CHAPTER 11

Applications of the theorem of Menelaus

This chapter contains a sampling of the many applications of the theorem of Menelaus.

11.1 TANGENT LINES AND ANGLE BISECTORS

The first applications are simple results about how tangent lines and angle bisectors intersect the sidelines of the triangle. All applications in this section use the trigonometric form of Menelaus's theorem.

EXERCISES

***11.1.1.** Construct a triangle and its circumscribed circle. For each vertex of the triangle, construct the line that is tangent to the circumcircle at that point. Mark the point at which the line that is tangent at a vertex intersects the opposite sideline of the triangle. Verify that the three points you have marked are always collinear. Under what conditions is one or more of the intersection points an ideal point?

11.1.2. Let $\triangle ABC$ be a triangle. Prove that the lines that are tangent to the circumcircle of $\triangle ABC$ at the vertices of the triangle cut the opposite sidelines at three collinear points.
[Hint: Use Euclid's Proposition III.32.]

***11.1.3.** Construct a triangle $\triangle ABC$. At each vertex of the triangle, construct the line that bisects the two exterior angles at that vertex. Mark the point at which the bisector intersects the opposite sideline. Verify that the three points you have marked are always collinear. Under what conditions is one of more of the intersection points an ideal point?

11.1.4. Let $\triangle ABC$ be a triangle. Prove that the external angle bisectors of $\triangle ABC$ meet the opposite sidelines of the triangle in three collinear points.

***11.1.5.** Construct a triangle $\triangle ABC$. At each of vertices A and B construct the line that bisects the interior angle at that vertex and mark the point at which the bisector intersects the opposite sideline. At vertex C, construct the line that bisects the exterior angles and mark the point at which the bisector intersects the opposite sideline. Verify that the three points you have marked are always collinear. Under what conditions is one or more of the intersection points an ideal point?

11.1.6. Let $\triangle ABC$ be a triangle. Prove that internal angle bisectors at A and B and the external angle bisector at C meet the opposite sidelines of the triangle in three collinear points.

11.2 DESARGUES'S THEOREM

The theorem in this section is due to Girard Desargues (1591–1661). It has important applications to the theory of perspective drawing in art.

Definition. Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are said to be *perspective from the point O* if the three lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ joining corresponding vertices are concurrent at O. The point O is called the *perspector* or the *point of perspective* (see Figure 11.1).

Some authors use the term *copolar* to describe triangles that are perspective from a point.

Definition. Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are said to be *perspective from a line* if the three points L, M, and N at which corresponding sidelines $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$, $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$, and $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ intersect are collinear. The line containing L, M, and N is called the *perspectrix* (see Figure 11.1).

Some authors use the term *coaxial* to describe triangles that are perspective from a line.

Desargues's Theorem. *Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are perspective from a point if and only if they are perspective from a line.*

EXERCISES

***11.2.1.** Make a GSP sketch showing two triangles that are perspective from a point. Verify that they are perspective from a line. What happens when the perspector is an ideal point? What happens when one or more of the pairs of corresponding sidelines of the triangles are parallel? How many pairs of corresponding sides can be parallel?

***11.2.2.** Make a GSP sketch showing two triangles that are perspective from a line. Verify that they are perspective from a point.

