CHAPTER 14

The Poincaré disk

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In this final chapter we study the Poincaré disk model for hyperbolic geometry. While hyperbolic geometry is a non-Euclidean geometry and the subject matter of these notes is Euclidean geometry, the topic is nonetheless appropriate for inclusion here because the Poincaré disk is built within Euclidean geometry. The main technical tool used in the construction of the Poincaré disk model is inversion in Euclidean circles, so the tools you made in Chapter 13 will be put to use in this chapter to perform hyperbolic constructions. As mentioned in the Preface, many of the constructions in this chapter are based on those in [9].

It was Eugenio Beltrami (1835–1900) who originated the idea of representing hyperbolic geometry within Euclidean geometry. There are many ways in which to construct models of hyperbolic geometry, but the Poincaré disk model is probably the best known. One reason for its popularity is the great beauty of the diagrams that can be constructed in this model. The model is named for the French mathematician Henri Poincaré (1854–1912) who first introduced it.

14.1 THE POINCARÉ DISK MODEL FOR HYPERBOLIC GEOMETRY

A *model* for a geometry is an interpretation of the technical terms of the geometry (such as point, line, distance, angle measure, etc.) that is consistent with the axioms of that geometry. The usual model for Euclidean geometry is $\mathbb{R}^2$, the Cartesian plane, which consists of all ordered pairs of real numbers. That model has been more or less assumed throughout this course, although most of the proofs we have given have been *synthetic*, which means that they are based on the axioms of geometry and not on specific (analytic) properties of any particular model for geometry.

We are about to begin the study of one form of non-Euclidean geometry—-geometry in which the Euclidean parallel postulate is not assumed to hold.[1] We

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[1] See Playfair's Postulate on page 6.

will consider two distinct kinds of non-Euclidean geometry: *neutral geometry* is the geometry that is based on all the usual axioms of Euclidean geometry except that no parallel postulate is assumed while *hyperbolic geometry* is the geometry that is based on all the axioms of neutral geometry together with the hyperbolic parallel postulate. The *hyperbolic parallel postulate* asserts that for every line ℓ and for every point P that does not lie on ℓ, there exist multiple lines through P that are parallel to ℓ. Since all the axioms of neutral geometry are also assumed in both Euclidean and hyperbolic geometries, any theorem that can be proved in neutral geometry is a theorem in both the other geometries as well. This is significant because many of the elementary constructions at the beginning of Euclid's elements are neutral, so we can use them in our study of hyperbolic geometry.

Let us now describe the Poincaré disk model for hyperbolic geometry. Fix a circle Γ in the Euclidean plane. A "point" in the Poincaré disk is a Euclidean point that is inside Γ. There are two kinds of "lines" in the Poincaré disk. The first kind of line is a diameter of Γ; more specifically, a Poincaré line of the first kind consists of all the points on a diameter of Γ that lie inside Γ. A second kind of Poincaré line is a Euclidean circle that is orthogonal to Γ; more specifically, a Poincaré line of the second kind consists of all the points of a Euclidean circle orthogonal to Γ that lie inside Γ.

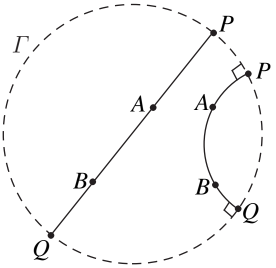

FIGURE 14.1: Two kinds of Poincaré lines and their ideal endpoints

The Poincaré distance between two points A and B in the Poincaré disk is defined by $d(A, B) = |\ln([AB, PQ])|$, where P and Q are the points at which the Poincaré line containing A and B intersects Γ. Observe that the points P and Q lie on Γ so they are not themselves points in the Poincaré disk, but they are still useful in defining the distance between points in the Poincaré disk. The points on Γ are ideal points for the Poincaré disk. (In hyperbolic geometry, each line has two ideal endpoints—see Figure 14.1.) The angle between two Poincaré lines is measured by measuring the Euclidean angle between their tangent lines.

