CHAPTER 4

Circumscribed circles, inscribed circles, and escribed circles

In this chapter we explore the construction of several circles that are associated with a triangle and also introduce more triangle centers. Again the emphasis is mainly on GSP exploration, with formal proofs of most results postponed until later. The chapter ends with a proof of Heron's formula for the area of a triangle.

4.1 THE CIRCUMSCRIBED CIRCLE AND THE CIRCUMCENTER

The first circle we will study is the circumscribed circle. It can be thought of as the smallest circle that contains a given triangle.

Definition. A circle that contains all three vertices of the triangle $\triangle ABC$ is said to *circumscribe* the triangle. The circle is called the *circumscribed circle* or simply the *circumcircle* of the triangle. The radius of the circumscribed circle is called the *circumradius*.

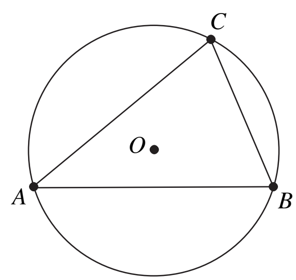

FIGURE 4.1: The circumcenter and the circumcircle

It is, of course, no accident that the term circumcenter was used as the name of one of the triangle centers that was introduced in Chapter 2. In fact, Exercise 2.4.4

shows that the three vertices of a triangle are equidistant from the circumcenter of the triangle, so all three vertices lie on a circle centered at the circumcenter. It follows that every triangle has a circumcircle and the circumcenter is the center of the circumcircle.

It should be noted that part of the definition of triangle is the assumption that the vertices are noncollinear. Thus the real assertion is that three noncollinear points determine a circle. Furthermore, the fact that we know how to construct the center of the circumcircle means we can construct the circumscribed circle itself.

EXERCISES

***4.1.1.** Make a tool that constructs the circumcircle of a triangle. The tool should accept three noncollinear points as given and produce a circle containing the three points as result. Make sure your tool is robust enough so that the circle you construct continues to go through all the vertices of the triangle even when the vertices of the triangle are moved.

***4.1.2.** Use the tool you made in the previous exercise to explore the circumscribed circle of various triangles. For which triangles is the circumcenter inside the triangle? For which triangles is the circumcenter on the edge of the triangle? What happens to the circumcircle when one vertex of the triangle is moved across the sideline determined by the other two vertices? Make notes on your observations.

4.1.3. Prove that the circumcircle is unique. In other words, prove that there can be at most one circle that passes through three given noncollinear points. How many circles pass through two given points?
[Hint: Begin by proving that any circle that passes through A, B, and C must have the circumcenter as its center.]

***4.1.4.** Construct a triangle $\triangle ABC$, its circumcircle γ, and measure the circumradius R of $\triangle ABC$. Verify the following result, which extends the usual Law of Sines that you learned in high school.
Extended Law of Sines. *If $\triangle ABC$ is a triangle with circumradius R, then*

$$\frac{BC}{\sin(\angle BAC)} = \frac{AC}{\sin(\angle ABC)} = \frac{AB}{\sin(\angle ACB)} = 2R.$$

4.1.5. Prove the extended law of sines.
[Hint: Let γ be the circumscribed circle of $\triangle ABC$ and let D be the point on γ such that $\overline{DB}$ is a diameter of γ. Prove that $\angle BAC \cong \angle BDC$. Use that result to prove that $\sin(\angle BAC) = BC/2R$. The other proofs are similar.]

4.2 THE INSCRIBED CIRCLE AND THE INCENTER

The second circle we study is called the *inscribed circle*, or simply the *incircle*. It is the opposite of the circumcircle in the sense that it is the largest circle that is contained in the triangle.

EXERCISES

***4.2.1.** Construct a triangle and the three bisectors of the interior angles of the triangle. Note that the three angle bisectors are concurrent. The point of concurrency is called the *incenter* of the triangle. Experiment with triangles of different shapes to determine whether the incenter can ever be on the triangle or outside the triangle. For which triangles is the incenter on the triangle? For which triangles is the incenter outside the triangle?

***4.2.2.** Construct another triangle and its incenter. Label the incenter I; keep it visible but hide the angle bisectors. Experiment with the triangle and the incenter in order to answer the following questions.

(a) Are there triangles for which the incenter equals the circumcenter? What shape are they?

(b) Are there triangles for which the incenter equals the centroid? What shape are they?

***4.2.3.** Construct another triangle and its incenter. Label the incenter I; keep it visible but hide the angle bisectors. For each side of the triangle, construct a line that passes through the incenter and is perpendicular to the sideline. Mark the feet of these perpendiculars and label them X, Y and Z as indicated in Figure 4.2. Measure the distances IX, IY, and IZ and observe that they are equal. This number is called the *inradius* of the triangle.

***4.2.4.** Hide the perpendiculars in your sketch from the last exercise, but keep the points I, X, Y, and Z visible. Construct the circle with center I and radius equal to the inradius. Observe that this circle is tangent to each of the three sides of the triangle. This circle is the inscribed circle for the triangle.