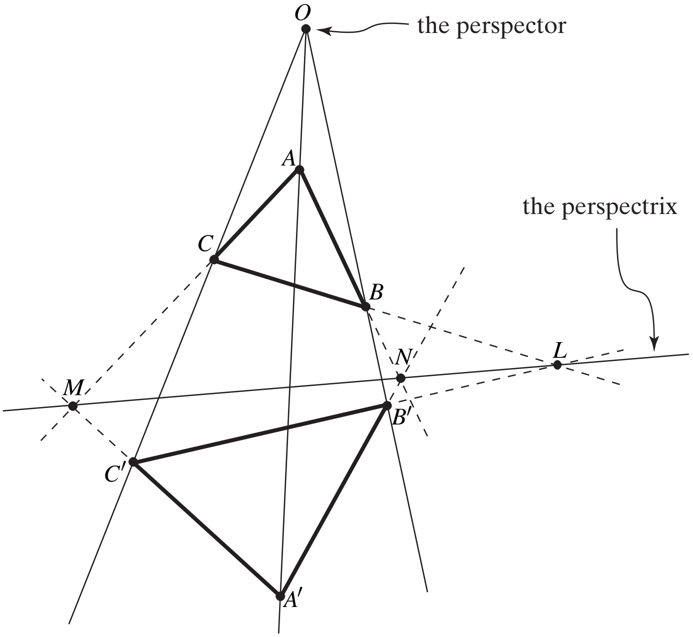

FIGURE 11.1: Desargues's Theorem

11.2.3. Prove that if the triangles $\triangle ABC$ and $\triangle A'B'C'$ are perspective from a point, then they are perspective from a line.
[Hint: Assume that the two triangles are perspective from O. The points L, M, and N may be defined as the intersections of corresponding sidelines of the two triangles. Explain why it is enough to prove that

$$\frac{\mathbf{AN}}{\mathbf{NB}} \cdot \frac{\mathbf{BL}}{\mathbf{LC}} \cdot \frac{\mathbf{CM}}{\mathbf{MA}} = -1.$$

Apply Menelaus's theorem to the triangle $\triangle OAB$ with collinear Menelaus points N, A', and B'. Apply it in a similar way to $\triangle OAC$ and $\triangle OBC$, and then multiply the resulting equations together.]

11.2.4. Prove that if the triangles $\triangle ABC$ and $\triangle A'B'C'$ are perspective from a line, then they are perspective from a point.
[Hint: Assume that the points L, M, and N are collinear. Define O to be the point of intersection of $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$. Explain why it is enough to prove that A, A', and O are collinear. Note that the triangles $\triangle MCC'$ and $\triangle NBB'$ are perspective from the point L, so you can apply the part of the theorem that you have already proved to them.]

11.3 PASCAL'S MYSTIC HEXAGRAM

Polygons inscribed in circles have some surprising properties. One of the most interesting was discovered by Blaise Pascal (1623–1662). Pascal discovered the next theorem when he was only sixteen years old and he gave it the colorful Latin title *mysterium hexagrammicum*. For that reason the theorem is still referred to as *Pascal's mystic hexagram*.

Definition. A *hexagon* is a polygon with six vertices $ABCDEF$. It is required that no three consecutive vertices (in cyclic order) are collinear. Just as is the case with quadrilaterals, we allow the sides to cross. A hexagon is *inscribed* in the circle γ if all the vertices lie on γ.

Pascal's Mystic Hexagram. *If a hexagon is inscribed in a circle, then the three points at which opposite sidelines intersect are collinear.*

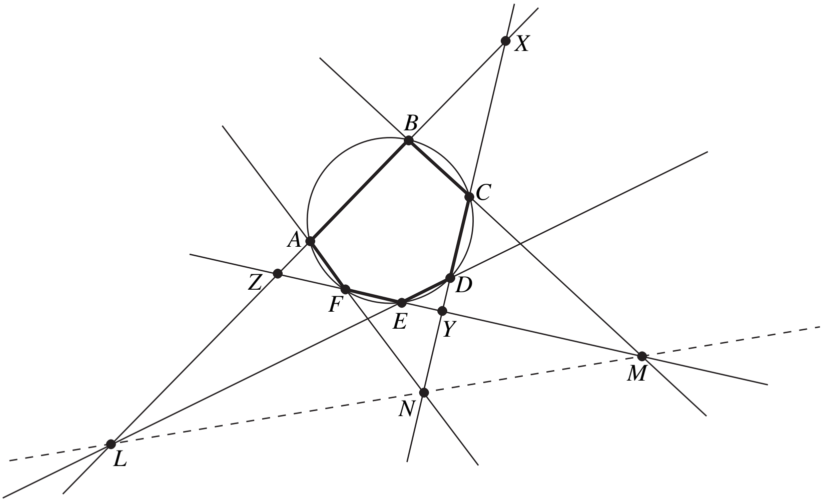

FIGURE 11.2: Pascal's Mystic Hexagram

EXERCISES

*__11.3.1.__ Construct a circle and six points A, B, C, D, E, and F, cyclically ordered around the circle. Draw the lines through the sides and mark the points at which the opposite sidelines intersect. Verify that these three points are collinear. Does the theorem hold for crossed hexagons whose vertices lie on a circle?