The Poincaré disk is a model for hyperbolic geometry. Proving this assertion means proving that, with the terms point, line, distance, etc. interpreted as above, all the axioms of hyperbolic geometry are satisfied. Since the model is described within Euclidean geometry, those proofs are all Euclidean proofs. For example, we will see in the next section that theorems about Euclidean inversions in circles can be used to construct a unique Poincaré line through any two points in the Poincaré disk.

Thus Euclid's first postulate holds in the model. Note also that Euclidean inversions preserve angles and cross-ratios, so they preserve Poincaré angle measure and Poincaré distance. The Euclidean inversions are therefore isometries of the model and they function as reflections across lines in hyperbolic geometry. Inversions are the main technical tool that are used in proofs of theorems about the Poincaré disk.

In this chapter we will focus on constructions in the Poincaré disk and will not supply proofs that the axioms of hyperbolic geometry are satisfied in the model. Such proofs can be found in most college geometry texts. In particular, §13.3 of [16] contains a more detailed description of the model and proofs that all the usual axioms of geometry are satisfied by this model.

14.2 THE HYPERBOLIC STRAIGHTEDGE

Our first objective is to make a hyperbolic straightedge. In other words, we want to make a GSP tool that constructs the unique Poincaré line through two points in the Poincaré disk. We also want to make variations on the tool that will construct the Poincaré ray or segment determined by two points. The tool you made in Exercise 13.3.3 is almost exactly what is needed in order to construct a Poincaré line; the only refinement required is to trim off the part of the Euclidean circle that lies outside of Γ.

When you make the tools in this and later sections, you may assume that the two given points do not lie on a common diameter of Γ. This means that the tools you make will only construct Poincaré lines of the second kind. As a result, some of the lines may disappear momentarily when you move one point across the diameter determined by another. Making the tools general enough that they can accept any points as givens and produce whichever kind of line is appropriate as result would significantly increase the complication of the constructions, so we will not bother with it and will live with the minor inconvenience that results. One justification for this is that randomly chosen points would not lie on a common diameter, so the case we are omitting is the exceptional case. Another justification is that the simple tools often produce a good approximation to the correct answer anyway since part of a circle of large radius is indistinguishable from a straight line.

EXERCISES

***14.2.1.** Make a tool that constructs the Poincaré line through two points. The tool should accept four points as givens (the center O of Γ, a point R on Γ, and the two points A and B) and return the hyperbolic line ℓ as its result. Assume in your construction that A, B, and O are noncollinear. What happens to ℓ when you move A to make the three points collinear?
[Hints: Use the construction of Exercise 13.3.3 to find a Euclidean circle α that contains A and B and is orthogonal to Γ. Mark the two points P and Q at which α intersects Γ. Hide α and then use the Arc Through 3 Points command in the Construct menu to construct the circular arc from P to Q through A. The suggestions at the end of §13.1 should be helpful in making tools that do not require you to reenter the information about O and Γ every time you use them.]

***14.2.2.** Use the tool from the last exercise to construct a Poincaré line ℓ. Construct a point P that does not lie on ℓ. Construct multiple lines through P that are all parallel to ℓ. Is there any limit to the number of parallel lines you can construct? [Hint: Remember that "parallel" simply means that the line do not intersect.]

***14.2.3.** Make a tool that constructs the Poincaré ray determined by two points. The tool should accept four points as givens (the center O of Γ, a point R on Γ, and the two points A and B) and return the hyperbolic ray $\overrightarrow{AB}$ as its result.
[Hints: The obvious construction is to start with the Poincaré line determined by the two points A and B and then to mark the two ideal points P and Q at which this line meets Γ (labeled as in Figure 14.1). Then hide the line and construct the arc of a circle determined by A, B, and Q. This construction produces a tool that usually returns the correct object, but does not do so reliably unless GSP is given a better rule for determining which of the two points P and Q to use. (You should do the construction and see for yourself what happens when you move A and B around.) One way to make a tool that reliably gives the correct answer is to construct the inverse B' of B in Γ, construct the circular segment determined by A, B, and B', and then to construct Q as the intersection of this arc with Γ. The circular arc determined by A, B, and Q will then be the Poincaré ray $\overrightarrow{AB}$.]

