On the spectra of periodic waves for infinite-dimensional Hamiltonian systems

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March 19, 2008

Abstract. We consider the problem of determining the spectrum for the linearization of an infinite-dimensional Hamiltonian system about a spatially periodic traveling wave. By using a Bloch-wave decomposition, we recast the problem as determining the point spectra for a family of operators $J_\gamma L_\gamma$, where $J_\gamma$ is skew-symmetric with bounded inverse and $L_\gamma$ is symmetric with compact inverse. Our main result relates the number of unstable eigenvalues of the operator $J_\gamma L_\gamma$ to the number of negative eigenvalues of the symmetric operator $L_\gamma$. The compactness of the resolvent operators allows us to greatly simplify the proofs, as compared to those where similar results are obtained for linearizations about localized waves. The theoretical results are general, and apply to a larger class of problems than those considered herein. The theory is applied to a study of the spectra associated with periodic and quasi-periodic solutions to the nonlinear Schrödinger equation, as well as periodic solutions to the generalized Korteweg-de Vries equation with power nonlinearity.

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1. INTRODUCTION

We are interested in certain spectral-stability aspects of spatially periodic waves in Hamiltonian nonlinear partial differential equations. Upon linearizing about a periodic wave one is left with studying an eigenvalue problem of the form

$$\mathcal{J}Lu = jhu,$$

where $\mathcal{L}$ is a symmetric operator with spatially periodic coefficients, and $\mathcal{J}$ is a skew-symmetric operator. The main purpose of this paper is to obtain a general result relating the spectra of the operators $\mathcal{L}$ and $\mathcal{J}\mathcal{L}$ in the spirit of the results in [6, 20, 21, 27] for localized waves, in which the total number of negative point eigenvalues of $\mathcal{L}$ is related to the total number of unstable eigenvalues of $\mathcal{J}\mathcal{L}$.

As is well-known, the spectra associated with periodic waves strongly depends upon the choice of the function space. While for spaces of periodic functions, e.g., $H \coloneqq L^2\per([-L,L];\mathbb{C}^n)$, the spectrum consists of isolated eigenvalues of finite multiplicity, for spaces of functions defined on the whole real line, e.g., $H \coloneqq L^2(\mathbb{R};\mathbb{C}^n)$, the spectrum is continuous, and in particular there are no isolated eigenvalues of finite multiplicity. Here, we are interested in the latter situation, which corresponds to studying the spectral stability of periodic waves with respect to localized perturbations. In this case, the fact that the spectrum of the operator is continuous is a serious impediment to implementing the results of [20, 21], which concern isolated eigenvalues of $\mathcal{L}$ and $\mathcal{J}\mathcal{L}$ only. However, this difficulty can be overcome by considering a Bloch-wave decomposition (see Section 3). Upon performing such a decomposition the eigenvalue problem (1.1) is replaced by a family of eigenvalue problems

$$\mathcal{J}_y\mathcal{L}_y u = jhu, \quad -\pi < y \leq \pi,$$

where for each $y$ the operator $\mathcal{L}_y$ is symmetric and has now a compact resolvent, while the operator $\mathcal{J}_y$ is skew-symmetric. As a consequence, $\mathcal{J}_y\mathcal{L}_y$ has point spectrum only, so that one expects that the results and ideas presented in [20, 21] will be immediately applicable. However, one runs into another problem which must be overcome here: $\text{Im}(\mathcal{J}_y\mathcal{L}_y) \neq 0$, in contrast to the situation in [20, 21] where $\text{Im}(\mathcal{J}\mathcal{L}) = 0$.

There is one crucial difference between the cases $\text{Im}(\mathcal{J}_y\mathcal{L}_y) = 0$ and $\text{Im}(\mathcal{J}_y\mathcal{L}_y) \neq 0$. In the former case the spectrum of $\mathcal{J}_y\mathcal{L}_y$ is symmetric with respect to both the real and imaginary axes of the complex plane, i.e., one has the eigenvalue quartets $\{\pm j, \pm j\}$, whereas in the latter case the symmetry is generically with respect to the imaginary axis only, i.e., one has the eigenvalue pairs $\{j, -j\}$. Consequently, this changes the form of the general statement in Theorem 2.13 below as compared to that of [21, Theorem 1]. Nevertheless, under certain symmetry assumptions one recovers the symmetry with respect to the real axis, and hence the expected form of the result (see Section 2.1 and Section 2.2).