*__11.3.2.__ Draw examples of inscribed hexagons for which 0, 1, and 3 of the points of intersection are ideal points. Is it possible for exactly two of the points to be ideal?

11.3.3. Explain how the result of Exercise 11.1.1 can be viewed as a limiting case of Pascal's theorem.
[Hint: Take the limit as pairs of adjacent vertices of the hexagon converge.]

Notation. Let γ be a circle, and let A, B, C, D, E, and F be six points on γ, cyclically ordered around the circle. Define six additional points as follows.

L is the point at which $\overleftrightarrow{AB}$ meets $\overleftrightarrow{DE}$
M is the point at which $\overleftrightarrow{BC}$ meets $\overleftrightarrow{EF}$
N is the point at which $\overleftrightarrow{CD}$ meets $\overleftrightarrow{AF}$
X is the point at which $\overleftrightarrow{AB}$ meets $\overleftrightarrow{CD}$
Y is the point at which $\overleftrightarrow{EF}$ meets $\overleftrightarrow{CD}$
Z is the point at which $\overleftrightarrow{AB}$ meets $\overleftrightarrow{EF}$

EXERCISES

***11.3.4.** Use GSP to make a sketch showing the circle γ and the twelve points listed above. Observe that L, M, and N are Menelaus points for $\triangle XYZ$.

11.3.5. Prove Pascal's theorem in case all three of the points X, Y, and Z are ordinary points.
[Hint: The proof is accomplished by using Menelaus's theorem to show that L, M, and N are collinear Menelaus points for the triangle $\triangle XYZ$. Apply Menelaus's theorem three times to $\triangle XYZ$ with collinear Menelaus points $\{B, C, M\}$, $\{A, F, N\}$, and $\{D, E, L\}$. Multiply the resulting equations together and apply Euclid's Proposition III.36 to reach the desired conclusion.]

11.3.6. Prove Pascal's theorem in case one of the points X, Y, Z is an ideal point.
[Hint: Instead of using X, Y, and Z, use the points of intersection of $\overleftrightarrow{BC}$ with $\overleftrightarrow{DE}$, $\overleftrightarrow{DE}$ with $\overleftrightarrow{FA}$ and $\overleftrightarrow{FA}$ with $\overleftrightarrow{BC}$. First check that if one of X, Y, or Z is ideal, then all three of these points are ordinary. Then proceed as in the previous proof.]

11.4 BRIANCHON'S THEOREM

As mentioned at beginning of chapter, Brianchon's theorem is dual to Pascal's theorem. It illustrates a new aspect of duality: a point on circle is dual to a line that is tangent to the circle. The theorem is included in this chapter because it is such a beautiful example of duality, even though the proof does not use Menelaus's theorem. Instead the proof is based on the radical center theorem. Brianchon's theorem is named for its discoverer, Charles Julien Brianchon (1783–1864).

Brianchon's Theorem. *If a hexagon is circumscribed about a circle, then the lines determined by pairs of opposite vertices are concurrent.*

EXERCISES

***11.4.1.** Make a GSP sketch that can be used to verify and illustrate Brianchon's theorem.

The proof relies on a construction that is relatively easy to reproduce with GSP, but which few people would think of for themselves. We will describe the construction first, and then look at the proof itself. As you read the next two paragraphs you should make a GSP sketch that reproduces Figure 11.3.

Let $\mathcal{C}$ be a circle and let $ABCDEF$ be a hexagon that is circumscribed about $\mathcal{C}$. More specifically, this means that all the vertices of $ABCDEF$ lie outside $\mathcal{C}$ and each side of $ABCDEF$ is tangent to $\mathcal{C}$ at an interior point of that side. Let P, Q, R, S, T, and U be the points of tangency, labeled as in Figure 11.3.

