CHAPTER 12

More topics in triangle geometry

This chapter examines a number of loosely connected topics regarding the geometry of the triangle. We will explore the the statements of the theorems using GSP, but will not prove them.

12.1 NAPOLEON'S THEOREM AND THE NAPOLEON POINT

The theorem in this section is commonly attributed to the French emperor Napoleon Bonaparte (1769-1821). Napoleon was an amateur mathematician, with a particular interest in geometry, who took pride in his mathematical talents. Thus this attribution may be based at least partially on historical fact. On the other hand, Coxeter and Greitzer [3, page 63] make the following comment regarding the possibility that the theorem might in fact be due to Napoleon: "...the possibility of his knowing enough geometry for this feat is as questionable as the possibility of his knowing enough English to compose the famous palindrome ABLE WAS I ERE I SAW ELBA."

EXERCISES

***12.1.1.** Construct a triangle $\triangle ABC$ and use the tool you made in Exercise 3.1.4 to construct an equilateral triangle on each side of $\triangle ABC$. Make sure that the three new triangles are on the outside of the original triangle. Label vertices so that the three new triangles are $\triangle A'BC$, $\triangle AB'C$, and $\triangle ABC'$. Construct the three centroids of triangles $\triangle A'BC$, $\triangle AB'C$, and $\triangle ABC'$ and label them U, V, and W, respectively. Verify the following.

(a) $\triangle UVW$ is equilateral.

(b) Lines $\overleftrightarrow{AU}$, $\overleftrightarrow{BV}$, and $\overleftrightarrow{CW}$ are concurrent.

***12.1.2.** Do the same construction as in the previous exercise, but this time construct the equilateral triangles so that they are oriented towards the inside of $\triangle ABC$. (The equilateral triangles will overlap and may even stick out of $\triangle ABC$.) Do the same two conclusions hold in this case?

Definition. The triangle $\triangle UVW$ in Exercise 12.1.1 is called the *Napoleon triangle* associated with $\triangle ABC$. *Napoleon's theorem* is the assertion that the Napoleon triangle is always equilateral, regardless of the shape of $\triangle ABC$. The point at which lines $\overleftrightarrow{AU}$, $\overleftrightarrow{BV}$, and $\overleftrightarrow{CW}$ concur is called the *Napoleon point* of $\triangle ABC$. It is another new triangle center.

12.1.1 The Torricelli point

There is still another triangle center that is closely related to the Napoleon point. This new point is called the Torricelli point of the triangle. Be careful not to confuse it with the Napoleon point. The point is named for the Italian Mathematician Evangelista Torricelli (1608–1647).

EXERCISES

***12.1.3.** Construct a triangle $\triangle ABC$ and construct external equilateral triangles on the three sides of the triangle. Label the vertices as in Exercise 12.1.1. Construct the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$. Note that they are concurrent. The point at which these three lines concur is called the *Torricelli point* of $\triangle ABC$.

***12.1.4.** For which triangles is the Torricelli point inside the triangle and for which is it outside?

***12.1.5.** For which triangles is the Torricelli point equal to the Napoleon point?

***12.1.6.** Construct the circumcircles for the three triangles $\triangle A'BC$, $\triangle AB'C$, and $\triangle ABC'$. Observe that all three of these circles pass through the Torricelli point. Thus an alternative way to define the Torricelli point would be to say that it is the point at which the three circumcircles of the external equilateral triangles intersect.

12.1.2 van Aubel's theorem

There is also a theorem for quadrilaterals that is closely related to Napoleon's theorem for triangles. The theorem for quadrilaterals is known as *van Aubel's theorem*. The *center of a square* is the point at which the diagonals intersect. The center of a square is obviously equidistant from the vertices.

EXERCISES

***12.1.7.** Construct a convex quadrilateral $\square ABCD$. Construct a square on each side of the quadrilateral. Make sure the fours squares are all on the outside of the quadrilateral. Construct the centers of the four squares and the two segments joining the centers of the squares based on opposite sides of $\square ABCD$. Measure the lengths of these segments. What do you observe? Measure the angles between the two segments. What do you observe?

***12.1.8.** Now try the same thing with the squares constructed towards the inside of the quadrilateral. Is the result still true in this case?

12.2 MIQUEL'S THEOREM AND MIQUEL POINTS

The theorem in this section is attributed to the nineteenth century French mathematician Auguste Miquel. We have proved many theorems regarding concurrent lines; by contrast, this theorem gives a condition under which three circles are concurrent.

EXERCISES

***12.2.1.** Construct a triangle $\triangle ABC$ and movable points D, E, and F on the sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Now construct the circumcircles for the three triangles $\triangle AEF$, $\triangle BDF$, and $\triangle CDE$. Observe that the three circles are concurrent regardless of the shape of $\triangle ABC$ and regardless of how the points D, E and F are chosen. Find examples for which the point of concurrence is inside the triangle as well as examples for which it is outside.

***12.2.2.** What happens if the points D, E, and F are chosen to lie on the sidelines of $\triangle ABC$ rather than the sides?

