

CHAPTER 0

A quick review of elementary Euclidean geometry

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This preliminary chapter lays out the basic results from elementary Euclidean geometry that will be assumed in the course. Those readers who are using this book as a supplement to a course in the foundations of geometry should omit the chapter and simply refer to it as needed for a summary of the notation and terminology that will be used in the remainder of the book. Readers who are using this book as a stand-alone text in Euclidean geometry should study the chapter carefully because the material in this chapter will be used in later chapters.

The theorems stated in this chapter are to be assumed without proof; they may be viewed as an extended set of axioms for the subject of advanced Euclidean geometry. The results in the exercises in the chapter should be proved using the theorems stated in the chapter. All the exercises in the chapter are results that will be needed later in the course.

We will usually refer directly to Euclid's *Elements* when we need a result from elementary Euclidean geometry. Several current editions of the *Elements* are listed in the bibliography (see [4], [5], or [10]). The *Elements* are in the public domain and are freely available on the world wide web. Euclid's propositions are referenced by book number followed by the proposition number within that book. Thus, for example, Proposition III.36 refers to the 36th proposition in Book III of the *Elements*.

0.1 MEASUREMENT AND CONGRUENCE

For each pair of points A and B in the plane there is a nonnegative number AB , called the *distance* from A to B . The *segment* from A to B , denoted \overline{AB} , consists of A and B together with all the points between A and B . The *length* of \overline{AB} is the distance from A to B . Two segments \overline{AB} and \overline{CD} are *congruent*, written $\overline{AB} \cong \overline{CD}$, if they have the same length. There is also a ray \overrightarrow{AB} and a line $\longleftrightarrow AB$.

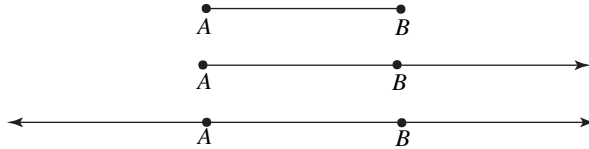


FIGURE 0.1: A segment, a ray, and a line

For each triple of points A , B , and C with $A \neq B$ and $A \neq C$ there is an *angle*, denoted $\angle BAC$, that is defined by $\angle BAC = \overrightarrow{AB} \cup \overrightarrow{AC}$. The *measure* of the angle is a number $\mu(\angle BAC)$. We will always measure angles in degrees and assume that $0 \leq \mu(\angle BAC) \leq 180^\circ$. The measure is 0° if the two rays \overrightarrow{AB} and \overrightarrow{AC} are equal; the measure is 180° if the rays are opposite; otherwise it is between 0° and 180° . An angle is *acute* if its measure is less than 90° , it is *right* if its measure equals 90° , and it is *obtuse* if its measure is greater than 90° . Two angles are *congruent* if they have the same measure.

The *triangle* with *vertices* A , B , and C consists of the points on the three segments determined by the three vertices; i.e.,

$$\triangle ABC = \overline{AB} \cup \overline{BC} \cup \overline{AC}.$$

The segments \overline{AB} , \overline{BC} , and \overline{AC} are called the *sides* of the triangle $\triangle ABC$. Two triangles are *congruent* if there is a correspondence between the vertices of the first triangle and the vertices of the second triangle such that corresponding angles are congruent and corresponding sides are congruent.

Notation. It is understood that the notation $\triangle ABC \cong \triangle DEF$ means that the two triangles are congruent under the correspondence $A \leftrightarrow D$, $B \leftrightarrow E$, and $C \leftrightarrow F$. The assertion that two triangles are congruent is really the assertion that there are six congruences, three angle congruences and three segment congruences. Specifically, $\triangle ABC \cong \triangle DEF$ means $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{AC} \cong \overline{DF}$, $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\angle CAB \cong \angle FDE$. In high school this is often abbreviated CPCTC (corresponding parts of congruent triangles are congruent).

0.2 PASCH'S AXIOM AND THE CROSSBAR THEOREM

The two results stated in this section specify how one-dimensional lines separate the two-dimensional plane. Neither of these results is stated explicitly in Euclid's

Elements. They are the kind of foundational results that Euclid took for granted. The first statement is named for Moritz Pasch (1843–1930).