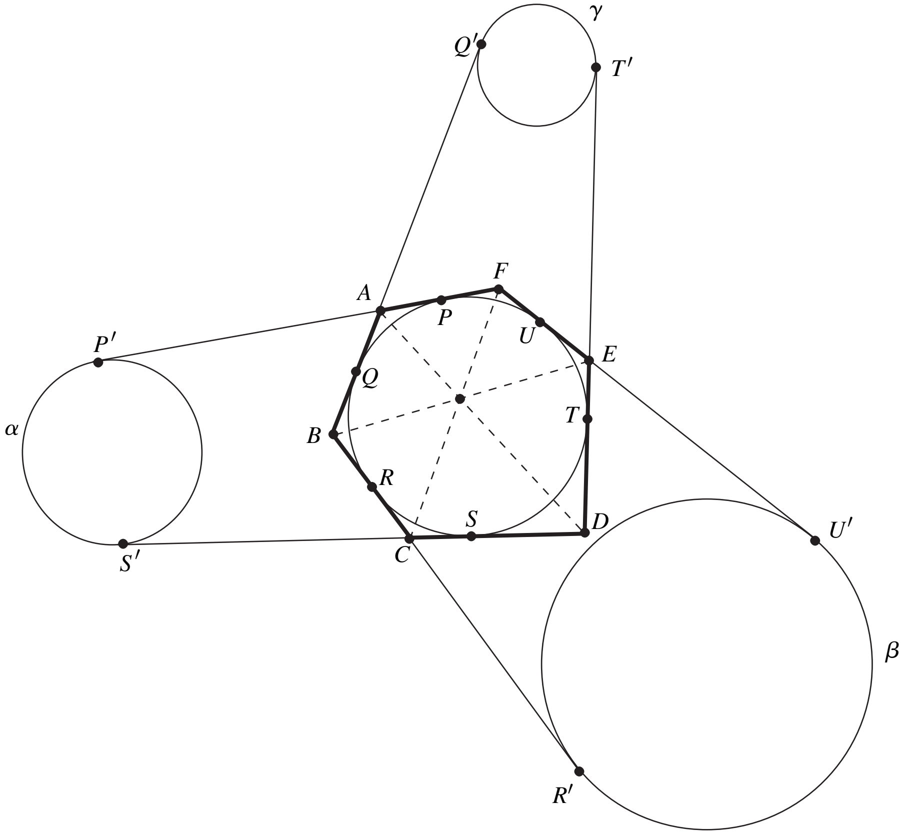

FIGURE 11.3: Construction for the proof of Brianchon's theorem

Construct points P' on $\overrightarrow{FA}$, Q' on $\overrightarrow{BA}$, R' on $\overrightarrow{BC}$, S' on $\overrightarrow{DC}$, T' on $\overrightarrow{DE}$, and U' on $\overrightarrow{FE}$ such that $PP' = QQ' = RR' = SS' = TT' = UU'$. (The precise value of these distances is not important, provided all six of the distances are equal.) Construct three circles α, β, and γ such that α is tangent to $\overleftrightarrow{AF}$ at P' and tangent to $\overleftrightarrow{CD}$ at S', β is tangent to $\overleftrightarrow{BC}$ at R' and tangent to $\overleftrightarrow{FE}$ at U', and γ is tangent to $\overleftrightarrow{AB}$ at Q' and tangent to $\overleftrightarrow{DE}$ at T'. These three circles exist by Exercise 11.4.2.

EXERCISES

11.4.2. Let $\mathcal{C}$ be a circle and let t and s be two lines that are tangent to $\mathcal{C}$ at points P and S, respectively. If P' is a point on t and $S' \neq P'$ is a point on s such that P'

and S' are on the same side of $\overleftrightarrow{PS}$ and $PP' = SS'$, then there exists a circle α that is tangent to t and s at P' and S'.
[Hint: The circles $\mathcal{C}$ and α in Figure 11.3 illustrate the exercise. First prove the theorem in case $\overleftrightarrow{PP'} \parallel \overleftrightarrow{SS'}$ and then use similar triangles to prove the other case.]