The results presented in [20, 21] rely heavily upon the previous work of [14, 15], in which it is assumed that the operator $\mathcal{L}$ has both point and continuous spectra. As it will be seen in Section 2, in our case the absence of continuous spectrum (due to the compactness of the resolvent) allows us to recover these results via more elementary means. The proofs are in the spirit of those presented in [6, 27] for localized waves in that they involve a detailed analysis of the indefinite quadratic form $\langle \cdot, \mathcal{L}_y \cdot \rangle$; however, the proofs presented herein allow us to remove the assumption that the point eigenvalues of $\mathcal{J}_y\mathcal{L}_y$ are semi-simple (see Remark 2.14(b)).

The paper is organized as follows. In Section 2 we consider general eigenvalue problems given in equation (1.2). Under the general Assumption 2.1 we obtain a relationship between the number of unstable eigenvalues of the operator $\mathcal{J}_y\mathcal{L}_y$ and the number of negative eigenvalues of the symmetric operator $\mathcal{L}_y$ (see Theorem 2.13). In particular, if $\text{Im}(\mathcal{J}_y\mathcal{L}_y) = 0$, we recover the results presented in [20, 21], although the proof is different. Some particular cases are discussed in Section 2.1 and Section 2.2. The proofs presented in this section are self-contained. A crucial ingredient of the proof, and a result which is of independent interest, is the result of the SCS Basis Lemma (Lemma 2.3). Therein it is shown that under general assumptions that the eigenvectors and generalized eigenvectors of the non-self-adjoint operator $\mathcal{J}_y\mathcal{L}_y$ form a basis. In Section 3 we apply these general results to a class of eigenvalue problems of NLS-type in which $\mathcal{F} = \mathcal{J}_y$ is a bounded operator. In particular, we study the spectra for periodic and quasi-periodic waves to nonlinear Schrödinger equations. In Section 4 we apply the results to eigenvalue problems of KdV-type in which now $\mathcal{J}_y$ is an unbounded operator with compact resolvent. As an example, we study the
spectra of periodic waves to the generalized Korteweg-de Vries equation (gKdV). Finally, in the Appendix we give the proof of the SCS Basis Lemma.

**Acknowledgments.** This work was partially supported by the Ministère de la Recherche through grant ACI JC 1039 (M.H.).

### 2. General Theory

We consider the general eigenvalue problem

\[ JL u = \lambda u \tag{2.1} \]

for a symmetric operator \( L \) and a skew-symmetric, invertible operator \( J \). The problem will be considered on a complex Hilbert space \( H \) with inner product \( \langle \cdot, \cdot \rangle \). We make the following assumptions:

**Assumption 2.1.** The operators \( J \) and \( L \) in equation (2.1) have the following properties:

(a) \( J \) is skew-adjoint (\( J^* = -J \)) and invertible with bounded inverse;

(b) \( L \) is self-adjoint (\( L^* = L \)) and invertible with compact inverse;

(c) \( \sigma(L) \cap \mathbb{R}^- \), where \( \sigma(L) \) denotes the spectrum of \( L \), is a finite set;

(d) there exists a skew-adjoint operator \( A_0 \) with compact resolvent such that:

(i) the operator \( B := JL - A_0 \) is \( A_0 \)-compact and satisfies

\[
\|Bu\| \leq a\|u\| + b\|A_0^ru\|. \quad \forall u \in D(JL),
\]

for some positive constants \( a, b \), and \( r \in [0, 1) \);

(ii) the sequence of purely imaginary eigenvalues \( \{i\omega_n\}_{n \in \mathbb{Z}} \) of \( A_0 \), where each eigenvalue is repeated as many times as the value of its multiplicity and \( \omega_n \leq \omega_{n+1} \), satisfies

\[
\sum_{n=-\infty}^{\infty} |\omega_n|^{-s} < \infty,
\]

for some \( s \geq 1 \);