Pasch's Axiom. Let $\triangle ABC$ be a triangle and let ℓ be a line such that none of the vertices A , B , and C lie on ℓ . If ℓ intersects \overline{AB} , then ℓ also intersects either \overline{BC} or \overline{AC} (but not both).

Let A , B , and C be three noncollinear points. A point P is in the interior of $\angle BAC$ if P is on the same side of \overleftrightarrow{AB} as C and on the same side of \overleftrightarrow{AC} as B .

Note that the interior of $\angle BAC$ is defined provided $0^\circ < \mu(\angle BAC) < 180^\circ$. It would be reasonable to define the interior of $\angle BAC$ to be the empty set in case $\mu(\angle BAC) = 0^\circ$, but there is no interior for an angle of measure 180° . The segment \overline{BC} is called a *crossbar* for $\angle BAC$.

Crossbar Theorem. If D is in the interior of $\angle BAC$, then there is a point G such that G lies on both \overrightarrow{AD} and \overline{BC} .

0.3 LINEAR PAIRS AND VERTICAL PAIRS

Angles $\angle BAD$ and $\angle DAC$ form a *linear pair* if A , B , and C are collinear and A is between B and C .

Linear Pair Theorem. If angles $\angle BAC$ and $\angle CAD$ form a linear pair, then $\mu(\angle BAC) + \mu(\angle CAD) = 180^\circ$.

Two angles whose measures add to 180° are called *supplementary angles* or *supplements*. The Linear Pair Theorem asserts that if two angles form a linear pair, then they are supplements.

Angles $\angle BAC$ and $\angle DAE$ form a *vertical pair* (or are *vertical angles*) if rays \overrightarrow{AB} and \overrightarrow{AE} are opposite and rays \overrightarrow{AC} and \overrightarrow{AD} are opposite or if rays \overrightarrow{AB} and \overrightarrow{AD} are opposite and rays \overrightarrow{AC} and \overrightarrow{AE} are opposite.

Vertical Angles Theorem. Vertical angles are congruent.

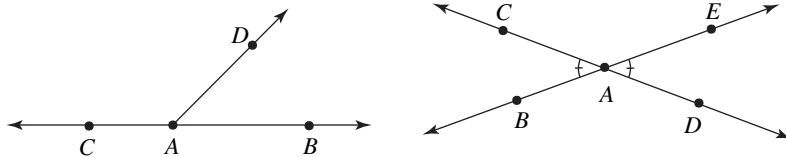


FIGURE 0.2: A linear pair and a vertical pair

The linear pair theorem is not found in the *Elements* because Euclid did not use angle measure; instead he simply called two angles “equal” if, in our terminology, they have the same measure. The vertical angles theorem is Euclid’s Proposition I.15.

0.4 TRIANGLE CONGRUENCE CONDITIONS

If you have two triangles and you know that three of the parts of one are congruent to the corresponding parts of the other, then you can usually conclude that the other three parts are congruent as well. That is the content of the triangle congruence conditions.

Side-Angle-Side Theorem (SAS). *If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\overline{AB} \cong \overline{DE}$, $\angle ABC \cong \angle DEF$, and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.*

Euclid used his “method of superposition” to prove SAS (Proposition I.4), but it is usually taken to be a postulate in modern treatments of geometry. The next two results (ASA and AAS) are both contained in Euclid’s Proposition I.26 and the third (SSS) is Euclid’s Proposition I.8.

Angle-Side-Angle Theorem (ASA). *If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\angle CAB \cong \angle FDE$, $\overline{AB} \cong \overline{DE}$, and $\angle ABC \cong \angle DEF$, then $\triangle ABC \cong \triangle DEF$.*

Angle-Angle-Side Theorem (AAS). *If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\overline{AC} \cong \overline{DF}$, then $\triangle ABC \cong \triangle DEF$.*

Side-Side-Side Theorem (SSS). *If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{CA} \cong \overline{FD}$, then $\triangle ABC \cong \triangle DEF$.*

There is no Angle-Side-Side condition, except in the special case in which the angle is a right angle.