***14.2.4.** Make a tool that constructs the Poincaré segment determined by two points. The tool should accept four points as givens (the center O of Γ, a point R on Γ, and the two points A and B) and return the hyperbolic segment $\overline{AB}$ as its result.
[Hint: Again you must be careful to make sure that your tool reliably returns the correct arc.]

***14.2.5.** Use the segment tool you just made to construct a triangle in the Poincaré disk. Move the vertices around to see what shapes are possible.

14.3 COMMON PERPENDICULARS

In the hyperbolic plane there are two kinds of parallel lines [16, §8.4]. One possibility is that two parallel lines m and n are *asymptotically parallel*, which means that in one direction the two lines get closer and closer together. This is illustrated in the Poincaré disk model by Poincaré lines that limit on a common point of Γ. Figure 14.2 shows three Poincaré lines that are pairwise asymptotically parallel. Even though the three lines converge at Γ, they are parallel because they do not intersect at a point of the Poincaré disk.

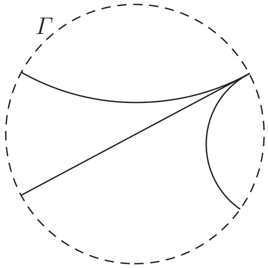

FIGURE 14.2: Three asymptotically parallel lines

The Classification of Parallels Theorem [16, Theorem 8.4.18] asserts that if m and n are two lines that are parallel but not asymptotically parallel, then m and n must admit a common perpendicular. This means that there is a line t that is perpendicular to both m and n (see Figure 14.3). In the exercises below you will construct this common perpendicular line.[2] The written exercises are exercises in Euclidean geometry.

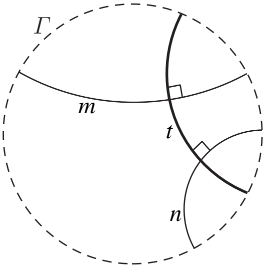

FIGURE 14.3: The Poincaré line t is the common perpendicular for m and n

EXERCISES

14.3.1. Let α be a circle with center A and let C be a point outside α. Construct the circle δ with diameter $\overline{AC}$ and let H be a point of $\alpha \cap \delta$. Define γ to be the circle with center C and radius $\overline{CH}$. Use Thales' Theorem to prove that α and γ are orthogonal.

***14.3.2.** Use Exercise 14.3.1 to make a tool that constructs the circle that is orthogonal to a specified circle α and has a specified point C as center. The tool should accept three points as givens (the center A of α, a point S on α, and a point C outside α) and should return a circle γ that has C as its center and is orthogonal to α as its result.

14.3.3. Let α and β be two orthogonal circles and let ℓ be their common secant line. Let γ be a circle whose center lies on ℓ. Use Theorem 13.3.1 to prove that γ is orthogonal to α if and only if γ is orthogonal to β.

14.3.4. Let α, β, and Γ be three circles such that α and β are both orthogonal to Γ. Let m be the common secant line for α and Γ, let n be the common secant line for β and Γ, and let P be the point at which m and n intersect. By Exercise 14.3.1 there is a circle τ with center P that is orthogonal to Γ. Use Exercise 14.3.3 to prove that τ is also orthogonal to both α and β.

***14.3.5.** Construct two Poincaré lines ℓ and m that are parallel but not asymptotically parallel. Use Exercise 14.3.4 to construct a Poincaré line t such that t is orthogonal to both ℓ and m. Under what conditions will the common perpendicular be a Poincaré line of the first kind (a diameter of Γ)?

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[2] The constructions in this section are based on the ideas in [13].