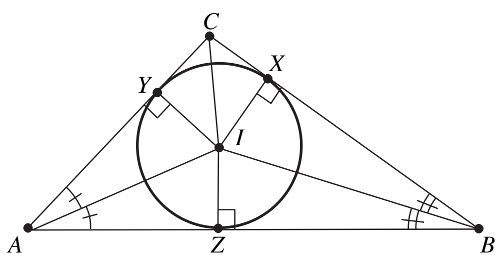

FIGURE 4.2: The incenter and the incircle

We now state a theorem and definition that formalize some of the observations you made in the exercises above.

Angle Bisector Concurrence Theorem. *If $\triangle ABC$ is any triangle, the three bisectors of the interior angles of $\triangle ABC$ are concurrent. The point of concurrency is equidistant from the sides of the triangle.*

Definition. The point of concurrency of the bisectors is called the *incenter* of the triangle. The distance from the incenter to sides of the triangle is the *inradius*. The circle that has its center at the incenter and is tangent to each of the sides of the triangle is called the *inscribed circle*, or simply the *incircle* of the triangle.

The concurrency part of the theorem will be assumed for now, but it will be proved in Chapter 8. You will prove the rest of the theorem in the exercises below.

EXERCISES

4.2.5. Prove that the point at which any two interior angle bisectors intersect is equidistant from all three sidelines of the triangle.

4.2.6. Use the definition of triangle interior to prove that the incenter is inside the triangle.

***4.2.7.** Make a tool that constructs the incenter of a given triangle.

***4.2.8.** Make a tool that constructs the inscribed circle of a given triangle.

4.3 THE ESCRIBED CIRCLES AND THE EXCENTERS

The incircle is not the only circle that is tangent to all three sidelines of the triangle. Any circle that is tangent to all three sidelines of a triangle is called an *equicircle* (or *tritangent circle*) for the triangle. In this section we will construct and study the remaining equicircles.

Definition. A circle that is outside the triangle and is tangent to all three sidelines of the triangle is called an *escribed circle* or an *excircle*. The center of an excircle is called an *excenter* for the triangle.

There are three excircles, one opposite each vertex of the triangle. The excircle opposite vertex A is shown in Figure 4.3; it is called the A*-excircle* and is usually denoted γ_A. There is also a B-excircle γ_B and a C-excircle γ_C.

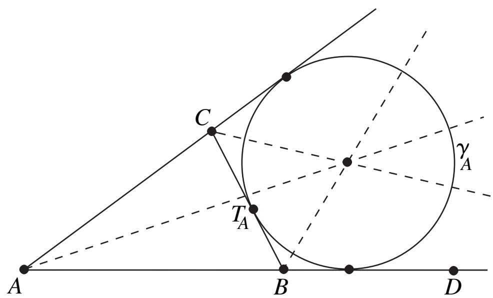

FIGURE 4.3: The A-excircle

Figure 4.3 indicates how the A-excircle is constructed. The A-excenter is the point at which the bisector of the interior angle at A and the bisectors of the exterior angles at B and C concur. (There are two exterior angles at B; when constructing the A-excenter use the exterior angle formed by extending $\overline{AB}$. At vertex C use the exterior angle formed by extending $\overline{AC}$.) Once the excenter has been located, a radius of the excenter is obtained by dropping a perpendicular from the excenter to one of the sidelines and marking the foot of the perpendicular.

EXERCISES

***4.3.1.** Construct a triangle $\triangle ABC$, the rays $\overrightarrow{AC}$ and $\overrightarrow{AB}$ and the three angle bisectors shown in Figure 4.3. Experiment with triangles of different shape to verify that the three angle bisectors are always concurrent.

***4.3.2.** Make a tool that constructs an excircle. Your tool should accept the three vertices of the triangle as givens and should construct the excircle opposite the first vertex selected as its result.
[Hint: Be careful with the construction of this tool. Start with the three vertices A, B, and C. Then choose a point D as shown in Figure 4.3. If you simply choose D to be a movable point on the sideline, it is not a well defined descendant of the vertices and GSP will not consistently place it where you want it. The result will be a tool that works correctly most of the time, but occasionally produces the incircle rather than the excircle. In order to specify the point D uniquely, you might define it to be the reflection of A across a line through B. Once D is defined, the excenter may be defined as the intersection the bisectors of $\angle CAB$ and $\angle CBD$.]

***4.3.3.** Construct a triangle, the three sidelines of the triangle, and the three excircles for the triangle. Explore triangles of different shapes to learn what the configuration of excircles looks like in different cases. What happens to the excircles when one vertex passes over the sideline determined by the other two vertices?

The fact that the three angle bisectors described above are always concurrent will be proved in Chapter 8. Assuming that result, the following exercise shows that the point of concurrency is an excenter.

EXERCISES

4.3.4. Prove that the point at which any two of the three angle bisectors shown in shown in Figure 4.3 intersect is equidistant from all three sidelines of the triangle.

4.4 THE GERGONNE POINT AND THE NAGEL POINT

There are two additional triangle centers that are associated with the inscribed and escribed circles.