Hypotenuse-Leg Theorem (HL). *If $\triangle ABC$ and $\triangle DEF$ are two right triangles with right angles at the vertices C and F , respectively, $\overline{AB} \cong \overline{DE}$, and $\overline{BC} \cong \overline{EF}$, then $\triangle ABC \cong \triangle DEF$.*

EXERCISES

0.4.1. Use SAS to prove the following theorem (Euclid’s Proposition I.5).

Isosceles Triangle Theorem. *If $\triangle ABC$ is a triangle and $\overline{AB} \cong \overline{AC}$, then $\angle ABC \cong \angle ACB$.*

0.4.2. Draw an example of two triangles that satisfy the ASS condition but are not congruent.

0.4.3. The *perpendicular bisector* of a segment \overline{AB} is a line ℓ such that ℓ intersects \overline{AB} at its midpoint and $\ell \perp \overline{AB}$. Prove the following theorem.

Pointwise Characterization of Perpendicular Bisector. *A point P lies on the perpendicular bisector of \overline{AB} if and only if $PA = PB$.*

0.4.4. The *angle bisector* of $\angle BAC$ is a ray \overrightarrow{AD} such that \overrightarrow{AD} is between \overrightarrow{AB} and \overrightarrow{AC} and $\mu(\angle BAD) = \mu(\angle DAC)$. The distance from a point to a line is measured along a perpendicular. Prove the following theorem.

Pointwise Characterization of Angle Bisector. *A point P lies on the bisector of $\angle BAC$ if and only if P is in the interior of $\angle BAC$ and the distance from P to \overleftrightarrow{AB} equals the distance from P to \overleftrightarrow{AC} .*

0.5 THE EXTERIOR ANGLE THEOREM

There is an inequality regarding the angles in a triangle that is of fundamental importance in many of the proofs of elementary geometry. The theorem is known as the Exterior Angle Theorem and it is Euclid's Proposition I.16.

Let $\triangle ABC$ be a triangle. At each vertex of the triangle there is an *interior angle* and two *exterior angles*. The interior angle at A is the angle $\angle BAC$. The two angles $\angle CAD$ and $\angle BAE$ shown in Figure 0.3 are the exterior angles at A . Note that the two exterior angles at a vertex form a vertical pair and are therefore congruent.

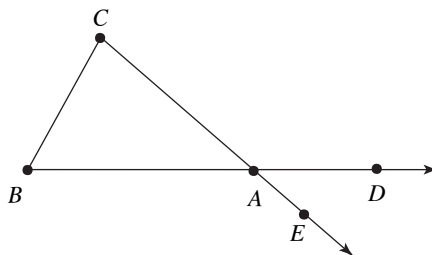


FIGURE 0.3: At each vertex there is one interior angle and there are two exterior angles

Exterior Angle Theorem. *The measure of an exterior angle for a triangle is strictly greater than the measure of either remote interior angle.*

0.6 PERPENDICULAR LINES AND PARALLEL LINES

Two lines ℓ and m are *perpendicular*, written $\ell \perp m$, if they intersect at right angles. If ℓ is a line and P is any point, then there is exactly one line m such that P lies on m and $m \perp \ell$. The point at which m intersects ℓ is called the *foot* of the perpendicular from P to ℓ . In case P lies on ℓ , P itself is the foot of the perpendicular. The process of constructing the perpendicular m is called *dropping a perpendicular*—see Figure 0.4. Euclid proved that it is possible to construct the unique perpendicular with compass and straightedge (Proposition I.12).

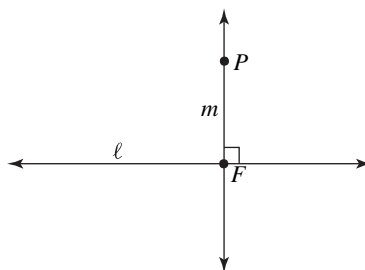


FIGURE 0.4: F is the foot of the perpendicular from P to ℓ

Two lines ℓ and m in the plane are *parallel*, written $\ell \parallel m$, if they do not intersect. It is the existence and uniqueness of parallels that distinguishes Euclidean geometry from non-Euclidean geometries. The Euclidean parallel property is stated most succinctly in the following postulate.