14.4 THE HYPERBOLIC COMPASS

Our next task is to make a hyperbolic compass tool. This tool should construct the Poincaré circle with a specified point A as center and and a specified point B on the circle. The Poincaré circle with center A and radius r is defined to be the set of all points X in the Poincaré disk such that $d(A, X) = r$. So the obvious way to attempt to construct a Poincaré circle is to make a tool that measures Poincaré distances and then construct the circle as the locus of all points at a fixed Poincaré distance from the center. But this method is not easy to implement and there is a much more elegant way to construct the hyperbolic circle.

The construction is based on two facts. The first is that every Poincaré circle is also a Euclidean circle [16, Exercises 13.7–13.9]. In general the Poincaré center of the circle is different from the Euclidean center and the Poincaré radius is different from the Euclidean radius, but for each Poincaré circle there is a point A' and a number r' such that the Poincaré circle is exactly equal to the Euclidean circle $\mathcal{C}(A', r')$. Thus we need only locate the Euclidean center A' and then construct the Euclidean circle with center A' that passes through B.

The second fact that the construction is based on is the fact that the Tangent Line Theorem (Chapter 0) is a theorem in neutral geometry. As a result, the theorem holds in the Poincaré disk and a Poincaré circle α must be perpendicular to every Poincaré line through the Poincaré center A. In particular, α must be perpendicular to the Euclidean line $\overleftrightarrow{OA}$, where A is the Poincaré center of the circle and O is the center of the Euclidean circle Γ that is used in the definition of the Poincaré disk (since it is a Poincaré line of the first kind). This means that the Euclidean center must lie on $\overleftrightarrow{OA}$. In addition, α must be perpendicular to the Poincaré line determined by A and B. This implies that the Euclidean center must lie on line that is tangent to the Poincaré line at B.

Construction. Assume that the Poincaré disk is defined by the Euclidean circle Γ with center O. Let points A and B inside Γ be given. Assume that A, B, and O are noncollinear. Construct the Euclidean line $\overleftrightarrow{OA}$. Construct the Poincaré line ℓ that contains A and B. Let t be the Euclidean line that is tangent to ℓ at B. Define A' to be the point at which t and $\overleftrightarrow{OA}$ intersect. (See Figure 14.4.) The Euclidean circle with center at A' and passing through B is the Poincaré circle with center A passing through B.

EXERCISES

***14.4.1.** Make a tool that constructs the Poincaré circle determined by two points. The tool should accept four points as givens (the center O of Γ, a point R on Γ, a point A that is to be the center of the Poincaré circle and a point B that is to lie on the Poincaré circle) and return the hyperbolic circle α as its result. Assume in your construction that A, B, and O are noncollinear. What happens to α when you move A to make the three points collinear? What happens when you move the "center" of the circle towards the boundary of the Poincaré disk?

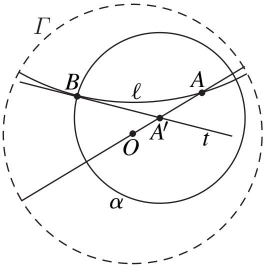

FIGURE 14.4: Construction of the Poincaré circle through B with center A

***14.4.2.** Use your hyperbolic compass to construct a hyperbolic circle and then use your hyperbolic straightedge to construct a triangle inscribed in the circle. Construct an inscribed triangle that has a diameter as one side. Based on your diagram, do you think Thales' Theorem is correct in hyperbolic geometry?

14.5 OTHER HYPERBOLIC TOOLS

Most of the compass and straightedge constructions you learned in high school are part of neutral geometry and are therefore valid in hyperbolic geometry. Since we now have a hyperbolic compass and straightedge, we can employ those tools in connection with the high school constructions to make other hyperbolic tools.

EXERCISES

***14.5.1.** Make a Poincaré perpendicular bisector tool. The tool should accept four points as givens (the center O of Γ, a point R on Γ, and the two endpoints A and B of the Poincaré segment) and it should return a Poincaré line as its result.
[Hint: The Pointwise Characterization of Perpendiculars (Chapter 0) is a neutral theorem. Thus you need only find two points that are equidistant from A and B and then take the Poincaré line through them. We have not yet measured Poincaré distances, but we do know how to construct Poincaré circles.]