(iii) there exists a subsequence of eigenvalues \( \{i\omega_{n_k}\}_{k \in \mathbb{Z}} \), and positive constants \( c > 0 \) and \( r' > r \), such that

\[
\begin{align*}
\omega_{n_k} &> 0 \text{ and } \omega_{n_k+1} - \omega_{n_k} \geq c\omega_{n_k+1}^r \text{ for } k \geq 0, \\
\omega_{n_k} &< 0 \text{ and } \omega_{n_k} - \omega_{n_k-1} \geq c|\omega_{n_k-1}|^{r'} \text{ for } k < 0.
\end{align*}
\]

**Remark 2.2.**

(a) Let \( \{e_k\}_{k \in \mathbb{Z}} \) be an orthonormal basis of \( H \) consisting of eigenvectors of \( A_0 \). Then, the operator \( |A_0|^r \) can be defined through

\[
|A_0|^ru = \sum_{k \in \mathbb{Z}} |\omega_k(A_0)|^ru_k e_k, \quad u = \sum_{k \in \mathbb{Z}} u_k e_k.
\]

(b) An immediate consequence of Assumption 2.1(b) is that the spectrum of \( L \) is a countable set, with no accumulation point in \( \mathbb{C} \), consisting of real eigenvalues with finite multiplicity. Furthermore, the set of its eigenvectors form an orthonormal basis for \( H \).

(c) Similarly, since \( A_0 \) is skew-adjoint with compact resolvent, by Assumption 2.1(d), its spectrum is a countable set, with no accumulation point in \( \mathbb{C} \), consisting of purely imaginary eigenvalues with finite multiplicity, and the set of its eigenvectors forms an orthonormal basis for \( H \). As we shall see in the next lemma, similar properties also hold for \( JL \).
(d) If \( s = 1 \) in Assumption 2.1(d)(ii), then the operator \( \mathcal{A}_0 \) is of trace class.

(e) Assumption 2.1(d) is easily verified in the examples considered in Section 3 and Section 4, in which \( \mathcal{A}_0 \) is an operator with constant coefficients. The eigenvalues of \( \mathcal{A}_0 \) can then be explicitly computed via Fourier analysis; in particular, for NLS we have that actually \( B \) is bounded so that equation \((2.2)\) holds with \( r = 0 \), and in equation \((2.3)\) we find \( r' = 1/2 \). For gKdV, we obtain \( r = 1/3 \) and \( r' = 2/3 \).

(f) Assumption 2.1(d)(ii) is required to show that the set of eigenfunctions and generalized eigenfunctions of \( \mathcal{J}L \) is complete in \( H \). The assumptions equation \((2.2)\) and equation \((2.3)\) are needed to show that the set actually forms a basis of \( H \).

This section will be devoted to comparing the spectra of the operators \( L \) and \( \mathcal{J}L \). We start by collecting a number of properties of the operator \( \mathcal{J}L \) which follow from the Assumption 2.1. We give the proof of the following lemma in the Appendix (the acronym follows from Skew-symmetric Composed with Self-adjoint).

**Lemma 2.3** (SCS Basis Lemma). Assume that Assumption 2.1 holds. Then the operator \( \mathcal{J}L \) has the following properties:

(a) \( \mathcal{J}L \) is invertible and has a compact inverse \( L^{-1}\mathcal{J}^{-1} \);

(b) the spectrum \( \sigma(\mathcal{J}L) \) of \( \mathcal{J}L \) is a countable set with no accumulation point in \( \mathbb{C} \) consisting of eigenvalues with finite algebraic multiplicity;

(c) the set of eigenfunctions and generalized eigenfunctions of \( \mathcal{J}L \) forms a basis of \( H \);

(d) given \( \epsilon > 0 \), the eigenvalues of \( (\mathcal{J}L)^{-1} \), with a possible exception of a finite number, belong to the sector

\[
\Delta_{\pi/2} := \{ \rho e^{i\theta} : \rho \in \mathbb{R}^+, |\theta - \pi/2| < \epsilon \text{ or } |\theta + \pi/2| < \epsilon \}.
\]

**Remark 2.4.** An important consequence of Lemma 2.3(d) is that \( \sigma(\mathcal{J}L) \cap \mathbb{R} \) is a finite set.