Playfair's Postulate. *For every line ℓ and for every point P that does not lie on ℓ there exists exactly one line m such that P lies on m and $m \parallel \ell$.*

In the presence of the other axioms of geometry, Playfair's Postulate is equivalent to Euclid's Fifth Postulate. The next two theorems relate parallelism to angle congruence. The two theorems are a standard part of high school geometry and are also Propositions I.27, and I.29 in Euclid. It is in the proof of Proposition I.29 that Euclid first uses his fifth postulate.

Let ℓ and ℓ' denote two lines in the plane. A *transversal* for the two lines is a line t such that t intersects ℓ and ℓ' in distinct points. The transversal makes a total of eight angles with the two lines—see Figure 0.5. The two pairs $\{\angle ABB', \angle BB'C'\}$ and $\{\angle A'B'B, \angle B'BC\}$ are called *alternate interior angles*. The angles $\{\angle ABB', \angle A'B'B''\}$ are *corresponding angles*. There are three other pairs of corresponding angles defined in the obvious way.

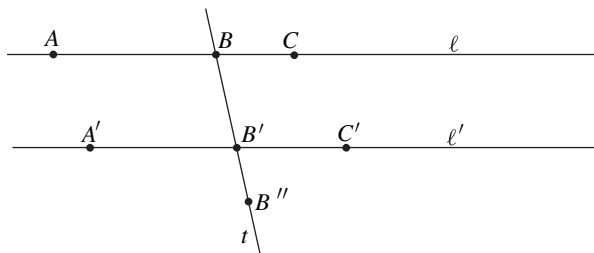


FIGURE 0.5: Angles formed by two lines and a transversal

Alternate Interior Angles Theorem. *If ℓ and ℓ' are two lines cut by a transversal t in such a way that a pair of alternate interior angles is congruent, then ℓ is parallel to ℓ' .*

Converse to the Alternate Interior Angles Theorem. *If two parallel lines are cut by a transversal, then both pairs of alternate interior angles are congruent.*

EXERCISES

0.6.1. Prove the following theorem (Euclid's Proposition I.28).

Corresponding Angles Theorem. *If ℓ and ℓ' are lines cut by a transversal t in such a way that two corresponding angles are congruent, then ℓ is parallel to ℓ' .*

0.6.2. Prove the following theorem (Euclid's Proposition I.32).

Angle Sum Theorem. *For every triangle, the sum of the measures of the interior angles of the triangle is 180° .*

[Hint: Let $\triangle ABC$ be a triangle. Draw a line through C that is parallel to the line through A and B . Then apply the Converse to Alternate Interior Angles.]

0.7 THE PYTHAGOREAN THEOREM

The Pythagorean theorem is probably the most famous theorem in all of geometry; it is the one theorem that every high school student remembers. For Euclid it was the culmination of Book I of the *Elements*. The theorem is named for Pythagoras of Samos who lived from about 569 to 475 BC. Few details about the life of Pythagoras are known, so it is difficult to determine whether Pythagoras really did prove the theorem that bears his name or what kind of proof he might have used.

Notation. Let $\triangle ABC$ be a triangle. It is standard to use lower case letters to denote the lengths of the sides of the triangle: $a = BC$, $b = AC$, and $c = AB$.

Pythagorean Theorem. *If $\triangle ABC$ is a right triangle with right angle at vertex C , then $a^2 + b^2 = c^2$.*

Euclid gave two kinds of proofs of the Pythagorean theorem; the first one based on area and then later another based on similar triangles.

0.8 SIMILAR TRIANGLES

The similar triangles theorem is one of the most useful in elementary Euclidean geometry. Euclid did not prove it, however, until Book VI of the *Elements*. (The similar triangles theorem is Euclid's Proposition VI.4.) The reason he waited so long is that the ancient Greeks had trouble dealing with the irrational ratios that can arise when similar triangles are compared. It is believed that Eudoxus of Cnidus (408–305 BC) was the first to give a complete proof of the theorem.

