C H A P T E R 13

Inversions in circles

In this chapter we will investigate a class of transformations of the Euclidean plane called inversions in circles. The study of inversions is a standard part of college geometry courses, so we will leave most of the details of the subject to those courses and will not attempt to give a complete treatment here. Instead we will concentrate on the construction of the GSP tools that will be needed in the study of the Poincaré disk model of hyperbolic geometry in the next chapter. We will state the basic definitions and theorems of the subject, but will not prove the theorems. Proofs may be found in many sources; we will use §12.7 of [16] as our basic reference. In the exercises you will be asked to use the theorems to verify that certain constructions work and then to make GSP tools based on those constructions.

13.1 INVERTING POINTS

Let us begin with the definition of inversion.

Definition. Let $\mathcal{C} = \mathcal{C}(O, r)$ be a circle. The *inverse of P in $\mathcal{C}$* is the point $P' = I_{O,r}(P)$ on $\overrightarrow{OP}$ such that $(OP)(OP') = r^2$.

Observe that $I_{O,r}(P)$ is defined for any $P \neq O$, but that the definition breaks down when $P = O$. As P approaches O, the point $I_{O,r}(P)$ will move farther and farther from O. In order to define $I_{O,r}$ at O, we extend the plane by adding a *point at infinity*. This new point is denoted by the symbol ∞. The plane together with this one additional point at infinity is called the *inversive plane*. We extend $I_{O,r}$ to the inversive plane by defining $I_{O,r}(O) = \infty$ and $I_{O,r}(\infty) = O$.

The inversive plane is quite different from the extended Euclidean plane that was introduced in Chapter 8. In particular, the extended Euclidean plane includes an infinite number of different ideal points, while the inversive plane includes just one point at infinity. The inversion $I_{O,r}$ is a transformation of the inversive plane.[1] There is just one inversive plane on which every inversion is defined (not a different inversive plane for each inversion).

[1] A *transformation* is a function of a space to itself that is one-to-one and onto.

Construction. Let P be a point inside $\mathcal{C}$. Construct the perpendicular to $\overleftrightarrow{OP}$ at P, define T and U to be the points at which the perpendicular intersects $\mathcal{C}$, and then construct P' as shown in Figure 13.1.

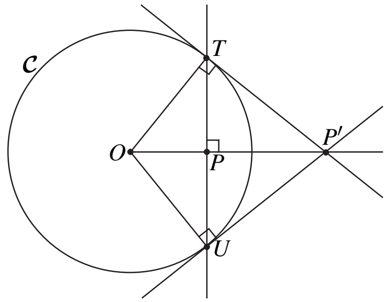

FIGURE 13.1: Contruction of P' in case P is inside $\mathcal{C}$

EXERCISES

***13.1.1.** Carry out the construction in GSP. Measure OP, OP', and OT and verify that $(OP)(OP') = (OT)^2$.

13.1.2. Prove that the point P' constructed above is the inverse of P.

***13.1.3.** Use the construction above to make a tool that finds $P' = I_{O,r}(P)$ for each point P inside $\mathcal{C}$. Your tool should accept three points as givens (the center O, a point on $\mathcal{C}$, and the point P) and it should return the point P' as its result.

Construction. Let P be a point outside $\mathcal{C}$. Find the midpoint M of $\overline{OP}$, construct a circle α with center M and radius MP, and let T and U be the two points of $\alpha \cap \mathcal{C}$. Then construct P' as shown in Figure 13.2.