In addition to the properties in the previous proposition, a key ingredient in the next arguments is the following symmetry property of the spectrum of the \( \mathcal{J}L \).

**Proposition 2.5.** If \( \lambda \in \sigma(\mathcal{J}L) \), then \( -\lambda \in \sigma(\mathcal{J}L) \). Furthermore, \( \lambda \) and \( -\lambda \) have the same algebraic multiplicity, \( m_\lambda(\lambda) = m_\lambda(-\lambda) \).

**Proof:** Suppose that \( \lambda \in \sigma(\mathcal{J}L) \) is an eigenvalue of \( \mathcal{J}L \) with associated eigenvector \( v \), so that \( \mathcal{J}Lv = \lambda v \). Since \( \mathcal{J} \) has a bounded inverse we have \( Lv = \lambda \mathcal{J}^{-1}v \), and then

\[
(\mathcal{J}L)\mathcal{J}^{-1} = -L = -\lambda \mathcal{J}^{-1}v = -\lambda v = -\lambda \mathcal{J}^{-1}v.
\]

Consequently \( -\lambda \in \sigma((\mathcal{J}L)^*) \), so that \( -\lambda \in \sigma(\mathcal{J}L) \). The equality of the algebraic multiplicities follows from the Fredholm alternative. \( \square \)

As a consequence of Proposition 2.5 the eigenvalues of \( \mathcal{J}L \) appear in pairs \( \{\lambda, -\lambda\} \), that is the spectrum of \( \mathcal{J}L \) is symmetric with respect to the imaginary axis. In addition, if \( \mathcal{J}L \) is a real operator, \( \text{Im}(\mathcal{J}L) = 0 \), then its spectrum is also symmetric with respect to the real axis, so that we have quartets of eigenvalues \( \{\pm\lambda, \pm\bar{\lambda}\} \). However, this property is typically not true in our applications in which \( \text{Im}(\mathcal{J}L) \neq 0 \).

**Notation 2.6.** For an eigenvalue \( \lambda \in \sigma(\mathcal{J}L) \), denote by \( E_\lambda \) the corresponding spectral subspace, and by \( I_\lambda \) the invariant subspace corresponding to the eigenvalue pair \( \{\lambda, -\lambda\} \), \( I_\lambda = E_\lambda \oplus E_{-\lambda} \).

The usual approach for comparing the spectra of \( \mathcal{J}L \) and \( L \) relies upon the study of the quadratic form \( \langle \cdot, \cdot \rangle_L \) associated to \( L \) in the basis of \( H \) given by the eigenfunctions and generalized eigenfunctions of \( \mathcal{J}L \). The following properties of the quadratic form \( \langle \cdot, \cdot \rangle_L \) are similar to the ones in [23, Lemmas 2-4]. The main difference is that here they concern pairs of eigenvalues \( \{\lambda, -\lambda\} \), while in [23] they were obtained for quartets of eigenvalues \( \{\pm\lambda, \pm\bar{\lambda}\} \).

**Lemma 2.7.** If \( v \in E_\lambda \) and \( w \in E_{\bar{\lambda}} \) with \( \lambda + \bar{\lambda} \neq 0 \), then \( \langle Lv, w \rangle = 0 \).
\textbf{Proof:} First suppose that \(v\) and \(w\) are not generalized eigenfunctions, so that \(JL v = \lambda v\) and \(JL w = \sigma w\). Then
\[
\langle Lv, w \rangle = \lambda \langle J^{-1} v, w \rangle, \quad \langle v, Lw \rangle = \sigma \langle v, J^{-1} w \rangle.
\]
Since \(L\) is symmetric and \(J^{-1}\) is skew-symmetric this implies that
\[
(\lambda + \sigma) \langle J^{-1} v, w \rangle = 0,
\]
from which one gets the desired result.

