''$.

***5.1.7.** Make a tool that constructs the anticomplementary triangle of a given triangle.

***5.1.8.** Let L be the orthocenter of the anticomplementary triangle $\triangle A''B''C''$. Use GSP to verify that L lies on the Euler line of the original triangle $\triangle ABC$. The point L is another triangle center for $\triangle ABC$; it is known as the *de Longchamps point* of $\triangle ABC$ for G. de Longchamps (1842-1906).

***5.1.9.** Let H be the orthocenter, O the circumcenter, and L the de Longchamps point of $\triangle ABC$. Use measurement to verify that O is the midpoint of the segment $\overline{HL}$.

5.2 THE ORTHIC TRIANGLE

Throughout this section $\triangle ABC$ is a triangle, A' is the foot of the altitude through A, B' is the foot of the altitude through B, and C' is the foot of the altitude through C. Note that A' is a point on the sideline $\overleftrightarrow{BC}$ and is not necessarily on the side $\overline{BC}$. Similar comments apply to B' and C'.

Definition. The triangle $\triangle A'B'C'$ is the *orthic triangle* of $\triangle ABC$.

[1] The term *antimedial* would be more appropriate since the construction is the opposite of the construction of the medial triangle. But the term anticomplementary is firmly established in the literature.

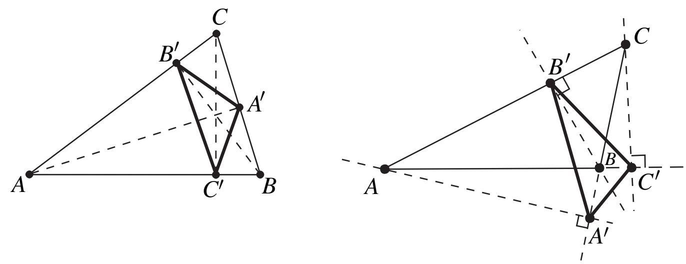

FIGURE 5.3: Two orthic triangles

EXERCISES

***5.2.1.** Make a tool that constructs the orthic triangle of a given triangle.

***5.2.2.** Construct a triangle $\triangle ABC$ and use the tool you just created to construct the orthic triangle $\triangle A'B'C'$. Move the vertices of $\triangle ABC$ and observe what happens to the orthic triangle.

- **(a)** Under what conditions is $\triangle A'B'C'$ contained in $\triangle ABC$?
- **(b)** Is it possible for the orthic triangle to be completely outside the original triangle?
- **(c)** Observe that in a strict sense the orthic triangle may not be a triangle at all since the vertices can be collinear. Under what conditions does the orthic triangle degenerate into a line segment?
- **(d)** Under what conditions is one of the vertices of the orthic triangle equal to one of the vertices of the original triangle?
- **(e)** Under what conditions will two of the vertices of the orthic triangle equal the same vertex of the original triangle?

Make notes on your observations.

5.2.3. Let $\triangle ABC$ be a triangle.

- **(a)** Prove that if $\angle BAC$ and $\angle ACB$ are acute, then B' lies in the interior of $\overline{AC}$.
- **(b)** Prove that every triangle must have at least one altitude with its foot on the triangle.
- **(c)** Prove that if $\angle ABC$ is obtuse, then both A' and C' lie outside $\triangle ABC$.

***5.2.4.** Use the Measure command to investigate the relationship between the measures of the angles $\angle ABC$, $\angle AB'C'$, and $\angle A'B'C$. Find two other triples of angles in the diagram whose measures are related in the same way.

5.2.5. Prove that sides of the orthic triangle cut off three triangles from $\triangle ABC$ that are all similar to $\triangle ABC$. Specifically, if $\triangle A'B'C'$ is the orthic triangle for $\triangle ABC$, then $\triangle ABC \sim \triangle AB'C' \sim \triangle A'BC' \sim \triangle A'B'C$.
[Hint: First prove that $\triangle AB'B \sim \triangle AC'C$ and then apply the SAS similarity criterion to prove $\triangle ABC \sim \triangle AB'C'$. Use similar proofs to show that $\triangle ABC \sim \triangle A'BC'$ and $\triangle ABC \sim \triangle A'B'C$.]