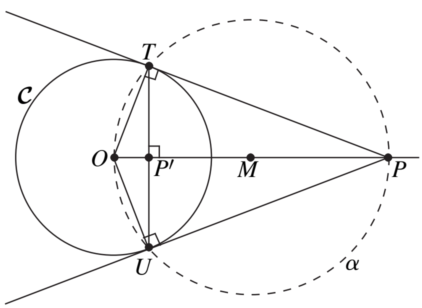

FIGURE 13.2: Contruction of P' in case P is outside of $\mathcal{C}$

EXERCISES

***13.1.4.** Carry out the construction in GSP. Measure OP, OP', and OT and verify that $(OP)(OP') = (OT)^2$.

13.1.5. Prove that the point P' constructed above is the inverse of P.

***13.1.6.** Use the construction above to make a tool that returns $P' = I_{O,r}(P)$ for each point P outside $\mathcal{C}$.

The drawback to the tools you made in the last several exercises is that one tool inverts points inside $\mathcal{C}$ but a different tool is required to invert points that are outside $\mathcal{C}$. In the next two exercises you will use a dilation to define one tool that inverts all points, whether they are inside $\mathcal{C}$ or outside $\mathcal{C}$. Note that $P' = I_{O,r}(P)$ is the point on $\overrightarrow{OP}$ that satisfies

$$OP' = \frac{r^2}{(OP^2)}(OP).$$

This does not make $I_{O,r}$ a dilation, because the scale factor relating OP' and OP is not constant, but it does allow us to invert an individual point using the Dilation command in GSP.

EXERCISES

***13.1.7.** Start with a circle $\mathcal{C} = \mathcal{C}(O, r)$ and a point $P \neq O$. Measure OP and r and then calculate $r^2/(OP)^2$. Select this quantity and choose Mark Ratio under the Transform menu. Next select O and then Mark Center. Now Dilate P using the Marked Ratio as the scale factor. Move P around in the plane to see that the resulting point P' is the inverse of P regardless of whether P is inside, on, or outside $\mathcal{C}$.

***13.1.8.** Use the construction in the last exercise to make a tool that inverts any point.

Suggestion. In this section you made several tools that find the inverse of a point in a circle. In subsequent sections you will make other tools that invert other objects, such as lines and circles. The givens for each of these tools consist of the center point O, a point R on $\mathcal{C}$ (which implicitly defines the radius r), and the object to be inverted. Since you will be inverting multiple objects in the same circle, it should not be necessary to reselect O and R every time you use one of your tools.

In order to make the tool easier to use, choose Show Script View under the Custom Tools menu. The Givens should be listed at the top of the script. Double click on the center point and a Properties box will appear. Type in a new label for the center and then click Automatically Match Sketch Object. The tool will still work the same way the first time you use it, but it will put the label you have chosen on the center. The next time you use the tool it will recognize this label and will not expect you to reselect the center. If you choose distinctive names that would never be assigned to other objects by GSP, you will only be required to enter the center and radius once.

Examples of distinctive names are "center of inversion" for O and "circle of inversion" for a point that determines $\mathcal{C}$. It is suggested that you consistently use the same names in all the tools you create in this chapter. If you use the same name for the center in all your tools, then you should only have to enter it once, even if you use several different tools to invert various objects. (Assuming, of course, that you want to invert all of them in the same circle.)

13.2 INVERTING CIRCLES AND LINES

One reason inversions are useful is that they preserve certain features of Euclidean geometry. Inversions are not isometries, and they do not preserve such basic properties as distance and collinearity, but they do preserve other geometric relationships. In this section we will see that inversions preserve circles and lines in the sense that the inverse of a circle is either a circle or a line and the inverse of a line is either a circle or a line. The following theorem makes that statement precise.