Now suppose that either one or both of \(v\) and \(w\) are generalized eigenfunctions. In this case the proof will follow by induction. One first has the Jordan chains
\[
JL v_{k+1} = J v_{k+1} + v_k \quad (u_0 = 0), \quad k = 0, \ldots, k',
\]
\[
JL w_{\ell+1} = \sigma w_{\ell+1} + w_\ell \quad (w_0 = 0) \quad \ell = 0, \ldots, \ell'.
\]
Arguing as above, we find
\[
\langle Lv_{k+1}, w_{\ell+1} \rangle = \lambda \langle J^{-1} v_{k+1}, w_{\ell+1} \rangle + \langle J^{-1} v_k, w_{\ell+1} \rangle = \sigma \langle v_{k+1}, J^{-1} w_{\ell+1} \rangle + \langle v_k, J^{-1} w_{\ell+1} \rangle,
\]
so that
\[
(\lambda + \sigma) \langle J^{-1} v_{k+1}, w_{\ell+1} \rangle + \langle J^{-1} v_k, w_{\ell+1} \rangle = 0.
\]
We now show that
\[
(\lambda + \sigma) \langle J^{-1} v_{k+1}, w_{\ell+1} \rangle = 0, \quad k = 0, \ldots, k', \quad \ell = 0, \ldots, \ell'.
\]
The result of equation (2.6) is known to be true for \((k, \ell) = (0, 0)\). First suppose that for some \(0 \leq n \leq k' - 1\) the equation (2.6) holds true for \(\ell = 0\) and \(k = 0, \ldots, n\). Upon using equation (2.5) and the fact that \(w_0 = 0\), one immediately gets that (2.6) holds for \(k = n + 1\). Thus, the result holds for \(\ell = 0\) and \(k = 0, \ldots, k'\). Similarly, equation (2.6) holds for \(k = 0\) and \(\ell = 0, \ldots, \ell'\). Now consider equation (2.5) with \((k, \ell) = (1, 1)\). Since equation (2.6) holds for \((k, \ell) = (1, 0)\) and \((k, \ell) = (0, 1)\), we conclude that equation (2.6) holds for \((k, \ell) = (1, 1)\). Another induction argument gives the result for \(\ell = 1\) and \(k = 0, \ldots, k'\), as well as for \(k = 1\) and \(\ell = 0, \ldots, \ell'\). Continuing in this fashion leads to the desired result.

Finally, since \(\lambda + \sigma \neq 0\) equation (2.6) gives
\[
\langle J^{-1} v_{k+1}, w_{\ell+1} \rangle = 0, \quad k = 0, \ldots, k', \quad \ell = 0, \ldots, \ell'.
\]
Substituting the above equalities into equation (2.4) completes the proof. \(\square\)

An immediate consequence of this lemma is the following property.

\textbf{Proposition 2.8.} Suppose that \(w = \sum v_j\), where \(v_j \in I_{\beta_j}\) for distinct \(\beta_j\). Then
\[
\langle Lw, w \rangle = \sum \langle Lv_j, v_j \rangle.
\]

Finally, following the argument presented for [23, Lemma 4] we obtain:

\textbf{Proposition 2.9.} Assume that \(\beta \in \sigma(JL)\). Then the restriction \(\langle L_{\beta}, \cdot, \cdot \rangle\) of the quadratic form \(\langle L, \cdot, \cdot \rangle\) to \(I_{\beta}\) is non-singular.

\textbf{Proof:} Recall that \(\beta \neq 0\), because \(JL\) has a bounded inverse. Then if \(\langle L_{\beta}, \cdot, \cdot \rangle\) is singular, there exists a nonzero \(v \in I_{\beta}\) such that \(\langle Lw, v \rangle = 0\) for all \(w \in I_{\beta}\). Together with Lemma 2.7 this implies that \(\langle Lw, v \rangle = 0\) for any eigenvector or generalized eigenvector \(w\) of \(JL\). By Lemma 2.3(c) this yields that \(v = 0\), which is a contradiction. \(\square\)