***5.2.6.** Construct a triangle $\triangle ABC$, the three altitudes, and the orthic triangle $\triangle A'B'C'$. Now add the incenter and incircle of $\triangle A'B'C'$ to your sketch Verify that the orthocenter of $\triangle ABC$ is the same as the incenter of $\triangle A'B'C'$ provided all the angles in $\triangle ABC$ are acute. What is the incenter of $\triangle A'B'C'$ in case $\angle BAC$ is obtuse? Try to determine how the orthocenter of $\triangle ABC$ is related to $\triangle A'B'C'$ in case $\angle BAC$ is obtuse.

5.2.7. Let $\triangle ABC$ be a triangle in which all three interior angles are acute and let $\triangle A'B'C'$ be the orthic triangle.
(a) Prove that the altitudes of $\triangle ABC$ are the angle bisectors of $\triangle A'B'C'$.
(b) Prove that the orthocenter of $\triangle ABC$ is the incenter of $\triangle A'B'C'$.
[Hint: Use Exercise 5.2.5.]

5.2.8. Let $\triangle ABC$ be a triangle such that $\angle BAC$ is obtuse and let $\triangle A'B'C'$ be the orthic triangle.
(a) Prove that A is the incenter of $\triangle A'B'C'$.
(b) Prove that the orthocenter of $\triangle ABC$ is the A'-excenter of $\triangle A'B'C'$.

5.3 CEVIAN TRIANGLES

The construction of the medial triangle and the orthic triangle are both examples of a more general construction. Start with a triangle $\triangle ABC$ and a point P that is not on any of the sidelines of $\triangle ABC$. Let L be the point at which $\overleftrightarrow{AP}$ intersects $\overleftrightarrow{BC}$, let M be the point at which $\overleftrightarrow{BP}$ intersects $\overleftrightarrow{AC}$, and let N be the point at which $\overleftrightarrow{CP}$ intersects $\overleftrightarrow{AB}$.

Definition. The triangle $\triangle LMN$ is the a *Cevian triangle* for $\triangle ABC$ associated with the point P.

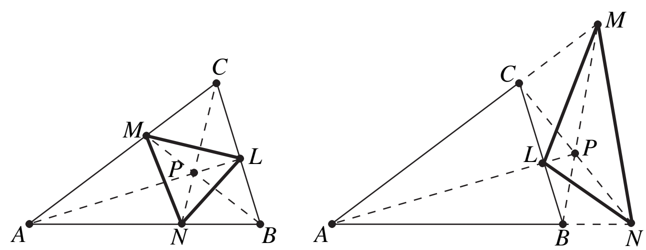

FIGURE 5.4: Two Cevian triangles associated with different points P

The medial triangle is the special case of a Cevian triangle in which P is the centroid of $\triangle ABC$ and the orthic triangle is the special case in which P it the orthocenter of $\triangle ABC$. Cevian triangles are named for Giovanni Ceva who studied the question of when three lines through the vertices of a triangle would be concurrent. Because of Ceva's work, such lines are called *Cevian lines* (or simply *cevians*) for the triangle. We will investigate Ceva's Theorem in Chapter 8.

EXERCISES

***5.3.1.** Make a tool that constructs the Cevian triangle associated with a given triangle $\triangle ABC$ and point P.

5.3.2. What happens if you attempt to construct a Cevian triangle associated with a point P that is on a sideline?

***5.3.3. Preview of Ceva's Theorem.** Construct a triangle $\triangle ABC$ and the Cevian triangle associated with the point P. Compute the quantity

$$d = \frac{AN}{NB} \cdot \frac{BL}{LC} \cdot \frac{CM}{MA}.$$

Observe that d has the same value regardless of the positions of A, B, C, and P (provided that none of the distances in the expression is 0).