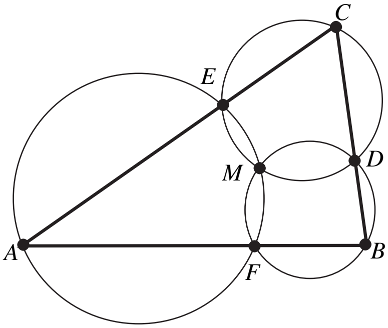

FIGURE 12.1: Miquel's Theorem

The assertion that the three circles in Exercise 12.2.1 are concurrent is known as *Miquel's theorem*. The point of concurrence is called a *Miquel point* for the triangle.

12.3 THE FERMAT POINT

The triangle center studied in this section is named after Pierre de Fermat (1601–1665).

Definition. The *Fermat point* of a triangle is the point F for which the sum of the distances from F to the vertices is as small as possible; i.e., the Fermat point of $\triangle ABC$ is the point F such that $FA + FB + FC$ is minimized.

In the exercises below you will see that for most triangles the Fermat point is the point that makes equal angles with of the vertices. For such triangles the Fermat point is equal to the Torricelli point.

EXERCISES

***12.3.1.** Construct a triangle $\triangle ABC$ such that all three angles in the triangle measure less than 120°. Construct a point P in the interior of the triangle and calculate $PA + PB + PC$. Move P around until you locate the Fermat point. Once you have found the point P for which $PA + PB + PC$ is minimized, measure the three angles $\angle APB$, $\angle BPC$, and $\angle CPA$. Observe that the sum of the distances is minimized at the same point where the three angles are congruent.

***12.3.2.** Now construct a triangle $\triangle ABC$ in which angle $\angle BAC$ has measure greater than 120°. Where is the Fermat point for this triangle located?

***12.3.3.** Construct a triangle $\triangle ABC$ and its Torricelli point. Verify that the Torricelli point T and the Fermat point F are the same in case all angles in the triangle have measure less than 120°. Find examples of triangles for which the Torricelli point and the Fermat point are different.

12.4 MORLEY'S THEOREM

Our final theorem requires that we trisect the angles of a triangle. To *trisect* an angle $\angle BAC$ means to find two rays $\overrightarrow{AD}$ and $\overrightarrow{AE}$ such that $\overrightarrow{AD}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and $\mu(\angle BAD) = (1/3)\mu(\angle BAC)$ while $\overrightarrow{AE}$ is between $\overrightarrow{AD}$ and $\overrightarrow{AC}$ and $\mu(\angle DAE) = (1/3)\mu(\angle BAC)$. It follows easily that $\mu(\angle EAC) = (1/3)\mu(\angle BAC)$ as well, so the original angle is divided into three congruent angles.

The ancient Greeks were interested in the problem of trisecting an angle using only a straightedge and compass. They never succeeded in trisecting an arbitrary angle, and it has been known since the nineteenth century that it is impossible to do so using only the Euclidean compass and straightedge. But it is quite easy to trisect angles using the measurement and calculation capabilities of GSP.

EXERCISES

***12.4.1.** Make a tool that trisects an angle. The tool should accept three points A, B, and C as givens and should produce as results the two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ together with the two additional rays in the interior of $\angle BAC$ that trisect the angle.

[Hint: Measure the angle $\angle BAC$ and calculate $(1/3)\mu(\angle BAC)$. Rotate $\overrightarrow{AB}$ through $(1/3)\mu(\angle BAC)$ to construct the first trisecting ray and then rotate again through the same angle to produce the second trisecting ray.]

***12.4.2.** Construct a triangle $\triangle ABC$. Construct the two angle trisectors for each of the three interior angles of $\triangle ABC$. These six rays intersect in a total of twelve points in the interior of $\triangle ABC$. Label the point at which the rays through B and C that are closest to $\overline{BC}$ intersect as A'. Similarly, label the intersection of the two trisectors closest to $\overline{AC}$ as B' and label the intersection of the two trisectors closest to $\overline{AB}$ as C'. The triangle $\triangle A'B'C'$ is called the *Morley triangle* for the triangle $\triangle ABC$. (See Figure 12.2.)

***12.4.3.** Hide all the angle trisectors and concentrate on the Morley triangle $\triangle A'B'C'$. Measure all the sides and the angles of $\triangle A'B'C'$ and verify that it is always equilateral.

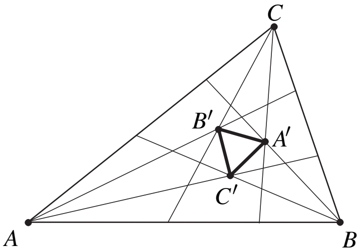

FIGURE 12.2: $\triangle A'B'C'$ is the Morley triangle for $\triangle ABC$

The theorem asserting that the Morley triangle is equilateral regardless of the shape of the original triangle is known as *Morley's theorem*. It was discovered around 1904 by the American mathematician Frank Morley (1869–1937). It is the most recently discovered theorem we have studied.