In other words, Proposition 2.8 and Proposition 2.9 state that the quadratic form associated to \(L\) when expanded over the eigenfunctions and generalized eigenfunctions of \(JL\) is the sum over the individual invariant subspaces \(I_{\beta}\), and that each term in the sum is a nonsingular quadratic form. Thus, in order to study \(\langle L, \cdot, \cdot \rangle\), on \(H\), it is enough to consider the restrictions \(\langle L_{\beta}, \cdot, \cdot \rangle\) for each \(\beta \in \sigma(JL)\).
Lemma 2.10. Assume that $\tilde{\lambda} \in \sigma(J\mathcal{L})$. Let $L(\tilde{\lambda})$ represent the Hermitian matrix associated with the quadratic form $\langle L \cdot, \cdot \rangle$ on $E_{\tilde{\lambda}}$, and denote by $n(L(\tilde{\lambda}))$ and $p(L(\tilde{\lambda}))$ the number of its negative and positive eigenvalues counted with algebraic multiplicities, respectively. If $\text{Re} \tilde{\lambda} \neq 0$, then $L(\tilde{\lambda}) \in \mathbb{C}^{2m_{s}(\tilde{\lambda}) \times 2m_{s}(\tilde{\lambda})}$, and
\[
 n(L(\tilde{\lambda})) = p(L(\tilde{\lambda})) = m_{s}(\tilde{\lambda}).
\]
If $\text{Re} \tilde{\lambda} = 0$, then $L(\tilde{\lambda}) \in \mathbb{C}^{m_{s}(\tilde{\lambda}) \times m_{s}(\tilde{\lambda})}$ and
\[
 n(L(\tilde{\lambda})) + p(L(\tilde{\lambda})) = m_{s}(\tilde{\lambda}).
\]

Proof: The properties on the dimension of the matrix $L(\tilde{\lambda})$ follow from the fact that $\dim(I_{\tilde{\lambda}}) = 2m_{s}(\tilde{\lambda})$, if $\text{Re} \tilde{\lambda} \neq 0$, whereas $\dim(I_{\tilde{\lambda}}) = m_{s}(\tilde{\lambda})$, if $\text{Re} \tilde{\lambda} = 0$.

Set $k := m_{s}(\tilde{\lambda})$. First assume that $\text{Re} \tilde{\lambda} \neq 0$. Let $\{u_{1}, \ldots, u_{k}\}$ be a basis for $E_{\tilde{\lambda}}$, and let $\{v_{1}, \ldots, v_{k}\}$ be a basis for $E_{-\tilde{\lambda}}$. One has that
\[
 L(\tilde{\lambda}) = \begin{pmatrix}
 L_{1} & L_{c} \\
 L_{c}^* & L_{2}
 \end{pmatrix},
\]
where

\[
 (L_{1})_{ij} = \langle L u_{i}, u_{j} \rangle, \quad (L_{2})_{ij} = \langle L v_{i}, v_{j} \rangle, \quad (L_{c})_{ij} = \langle L u_{i}, v_{j} \rangle.
\]

As a consequence of Lemma 2.7 one has that $L_{1} = L_{2} = 0$. Since $\mu^{2} \in \sigma(L_{c}^{2})$ if and only if $\pm \mu \in \sigma(L(\tilde{\lambda}))$, and since $L_{c}^{2}$ is symmetric, positive semi-definite, and nonsingular, the first result now follows. The result for $\text{Re} \tilde{\lambda} = 0$ follows immediately as a consequence of Proposition 2.9. \hfill \Box

Definition 2.11. Consider the spectral subsets
\[
 \sigma_{c} := \{ \tilde{\lambda} \in \sigma(J\mathcal{L}) : \text{Re} \tilde{\lambda} > 0, \ \text{Im} \tilde{\lambda} = 0 \}, \quad \sigma_{c} := \{ \tilde{\lambda} \in \sigma(J\mathcal{L}) : \text{Re} \tilde{\lambda} > 0, \ \text{Im} \tilde{\lambda} \neq 0 \},
\]
\[
 \sigma_{c} := \{ \tilde{\lambda} \in \sigma(J\mathcal{L}) : \text{Re} \tilde{\lambda} = 0 \}.
\]

For an eigenvalue $\tilde{\lambda} \in \sigma(J\mathcal{L})$, with $\text{Re} \tilde{\lambda} \geq 0$, we define the following quantities:

- $k_{c}(\tilde{\lambda}) := m_{c}(\tilde{\lambda})$, if $\tilde{\lambda} \in \sigma_{c}$,
- $k_{c}(\tilde{\lambda}) := m_{c}(\tilde{\lambda})$, if $\tilde{\lambda} \in \sigma_{c}$,
- $k_{c}^{-}(\tilde{\lambda}) := n(L(\tilde{\lambda}))$, if $\tilde{\lambda} \in \sigma_{c}$,

and set
\[
 k_{c} := \sum_{\tilde{\lambda} \in \sigma_{c}} k_{c}(\tilde{\lambda}), \quad k_{c} := \sum_{\tilde{\lambda} \in \sigma_{c}} k_{c}(\tilde{\lambda}), \quad k_{c}^{*} := \sum_{\tilde{\lambda} \in \sigma_{c}} k_{c}^{-}(\tilde{\lambda}). \quad (2.8)
\]

Remark 2.12.