Theorem 13.2.1. *Let $I_{O,r}$ be inversion in the circle $\mathcal{C} = \mathcal{C}(O, r)$.*
Part 1. *If α is a circle that does not pass through O, then $I_{O,r}(\alpha)$ is a circle that does not pass through O.*
Part 2. *If α is a circle that passes through O, then $I_{O,r}(\alpha \setminus \{O\})$ is a line that does not pass through O.*
Part 3. *If ℓ is a line that does not pass through O, then $I_{O,r}(\ell \cup \{\infty\})$ is a circle that passes through O.*
Part 4. *If ℓ is a line that passes through O, then $I_{O,r}(\ell \cup \{\infty\}) = \ell \cup \{\infty\}$.*

Proof. See [16], Theorems 12.7.4 through 12.7.7. □

EXERCISES

***13.2.1.** Construct a circle $\mathcal{C}(O, r)$ and a circle α such that $O \notin \alpha$. Construct a movable point P on α and its inverse P'. Use the Locus command in the Construct menu to confirm that the locus of points P' is a circle.
(a) Where is the circle $I_{O,r}(\alpha)$ in case α is completely outside $\mathcal{C}$?
(b) Where is the circle $I_{O,r}(\alpha)$ in case α intersects $\mathcal{C}$ in two points?
(c) Where is the circle $I_{O,r}(\alpha)$ in case α is completely inside $\mathcal{C}$?
[Hint: To use the Locus command, select α, P, and P'; then choose Locus.]

***13.2.2.** Construct a circle $\mathcal{C}(O, r)$ and a circle α such that $O \in \alpha$. Construct a movable point P on α and its inverse P'. Use the Locus command in the Construct menu to confirm that the locus of points P' is a line.
(a) How is the line $I_{O,r}(\alpha \setminus \{O\})$ related to $\mathcal{C}$ and α in case $\mathcal{C}$ and α intersect in two points? (Draw a sketch.)
(b) How is the line $I_{O,r}(\alpha \setminus \{O\})$ related to $\mathcal{C}$ and α in case $\mathcal{C}$ and α intersect in one point? (Draw a sketch.)
(c) How is the line $I_{O,r}(\alpha \setminus \{O\})$ related to $\mathcal{C}$ and α in case α is contained inside $\mathcal{C}$?.

***13.2.3.** Construct a circle $\mathcal{C}(O, r)$ and a line ℓ such that $O \notin \ell$. Construct a movable point P on ℓ and its inverse P'. Use the Locus command to confirm that the locus of points P' is a circle through O.
(a) How is the circle $I_{O,r}(\ell \cup \{\infty\})$ related to $\mathcal{C}$ in case ℓ is outside $\mathcal{C}$?
(b) How is the circle $I_{O,r}(\ell \cup \{\infty\})$ related to $\mathcal{C}$ in case $\mathcal{C}$ and ℓ intersect in two points?

[Hint: The Locus command will not show you the inverse of every point on the (infinitely long) line ℓ, but only the inverses of the points on ℓ that you can see in your sketch. You can see a larger portion of the circle if you make $\mathcal{C}$ relatively small.]

***13.2.4.** Make a tool that inverts a circle α. The tool should accept five points as givens (the center O of inversion, a point R on the circle of inversion, and three points that determine α) and should return the circle α' as its result. You may assume that $O \notin \alpha$ when you make the tool, so that α' is simply the circumcircle of the inverses of three given points on α. What happens to α' when you move α through O?

***13.2.5.** Make a tool that inverts a line ℓ. The tool should accept four points as givens (the center O of inversion, a point R on the circle of inversion, and two points that determine ℓ) and should return the circle ℓ' as its result. You may assume that $O \notin \ell$ when you make the tool, so that ℓ' is simply the circumcircle of O and the inverses of two given points on ℓ. What happens to ℓ' when you move ℓ through O?

13.3 OTHOGONALITY

Another geometric relationship preserved by inversions is orthogonality.

Definition. Two circles are said to be *orthogonal* if they intersect and their tangent lines are perpendicular at the points of intersection.