EXERCISES

***4.4.1.** Construct a triangle $\triangle ABC$ and its incircle. Mark the three points at which the incircle is tangent to the triangle. The point of tangency opposite to vertex A should be labeled X, the point of tangency opposite B should be labeled Y, and the one opposite C should be labeled Z. Construct the segments $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$. Observe that the three segments are always concurrent, regardless of the shape of the triangle. The point of concurrency is call the *Gergonne point* of the triangle. It is denoted Ge. (Don't confuse it with the centroid G.)

***4.4.2.** Construct a triangle, the incenter I, and the Gergonne point Ge. For which triangles is $I = Ge$?

***4.4.3.** Construct a triangle and its three excircles. Mark the three points at which the three excircles are tangent to the triangle. Use the label T_A for the point at which the A-excircle is tangent to $\overline{BC}$, T_B for the point at which the B-excircle is tangent to $\overline{AC}$, and T_C for the point at which the C-excircle is tangent to $\overline{AB}$. (In GSP you can use T−A as a substitute for T_A.) Construct the segments $\overline{AT_A}$, $\overline{BT_B}$, and $\overline{CT_C}$. Observe that the three segments are always concurrent, regardless of the shape of the triangle. The point of concurrency is call the *Nagel point* of the triangle. It is denoted Na.

***4.4.4.** Is the Nagel point ever equal to the Gergonne point? If so, for which triangles?

***4.4.5.** Which of the three points I, Ge, or Na lies on the Euler line? Do the points you have identified lie on the Euler line for every triangle, or only for some triangles?

The Nagel point is named for the German high school teacher and geometer Christian Heinrich von Nagel (1803–1882) while the Gergonne point is named for the French mathematician Joseph Diaz Gergonne (1771–1859). Each of the Nagel and Gergonne points is defined as the point at which three segments are concurrent. You have accumulated GSP evidence that these segments do concur, but we have not actually proved that. The two concurrence theorems that allow us to define the Gergonne and Nagel points will be proved in Chapter 8. In that chapter we will also show that the two points are "isotomic conjugates" of one another.

4.5 HERON'S FORMULA

In this section we use the geometry of the incircle and the excircle to derive a famous formula that expresses the area of a triangle in terms to the lengths of the sides of the triangle. This formula is named for the ancient Greek geometer Heron of Alexandria who lived from approximately AD 10 until about AD 75. Even though the formula is commonly attributed to Heron, it was probably already known to Archimedes (287–212 BC).

Let us begin with some notation that will be assumed for the remainder of this section. Fix $\triangle ABC$. Let I be the center of the incircle and let E be the center of the A-excircle. The feet of the perpendiculars from I and E to the sidelines of $\triangle ABC$ are labeled X, Y, Z, T, U, and V, as indicated in Figure 4.4.

Define $a = BC$, $b = AC$, and $c = AB$. We will use r to denote the inradius of $\triangle ABC$ and r_a to denote the radius of the A-excircle.

Definition. The *semiperimeter* of $\triangle ABC$ is $s = (1/2)(a + b + c)$.

EXERCISES

4.5.1. Prove that the area of $\triangle ABC$ satisfies $\alpha(\triangle ABC) = sr$.

4.5.2. By the External Tangents Theorem, $AZ = AY$, $BZ = BX$, $CX = CY$, $BV = BT$, and $CT = CU$. In order to simplify notation, let $z = AZ$, $x = BX$, $y = CY$, $u = CU$, and $v = BV$.

(a) Prove that $x + y + z = s$, $x + y = a$, $y + z = b$, and $x + z = c$.

(b) Prove that $x = s - b$, $y = s - c$, and $z = s - a$.

(c) Prove that $u + v = x + y$ and $x + z + v = y + z + u$.

(d) Prove that $u = x$ and $v = y$.

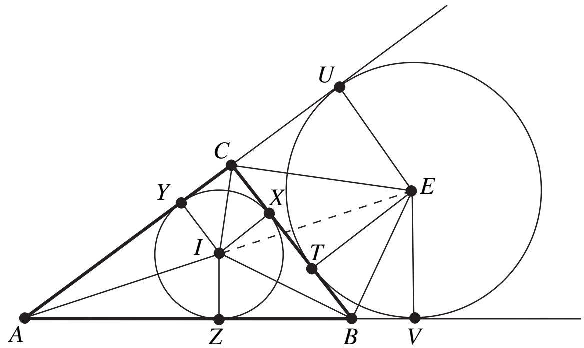

FIGURE 4.4: Notation for proof of Heron's formula

4.5.3. Prove that $AV = s$.
4.5.4. Use similar triangles to prove that $r_a/s = r/(s - a)$.
4.5.5. Prove that $\mu(\angle ZBI) + \mu(\angle EBV) = 90^\circ$. Conclude that $\triangle ZBI \sim \triangle VEB$.
4.5.6. Prove that $(s - b)/r = r_a/(s - c)$.
4.5.7. Combine Exercises 4.5.1, 4.5.4, and 4.5.6 to prove Heron's Formula:

$$\alpha(\triangle ABC) = \sqrt{s(s - a)(s - b)(s - c)}.$$