***14.5.2.** Make a Poincaré midpoint tool.

***14.5.3.** Make a Poincaré perpendicular line tool; i.e., a tool that drops a perpendicular from a point P to a Poincaré line ℓ. The tool should accept five points as givens (the center O of Γ, a point R on Γ, two points A and B that determine a hyperbolic line ℓ, and a point P that is to lie on the perpendicular) and return a Poincaré line as result.
[Hint: Find two points on ℓ that are equidistant from P (using Poincaré distance). The perpendicular line is the perpendicular bisector of the segment connecting the two points.]

***14.5.4.** Make a Poincaré angle bisector tool. The tool should accept five points as givens (the center O of Γ, a point R on Γ, and three points A, B, and C that define a hyperbolic angle $\angle ABC$) and should return the angle bisector, which is a hyperbolic ray.

[Hint: The Pointwise Characterization of Angle Bisectors (Chapter 0) is a neutral theorem. You should be able to locate a point on the ray as the intersection of two hyperbolic circles.]

***14.5.5.** Use your hyperbolic tools to determine whether or not the Secant Theorem correct in hyperbolic geometry.

***14.5.6.** Make a hyperbolic compass tool that improves on the one you made in Exercise 14.4.1. The new tool should accept three points A, B, and C as givens and construct the Poincaré circle with center A and radius equal to the Poincaré distance from B to C.
[Hint: Find the perpendicular bisector of the Poincaré segment $\overline{AC}$. It is part of a Euclidean circle γ. Reflect B across γ to B'. Convince yourself that the circle with Poincaré center A that passes through B' is the circle you want.]

14.6 TRIANGLE CENTERS IN HYPERBOLIC GEOMETRY

We will use the tools constructed in the last several sections to explore triangle centers in hyperbolic geometry. Some of the classical triangle centers exist in hyperbolic geometry and others do not.

EXERCISES

***14.6.1.** The hyperbolic incenter and incircle.
- **(a)** Construct a Poincaré triangle and the three bisectors of the interior angles of the triangle. Verify that the angle bisectors are always concurrent, regardless of the shape of the triangle.
- **(b)** Construct the hyperbolic incircle for your triangle.

***14.6.2.** Hyperbolic median concurrence.
- **(a)** Construct a Poincaré triangle, the three midpoints of the sides, and the three medians. Verify that the three medians are concurrent regardless of the shape of the triangle.
- **(b)** The point of concurrence is called the hyperbolic centroid of the triangle. Make a tool that constructs the hyperbolic centroid.
- **(c)** Construct the hyperbolic medial triangle.

***14.6.3.** The hyperbolic circumcenter and circumcircle.
- **(a)** Construct a Poincaré triangle and the hyperbolic perpendicular bisectors of the three sides. Move the vertices of the triangle to vary the shape of the triangle. Verify that the perpendicular bisectors are concurrent for some triangles and not for others.
- **(b)** Can you find a triangle for which exactly two of the perpendicular bisectors intersect? If so, describe it.
- **(c)** Can you find a triangle for which all three of the perpendiculars are asymptotically parallel? If so, describe it.
- **(d)** Can you find a triangle for which all three of the perpendiculars admit a common perpendicular? If so, describe it.
- **(e)** Can you find a triangle for which exactly two of the perpendicular bisectors are asymptotically parallel? If so, describe it.
- **(f)** Verify that the triangle has a hyperbolic circumcircle if the three perpendicular bisectors are concurrent.

- **(g)** Make a tool that constructs the hyperbolic circumcenter and circumcircle, provided they exist.

14.6.4. Prove that if two of the hyperbolic perpendicular bisectors of the sides of a hyperbolic triangle intersect, then the third perpendicular bisector also passes through the point of intersection.