11.4.3. Prove that $FP' = FU'$, $CS' = CR'$, $BQ' = BR'$, $ET' = EU'$, $AQ' = AP'$, and $DT' = DS'$ in Figure 11.3.

11.4.4. Prove that $r(\alpha, \beta) = \overleftrightarrow{CF}$, $r(\beta, \gamma) = \overleftrightarrow{BE}$, and $r(\alpha, \gamma) = \overleftrightarrow{AD}$.

11.4.5. Use the preceding exercise and the radical center theorem to prove Brianchon's theorem.

11.5 PAPPUS'S THEOREM

The next theorem was discovered by Pappus of Alexandria. Pappus, who live from approximately AD 290 until about 350, was one of the last great geometers of antiquity. Much later his theorem became an important result in the foundations of projective geometry. Both the statement and proof of Pappus's theorem are reminiscent of Pascal's theorem. Again there is a hexagon involved. In this case the vertices of the hexagon lie on two lines rather than on a circle. The hexagon is necessarily a crossed hexagon.

Pappus's Theorem. *Let A, B, C, D, E, and F be six points. Define L to be the point at which* $\overleftrightarrow{AB}$ *meets* $\overleftrightarrow{DE}$*, define M to be the point at which* $\overleftrightarrow{BC}$ *meets* $\overleftrightarrow{EF}$*, and define N to be the point at which* $\overleftrightarrow{CD}$ *meets* $\overleftrightarrow{AF}$*. If A, C, E lie on one line and B, D, F lie on another line, then the points L, M, and N are collinear.*

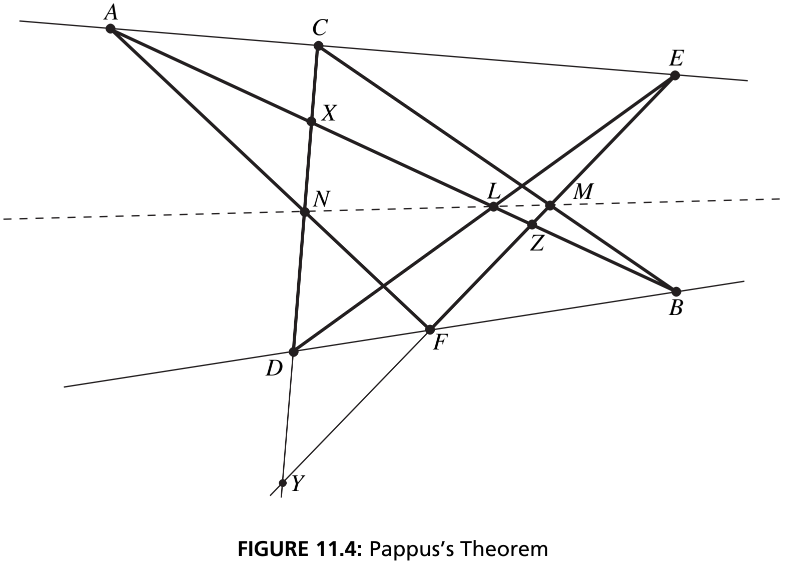

FIGURE 11.4: Pappus's Theorem

EXERCISES

***11.5.1.** Make a sketch that can be used to verify and illustrate Pappus's theorem.

Before we begin the proof of the theorem, let us establish some notation (see Figure 11.4).

Notation. Start with six points A, B, C, D, E, and F. Assume A, C, and E lie on the line ℓ and that B, D, and F lie on a line m. Define six additional points as follows.