Triangles $\triangle ABC$ and $\triangle DEF$ are *similar* if $\angle ABC \cong \angle DEF$, $\angle BCA \cong \angle EFD$, and $\angle CAB \cong \angle FDE$. Write $\triangle ABC \sim \triangle DEF$ if $\triangle ABC$ is similar to $\triangle DEF$. As with congruence of triangles, the order in which the vertices are listed is significant.

Similar Triangles Theorem. *If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\triangle ABC \sim \triangle DEF$, then*

$$\frac{AB}{AC} = \frac{DE}{DF}.$$

EXERCISES

0.8.1. Prove the following theorem. It is a special case of the Parallel Projection Theorem [16, Theorem 7.3.1] and will prove to be very useful later.

Euclid's Proposition VI.2. *Let $\triangle ABC$ be a triangle, and let D and E be points on the sides \overline{AB} and \overline{AC} , respectively. Then $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$ if and only if $AD/AB = AE/AC$. [Hint for converse: Assume $AD/AB = AE/AC$. Let ℓ be the line through D such that $\ell \parallel \overleftrightarrow{BC}$. Use Pasch's Axiom to prove that there is a point E' where ℓ intersects \overline{AC} . Prove that $E' = E$.]*

0.8.2. Prove the following theorem (Euclid's Proposition VI.6).

SAS Similarity Criterion. *If $\triangle ABC$ and $\triangle DEF$ are two triangles such that $\angle CAB \cong \angle FDE$ and $AB/AC = DE/DF$, then $\triangle ABC \sim \triangle DEF$.*

[Hint: If $AB = DE$, the proof is easy. Otherwise it may be assumed that $AB > DE$ (explain). Choose a point B' between A and B such that $AB' = DE$ and let m be the line through B' that is parallel to \overleftrightarrow{BC} . Prove that m intersects \overline{AC} in a point C' such that $\triangle AB'C' \cong \triangle DEF$.]

0.9 QUADRILATERALS

Four points A , B , C , and D such that no three of the points are collinear determine a *quadrilateral*, which we will denote by $\square ABCD$. Specifically,

$$\square ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}.$$

It is usually assumed that the sides \overline{AB} , \overline{BC} , \overline{CD} , and \overline{DA} intersect only at their endpoints, but we will relax that requirement later in the course.

The four segments are called the *sides* of the quadrilateral and the points A , B , C , and D are called the *vertices* of the quadrilateral. The sides \overline{AB} and \overline{CD} are called *opposite sides* of the quadrilateral as are the sides \overline{BC} and \overline{AD} . Two quadrilaterals are *congruent* if there is a correspondence between their vertices so that all four corresponding sides are congruent and all four corresponding angles are congruent.

There are several special kinds of quadrilaterals that have names. A *trapezoid* is a quadrilateral in which at least one pair of opposite sides is parallel. A *parallelogram* is a quadrilateral in which both pairs of opposite sides are parallel. It is obvious that every parallelogram is a trapezoid, but not vice versa. A *rhombus* is a quadrilateral in which all four sides are congruent. A *rectangle* is a quadrilateral in which all four angles are right angles. A *square* is a quadrilateral that is both a rhombus and a rectangle.

EXERCISES

0.9.1. Prove the following theorem.

Euclid's Proposition I.34. *The opposite sides of a parallelogram are congruent.*

[Hint: Draw a diagonal and use ASA.]

0.10 CIRCLES AND INSCRIBED ANGLES

Let r be a positive number and let O be a point. The *circle* with *center* O and *radius* r is defined by $\mathcal{C}(O, r) = \{P \mid OP = r\}$. The *diameter* of the circle is $d = 2r$. While the radius of a circle is usually thought of as a number, it is often convenient to refer to one of the segments \overline{OP} , $P \in \mathcal{C}(O, r)$, as a radius of the circle $\mathcal{C}(O, r)$. In the same way, a segment \overline{PQ} such that P and Q lie on the circle and $O \in \overline{PQ}$ is called a diameter of $\mathcal{C}(O, r)$.

Let γ be a circle and let P be a point on γ . A line t is *tangent to γ at P* if $t \cap \gamma = \{P\}$.