***14.6.5.** The hyperbolic orthocenter.

- **(a)** Construct a Poincaré triangle and the three hyperbolic altitudes. Verify that the altitudes are concurrent for some triangles and not for others.
- **(b)** Can you find a triangle for which exactly two of the altitudes intersect? If so, describe it.
- **(c)** Can you find a triangle for which all three of the altitudes are asymptotically parallel? If so, describe it.
- **(d)** Can you find a triangle for which all three of the altitudes admit a common perpendicular? If so, describe it.
- **(e)** Can you find a triangle for which exactly two of the altitudes are asymptotically parallel? If so, describe it.
- **(f)** Make a tool that constructs the hyperbolic orthocenter, provided it exists.
- **(g)** Construct the hyperbolic orthic triangle.

***14.6.6.** The hyperbolic Euler line.

Construct a Poincaré triangle that has both a hyperbolic circumcenter and a hyperbolic orthocenter. Let us agree to call the Poincaré line determined by the circumcenter and the orthocenter the hyperbolic Euler line for the triangle (provided the circumcenter and orthocenter exist). Now construct the hyperbolic centroid of the triangle. Does the hyperbolic centroid always lie on the hyperbolic Euler line?

14.7 MEASURING HYPERBOLIC ANGLES AND DISTANCES

In this section you will make tools that measure hyperbolic angles and hyperbolic distances. We will refer to the tools as the hyperbolic protractor and hyperbolic ruler, since those are the names of the tools we ordinarily use to measure the corresponding Euclidean quantities.

The formula for the Poincaré distance between two points A and B is given on page 97. It is

$$d(A, B) = \left|\ln[AB, PQ]\right| = \left|\ln \frac{(AP)(BQ)}{(AQ)(BP)}\right|,$$

where P and Q are the ideal endpoints of the Poincaré line determined by A and B (see Figure 14.1).

The definition of Poincaré angle measure on page 97 requires a little elaboration. Let A, B, and C be three points in the Poincaré disk that are noncollinear in the hyperbolic sense. The hyperbolic angle $\angle BAC$ is the union of the two hyperbolic rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$. If $A = O$, then the hyperbolic angle is also a Euclidean angle and the hyperbolic measure is defined to be the Euclidean measure.

Another possibility is that $\overrightarrow{AB}$ is part of a Euclidean line and $\overrightarrow{AC}$ is part of a Euclidean circle γ with center G. Let t be the line that is tangent to γ at A and choose a point C^* on t such that C and C^* are on the same side of m, where m is the Euclidean line determined by A and G (see the left half of Figure 14.5).[3] The Poincaré measure of the hyperbolic angle $\angle BAC$ is defined to be the ordinary measure of the Euclidean angle $\angle BAC^*$.

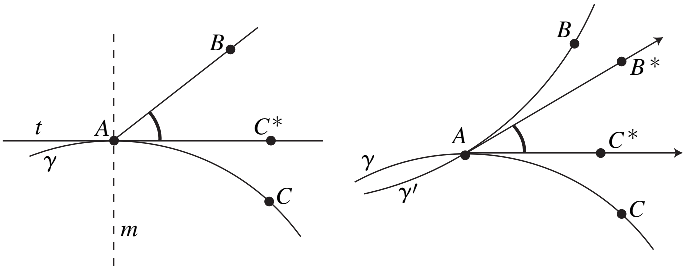

FIGURE 14.5: Definition of Poincaré angle measure

The final possibility is that $\overrightarrow{AC}$ is part of a Euclidean circle γ and $\overrightarrow{AB}$ is part of a Euclidean circle γ'. Choose points B^* and C^* on the tangent lines to γ' and γ as indicated in the right half of Figure 14.5. The Poincaré measure of the hyperbolic angle $\angle BAC$ is defined to be the ordinary measure of the Euclidean angle $\angle B^*AC^*$.

GSP allows the Euclidean angle to be measured in either radians or degrees, whichever is preferred.

EXERCISES

***14.7.1.** Make a tool that measures Poincaré distance between two points (a hyperbolic ruler). You may assume, when you make the tool, that the two points do not lie on a diameter of Γ.

- **(a)** How does your tool respond when you move the two points so that they do lie on a diameter?
- **(b)** What happens when you move one of the points so that it approaches Γ? Did you expect the Poincaré distance to approach infinity? Can you explain why it does not?
- **(c)** Change the size of the Poincaré disk you are using. Does this change the limiting value for $d(A, B)$ as B approaches the boundary of the disk?