***5.3.4. Preview of Menelaus's Theorem.** Construct a triangle $\triangle ABC$ and the Cevian triangle associated with the point P. Let A'' be the point of intersection of the lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{MN}$, let B'' be the point of intersection of $\overleftrightarrow{AC}$ and $\overleftrightarrow{LN}$, and let C'' be the point of intersection of the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{LM}$. (These points may not exist if any two of the lines are parallel, but you can always move the vertices of $\triangle ABC$ slightly so that the lines do intersect.) Verify that A'', B'', and C'' are collinear.

5.4 PEDAL TRIANGLES

There is a second way in which to generalize the construction of the medial and orthic triangles that leads to another useful class of triangles. Again start with a triangle $\triangle ABC$ and point P that is not on any of the sidelines of the triangles. Drop perpendiculars from P to each of the sidelines of the triangle. Let A' be the foot of the perpendicular to the sideline opposite A, let B' be the foot of the perpendicular to the sideline opposite B, and let C' be the foot of the perpendicular to the sideline opposite C.

Definition. The triangle $\triangle A'B'C'$ is the *pedal triangle* associated with $\triangle ABC$ and P.

It is obvious that the orthic triangle is the pedal triangle associated with the orthocenter. Since the midpoints of the sides of the triangle are the feet of the perpendiculars from the circumcenter, we see that the medial triangle is also a pedal triangle; it is the pedal triangle associated with the circumcenter.

EXERCISES

***5.4.1.** Make a tool that constructs the pedal triangle associated with a given triangle $\triangle ABC$ and point P.

***5.4.2.** Use your incenter tool and the pedal triangle tool to construct the pedal triangle associated with the incenter of the triangle $\triangle ABC$. Verify that the inscribed circle for the original triangle is the same as the circumscribed circle for the pedal triangle.

***5.4.3.** Construct a triangle and choose a point P in the interior of the triangle. Use your pedal triangle tool to construct the pedal triangle with respect to P. Now construct the pedal triangle of the pedal triangle with respect to the same point P. This new triangle is called the *second pedal* triangle. There is also a *third pedal triangle*, which is the pedal triangle of the second pedal triangle (with respect to the same point P). Verify that the third pedal triangle is similar to the original triangle.
[Hint: This construction will work best if your pedal triangle returns the vertices of the pedal triangle as results, not just the triangle itself.]

***5.4.4.** **Preview of Simson's Theorem.** Construct a triangle $\triangle ABC$, the circumscribed circle γ for $\triangle ABC$, and a point P that is not on any of the sidelines of the triangle. Verify that the vertices of the pedal triangle are collinear if and only if P lies on γ.

5.4.5. Let $\triangle ABC$ be a triangle, let P be a point, and let A', B', and C' be the feet of the perpendiculars from P to the sidelines opposite A, B, and C, respectively. Prove that

$$B'C' = \frac{BC \cdot AP}{2R}, \quad A'C' = \frac{AC \cdot BP}{2R}, \quad \text{and} \quad A'B' = \frac{AB \cdot CP}{2R}.$$

[Hint: Use the converse to Thales' Theorem to prove that B' and C' lie on the circle with diameter $\overline{AP}$. Conclude that P is on the circumcircle of $\triangle AB'C'$. Apply the extended law of sines (page 31) to the triangles $\triangle ABC$ and $\triangle AB'C'$ to get $BC/\sin(\angle BAC) = 2R$ and $B'C'/\sin(\angle BAC) = AP$. Solve for $B'C'$ to obtain the first equation. The other two equations are proved similarly.]

The result in Exercise 5.4.3 was discovered by J. Neuberg in 1892. The line containing the three collinear feet of the perpendiculars in the Exercise 5.4.4 is called a *Simson line* for the triangle. It is named for Robert Simson (1687–1768). We will prove Simson's Theorem in Chapter 11. Exercise 5.4.5 is a technical fact that will be needed in the proof of Ptolemy's Theorem in that same chapter.