L is the point at which $\overleftrightarrow{AB}$ meets $\overleftrightarrow{DE}$
M is the point at which $\overleftrightarrow{BC}$ meets $\overleftrightarrow{EF}$
N is the point at which $\overleftrightarrow{CD}$ meets $\overleftrightarrow{AF}$
X is the point at which $\overleftrightarrow{AB}$ meets $\overleftrightarrow{CD}$
Y is the point at which $\overleftrightarrow{EF}$ meets $\overleftrightarrow{CD}$
Z is the point at which $\overleftrightarrow{AB}$ meets $\overleftrightarrow{EF}$

In order to keep the proof relatively simple, we will only prove Pappus's theorem in case the three points X, Y, and Z are ordinary points.

EXERCISES

***11.5.2.** Use GSP to make a sketch showing the ℓ and m along with the twelve points listed above. Observe that L, M, and N are Menelaus points for $\triangle XYZ$.

11.5.3. Prove Pappus's theorem in case the three points X, Y, and Z defined above are ordinary points.
[Hint: The proof is accomplished by using Menelaus's theorem to show that L, M, and N are collinear Menelaus points for the triangle $\triangle XYZ$. Apply Menelaus's theorem five times to $\triangle XYZ$ with collinear Menelaus points $\{B, C, M\}$, $\{A, F, N\}$, $\{D, E, L\}$, $\{A, C, E\}$, and $\{B, D, F\}$. Multiply the first three equations, divide by the product of the last two, and cancel lots of terms.]

11.6 SIMSON'S THEOREM

This theorem is usually attributed to the Scottish mathematician Robert Simson (1687–1768), but there is no record that he ever published it. The theorem was published in 1799 by William Wallace (1768–1843). The result was discussed earlier in the context of pedal triangles.

Simson's Theorem. *A point P is on the circumscribed circle of triangle $\triangle ABC$ if and only if the feet of the perpendiculars from P to the sidelines of $\triangle ABC$ are collinear.*

Definition. A line that contains the feet of three perpendiculars from a point P to the triangle $\triangle ABC$ is called a *Simson line* for $\triangle ABC$. The point P is called the *pole* of the Simson line.

We will prove Simson's theorem below, but first we explore several interesting properties of Simson lines.

EXERCISES

***11.6.1.** Construct a triangle $\triangle ABC$ and its circumcircle. Choose a point P and construct the feet of the three perpendiculars from P to the sidelines of the triangle. Put a line through two of the feet and verify that the third foot is on that line if and only if P is on the circumcircle.
[Hint: In order to make sure that you consider all possible shapes for $\triangle ABC$, it is best to construct the circumcircle first and then construct A, B, and C to be three movable points on the circle.]

***11.6.2.** Find triangles $\triangle ABC$ and points P on the circumcircle of $\triangle ABC$ such that the Simson line with pole P intersects $\triangle ABC$ in exactly two points. Now find examples for which the Simson line is disjoint from the triangle. Make observations about the shape of the triangles and the location of P for which the latter possibility occurs.

***11.6.3.** Let Q be the point at which the altitude through A meets the circumcircle. Verify that the Simson line with pole Q is parallel to the tangent to the circle at A.

***11.6.4.** Let P and Q be two points on the circumcircle of $\triangle ABC$. Verify that the measure of the angle between the Simson lines with poles P and Q is half the measure of $\angle POQ$ (where O is the circumcenter).

***11.6.5.** Verify that if $\overline{PQ}$ is a diameter for the circumcircle, then the Simson lines with poles P and Q intersect at a point on the nine-point circle.

***11.6.6.** Let H be the orthocenter of $\triangle ABC$. Verify that if P is any point on the circumcircle, then the midpoint of $\overline{HP}$ lies on the Simson line with pole P.

11.6.7. Prove that the feet of the perpendiculars are collinear if and only if P lies on the circumcircle.
[Hint: Apply Menelaus's theorem along with similar triangles and Euclid's Proposition III.22.]

***11.6.8.** Construct a circle γ and three movable points A, B, and C on γ. Construct an additional point P that lies on the arc of γ from A to C that does not contain B. Verify that B' is always between A' and C', whether the Simson line intersects $\triangle ABC$ or not.