***14.7.2.** Construct a Poincaré segment $\overline{AB}$, its perpendicular bisector ℓ, and a movable point C on ℓ. Now measure the distances $d(A, C)$ and $d(B, C)$ and compare the two. What happens when C is moved along ℓ?

***14.7.3.** Construct two Poincaré lines that meet at right angles. Let C be the point at which the two lines intersect, and choose points A and B on the lines so that $\triangle ABC$ is a Poincaré right triangle. Measure the lengths of the sides of the triangle and determine whether or not the Pythagorean Theorem is valid in hyperbolic geometry.

***14.7.4.** Construct a hyperbolic triangle $\triangle ABC$, the midpoints of the sides, and the three medians. The Euclidean Median Concurrence Theorem (Chapter 2) asserts

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[3] It is easy to check that the entire hyperbolic ray $\overrightarrow{AC}$ is on one side of m.

that the three medians are concurrent and that the point of concurrence divides each median into two segments, one exactly twice as long as the other. In Exercise 14.6.2 you verified that the concurrence part of the theorem is valid in hyperbolic geometry. Measure distances and determine whether or not the second part of the theorem (the part regarding the distances) is valid in hyperbolic geometry.

***14.7.5.** Make a tool that constructs the tangent Euclidean ray $\overrightarrow{AB^*}$ associated with a hyperbolic ray $\overrightarrow{AB}$ as in Figure 14.5.
[Hints: Again you must be careful when you make this tool to be certain that it always constructs the correct ray; if the point B^* is not specified carefully enough, the tool will sometimes give the opposite ray. The hyperbolic ray is part of a Euclidean circle γ. Locate the center G of γ and construct the tangent line t at A. Let D be the point at which ray $\overrightarrow{GB}$ intersects t. The point D could serve as B^* except for the fact that it might not be a point in the Poincaré disk. A more pleasing tool results if you trim the ray $\overrightarrow{AD}$ so that it lies inside the Poincaré disk and then choose a point B^* on the trimmed ray.]

***14.7.6.** Use the tool from the last exercise to make another tool that measures Poincaré angles (a hyperbolic protractor). The tool should accept five points as givens (the center O of Γ, a point R on Γ, and three points A, B, and C that define a hyperbolic angle $\angle ABC$) and should return the measure of the angle, which is a number. You may assume, when you make the tool, that A and B do not lie on a diameter of Γ and that A and C do not lie on a diameter of Γ.

- **(a)** How does your tool respond when you move A and B so that they do lie on a diameter?
- **(b)** What happens if you move A so that it is at the center of Γ?

***14.7.7.** Construct a Poincaré triangle and calculate its hyperbolic angle sum. How large can you make the angle sum? How small can you make it?

***14.7.8.** Construct a Poincaré circle α with Poincaré center A and construct two movable points B and C on α. Then $\triangle ABC$ is a hyperbolic isosceles triangle. Measure the base angles. Does the Isosceles Triangle Theorem hold in hyperbolic geometry?

***14.7.9.** Use Euclid's construction (Exercise 1.3.3) to construct an equilateral Poincaré triangle. Measure the perimeter of the triangle and the angle sum. Now change the lengths. How are the angle sum and perimeter of the triangle related?

***14.7.10.** Construct a quadrilateral $\square ABCD$ that has three right angles. (Such a quadrilateral is called a *Lambert quadrilateral.*) Measure the fourth angle. Can you adjust the vertices so that the fourth angle is right? How do the lengths of $\overline{AB}$ and $\overline{CD}$ compare?

***14.7.11.** Construct a quadrilateral $\square ABCD$ that has right angles at A and B and such that $d(B, C) = d(A, D)$. (Such a quadrilateral is called a *Saccheri quadrilateral.*) Measure the angles at C and D. Can you adjust the vertices so that all the angles are right? How does the length of $\overline{AB}$ compare with the length of $\overline{CD}$?