Theorem 13.3.1. *Let $\mathcal{C} = \mathcal{C}(O, r)$ and α be two circles.*
Part 1. *If α is orthogonal to $\mathcal{C}$, then $I_{O,r}(P) \in \alpha$ for every $P \in \alpha$.*
Part 2. *If there exists a point $P \in \alpha$ such that $I_{O,r}(P) \in \alpha$ and $I_{O,r}(P) \neq P$, then α is orthogonal to $\mathcal{C}$.*

Proof. See [16], Theorems 12.7.9 through 12.7.11. □

The theorem will be used in the exercises below to construct two circles that are orthogonal to $\mathcal{C}$. The first construction allows us to specify two points that are to lie on the circle. The second construction allows us to specify one point on the circle and the tangent line at that point.

Construction. Let $\mathcal{C} = \mathcal{C}(O, r)$ be a circle. Let A and B be two points such that O, A, B are noncollinear and A and B are not on $\mathcal{C}$. Construct $A' = I_{O,r}(A)$ and then construct α, the circumcircle for A, B, and A'.

EXERCISES

***13.3.1.** Carry out the construction above and verify that α is orthogonal to $\mathcal{C}$.

13.3.2. Use Theorem 13.3.1 to prove that α is orthogonal to $\mathcal{C}$.

***13.3.3.** Make a GSP tool that constructs a circle through two points that is orthogonal to $\mathcal{C}$. The tool should accept four points as givens (the center O of inversion, a point R on the circle of inversion, and the two points A and B), and return the circle α as result.

(a) What happens to α when you move the two points so that they lie on a diameter of $\mathcal{C}$?

(b) What happens to α if one of A or B lies on $\mathcal{C}$?

Construction. Let $\mathcal{C} = \mathcal{C}(O, r)$ be a circle. Let t be a line and let P be a point on t. Construct the line ℓ that is perpendicular to t at P. Construct $P' = I_{O,r}(P)$ and $Q = \rho_\ell(P)$, the reflection of P in ℓ. Construct the circumcircle α of P, P', and Q (see Figure 13.3).

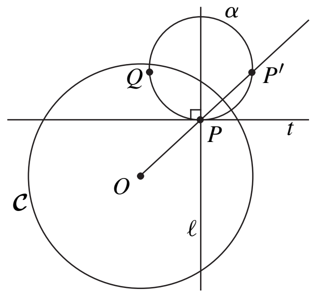

FIGURE 13.3: Construction of circle tangent to t at P

EXERCISES

***13.3.4.** Carry out the construction above and verify that α is a circle that is tangent to t and orthogonal to $\mathcal{C}$.

13.3.5. Prove that α is tangent to t and orthogonal to $\mathcal{C}$.

***13.3.6.** Make a GSP tool that constructs a circle that is tangent to t at P and is orthogonal to $\mathcal{C}$. The tool should accept four points as givens (the center O of inversion, a point R on the circle of inversion, the point P and a second point on t), and return the circle α as result. What happens to α if O lies on ℓ?

13.4 ANGLES AND DISTANCES

Inversions preserve angles in the following sense: Given two intersecting lines in the plane, the inverses of these lines will be two intersecting circles. The angles between the lines are congruent to the angles between the tangent lines of the circles. Inversions do not preserve individual distances, but they do preserve the following combination of distances.

Definition. Let A, B, P, and Q be four distinct points. The *cross-ratio* $[AB, PQ]$ of the four points is defined by

$$[AB, PQ] = \frac{(AP)(BQ)}{(AQ)(BP)}.$$

EXERCISES

***13.4.1.** Construct two intersecting lines ℓ and m. Use the tool of Exercise 13.2.5 to invert the two lines. The result should be two circles that intersect at two points, one of which is O. Construct the two tangent lines at the other point of intersection. Measure the angles between ℓ and m and then measure the angles between the tangent lines to ℓ' and m'. Are the measures equal?

***13.4.2.** Construct four points A, B, P, and Q. Measure the distances and calculate $[AB, PQ]$. Now invert the four points in a circle $\mathcal{C}(O, r)$ and calculate $[A'B', P'Q']$. Verify that $[AB, PQ] = [A'B', P'Q']$.

***13.4.3.** Make a tool that calculates the cross ratio of four points.