11.6.9. Prove the result you verified in the last exercise.

11.7 PTOLEMY'S THEOREM

Ptolemy's theorem is a very useful result regarding cyclic quadrilaterals that is attributed to Claudius Ptolemaeus of Alexandria (85–165). It asserts that the product of the lengths of the two diagonals of a cyclic quadrilateral is equal to the sum of the product of the lengths of the opposite sides. Here is a precise statement of the theorem.

Ptolemy's Theorem. *If $\square ABCD$ is a cyclic quadrilateral, then*

$$AB \cdot CD + BC \cdot AD = AC \cdot BD.$$

EXERCISES

***11.7.1.** Make a GSP sketch that illustrates and verifies Ptolemy's theorem. Part of the definition of cyclic quadrilateral requires that the quadrilateral be convex; is the theorem valid for crossed quadrilaterals whose vertices lie on a circle?

11.7.2. Prove Ptolemy's theorem.
[Hint: Consider the Simson line for $\triangle ABC$ with pole D. By Exercise 11.6.9, B' is between A' and C', so $A'B' + B'C' = A'C'$. Apply Exercise 5.4.5 to reach the desired conclusion.]

11.8 THE BUTTERFLY THEOREM

The next theorem is known as the butterfly theorem. The reason for the name is evident from Figure 11.5.

Butterfly Theorem. *Assume γ is a circle, $\overline{PQ}$ is a chord of γ, and M is the midpoint of $\overline{PQ}$. Let $\overline{AB}$ and $\overline{CD}$ be two chords of γ such that A and C are on the same side of $\overleftrightarrow{PQ}$ and both $\overline{AB}$ and $\overline{CD}$ intersect $\overline{PQ}$ at M. If $\overline{AD}$ intersects $\overline{PQ}$ at X and $\overline{BC}$ intersects $\overline{PQ}$ at Y, then M is the midpoint of $\overline{XY}$.*

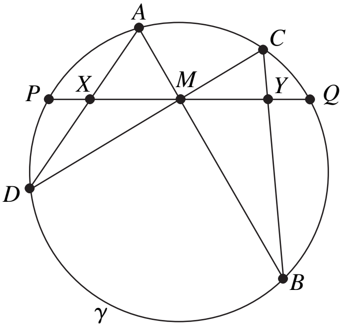

FIGURE 11.5: The Butterfly Theorem

EXERCISES

***11.8.1.** Make a GSP sketch that illustrates and verifies the butterfly theorem.

***11.8.2.** Verify that the butterfly theorem remains true even when the assumption that A and C are on the same side of $\overleftrightarrow{PQ}$ is dropped. In that case the segment $\overline{PQ}$ must be extended to the line $\overleftrightarrow{PQ}$ and the points X and Y lie outside γ. Can you find a butterfly in this diagram? While the result is true in this generality, we will only prove the theorem as stated above.

11.8.3. Prove the butterfly theorem in the special case in which $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$.
[Hint: Prove that M is the center of γ (in this case) and then use ASA to prove that $\triangle DMX \cong \triangle CMY$.]

11.8.4. Prove the following simple algebra result that that is needed in the proof of the butterfly theorem: *If x, y, and m are three positive numbers such that*

$$\frac{x^2(m^2 - y^2)}{y^2(m^2 - x^2)} = 1,$$

then $x = y$.

11.8.5. Prove the butterfly theorem in case $\overleftrightarrow{AD}$ intersects $\overleftrightarrow{BC}$.

[Hint: Let R be the point at which $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ intersect. To simplify the notation, let $x = MX$, $y = MY$, and $m = PM$. Apply Menelaus's theorem twice to $\triangle RXY$, first with collinear Menelaus points $\{A, M, B\}$ and then with collinear Menelaus points $\{C, M, D\}$. Multiply the resulting equations. Apply Euclid's Proposition III.36 three times to conclude that $RA \cdot RD = RB \cdot RC$, $XA \cdot XD = XP \cdot XQ$, and $YB \cdot YC = YP \cdot YQ$. All of that should result in the equation

$$\frac{x^2(m^2 - y^2)}{y^2(m^2 - x^2)} = 1.$$

Use Exercise 11.8.4 to complete the proof.]