Tangent Line Theorem. Let $\gamma = \mathcal{C}(O, r)$ be a circle and let ℓ be a line that intersects γ at P . Then ℓ is tangent to γ at P if and only if $\ell \perp \overleftrightarrow{OP}$.

Let $\gamma = \mathcal{C}(O, r)$ be a circle. An *inscribed angle* for γ is an angle of the form $\angle PQR$, where P , Q , and R all lie on γ . The *arc intercepted* by the inscribed angle $\angle PQR$ is the set of points on γ that lie in the interior of $\angle PQR$.

Inscribed Angle Theorem. If two inscribed angles intercept the same arc, then the angles are congruent.

The inscribed angle theorem is Euclid's Proposition III.21.

EXERCISES

0.10.1. Prove the following theorem. It is Euclid's Proposition III.3.

Secant Line Theorem. If $\gamma = \mathcal{C}(O, r)$ is a circle and ℓ is a line that intersects γ at distinct points P and Q , then O lies on the perpendicular bisector of the chord \overline{PQ} .

0.10.2. Prove the following theorem.

External Tangents Theorem. If $\gamma = \mathcal{C}(O, r)$ is a circle and ℓ and m are two nonparallel lines that are tangent to γ at the points P and Q , and A is the point of intersection of ℓ and m , then $PA = QA$.

0.10.3. The following theorem can be viewed as a special case of the Inscribed Angle Theorem. Give a proof that does not use the Inscribed Angle Theorem. The theorem is named for Thales of Miletus (624–547 BC); it is Euclid's Proposition III.31.

Thales' Theorem. If the vertices of $\triangle ABC$ lie on a circle and \overline{AB} is a diameter of that circle, then $\angle ACB$ is a right angle.

[Hint: Let O be the midpoint of \overline{AB} . Observe that $AO = BO = CO$ and apply the Isosceles Triangle Theorem along with the Angle Sum Theorem.]

0.10.4. Prove the following theorem.

Converse to Thales' Theorem. If $\angle ACB$ is a right angle, then the vertices of $\triangle ABC$ lie on a circle and \overline{AB} is a diameter of that circle.

[Hint: Again let O be the midpoint of \overline{AB} . There is a point C' such that C' lies on \overrightarrow{OC} and $OC' = OA$. Show that the assumption $C \neq C'$ leads to a contradiction.]

0.10.5. Use the angle sum theorem and the linear pair theorem to prove the following result: If $\triangle ABC$ is a right triangle with right angle at C and O is the midpoint of \overline{AB} , then $\mu(\angle BOC) = 2\mu(\angle BAC)$. This theorem is a special case of the Central Angle Theorem [16, Theorem 10.4.9]. It will be used repeatedly in later chapters.

0.11 AREA

A *polygon* is a generalization of triangle and quadrilateral. A polygon P has a finite set of *vertices* A_1, A_2, \dots, A_n . The polygon is defined by

$$P = \overline{A_1A_2} \cup \overline{A_2A_3} \cup \dots \cup \overline{A_{n-1}A_n} \cup \overline{A_nA_1}.$$

The segments $\overline{A_1A_2}$, etc., are called the *sides* of the polygon. The sides of a polygon are one-dimensional and have no area. Corresponding to each polygon in the plane there is a *region*, which consists of the points of the polygon itself together with the

points inside the polygon. It is the region that is two-dimensional and has area. The distinction between the polygon and the corresponding polygonal region will be important in this course because Geometer's Sketchpad treats the two as different objects that must be constructed separately.

For each polygonal region in the plane there is a nonnegative number called the *area* of the region. The area of a region R is denoted by $\alpha(R)$. The area of a triangular region is given by the familiar formula

$$\text{area} = (1/2) \text{ base} \times \text{height}.$$

The other important property of area is that it is *additive*, which means that the area of a region that is the union of two nonoverlapping subregions is the sum of the areas if the subregions.

EXERCISES

0.11.1. Prove that the area of a triangle $\triangle ABC$ is given by the formula

$$\alpha(\triangle ABC) = \frac{1}{2} AB \cdot AC \cdot \sin(\angle BAC).$$