(a) The union $\sigma_{c} \cup \sigma_{c} \cup \sigma_{c}$ is the set of eigenvalues of $J\mathcal{L}$ with $\text{Re} \tilde{\lambda} \geq 0$. Since the spectrum of $J\mathcal{L}$ is symmetric with respect to the imaginary axis we have $\sigma(J\mathcal{L}) = \{ \tilde{\lambda}, \ -\tilde{\lambda} : \tilde{\lambda} \in \sigma_{c} \cup \sigma_{c} \cup \sigma_{c} \}$.

(b) The numbers $k_{c}$, $k_{c}$, and $k_{c}^{-}$ have been used in [20, 21] with a slightly different meaning. While in [20, 21] they were associated to quartets of eigenvalues $\{ \pm \tilde{\lambda}, \pm \tilde{\lambda} \}$, here they are associated to the pairs $\{ \tilde{\lambda}, \ -\tilde{\lambda} \}$. Of course this is due to the absence of the symmetry of the spectrum with respect to the real axis. However, if in addition the spectrum of $J\mathcal{L}$ is also symmetric with respect to the real axis (for example if $\text{Im}(J\mathcal{L}) = 0$), so that we also have quartets of eigenvalues $\{ \pm \tilde{\lambda}, \pm \tilde{\lambda} \}$, then $k_{c}$ has the same significance, whereas $k_{c}$ and $k_{c}^{*}$ are even numbers, and are precisely twice the numbers considered in [20, 21].

The following general result connecting the number of unstable eigenvalues of $J\mathcal{L}$ and the number of negative eigenvalues of $\mathcal{L}$ can now be proven.
Theorem 2.13. Consider the eigenvalue problem of equation (2.1) under Assumption 2.1. One has that

\[ k_+ + k_c + k_- = n(L). \]

where \( k_+ \), \( k_c \), \( k_- \) are the numbers introduced in Definition 2.11, and \( n(L) \) denotes the number of negative eigenvalues (including multiplicities) of \( L \). Furthermore, if \( \text{Im}(\mathcal{J}L) = 0 \), then the spectrum of \( \mathcal{J}L \) is symmetric with respect to the real axis and the numbers \( k_+ \) and \( k_- \) are even.

Proof: Consider the quadratic form \( \langle L\cdot, \cdot \rangle \) and define the cone \( C(L) \) by

\[ C(L) := \{ u \in H^1 : \langle L u, u \rangle < 0 \} \cup \{ 0 \}, \]

where \( H^1 \) is the domain of \( \langle L\cdot, \cdot \rangle \), \( D(L) \subset H^1 \subset H \). Denote by \( \dim(C(L)) \) the dimension of the maximal linear subspace \( E \subset C(L) \). Since \( L \) is a symmetric operator with point spectrum only, upon using an orthogonal basis for \( H \) consisting of eigenvectors of \( L \) we find that \( \dim(C(L)) = n(L) \). Indeed, the linear subspace spanned by the eigenvectors of \( L \) associated to the negative eigenvalues of \( L \) belongs to \( C(L) \), so that \( \dim(C(L)) \geq n(L) \). Next, assuming that \( C(L) \) contains a linear subspace of dimension \( n(L) + 1 \), at least, this subspace necessarily contains a vector \( v = \sum a_j v_j \), with \( v_j \) eigenvectors associated to the positive eigenvalues of \( L \). Then \( \langle L v, v \rangle = \sum |a_j|^2 \langle L v_j, v_j \rangle > 0 \), which is in contradiction with \( v \in C(L) \). This shows that \( \dim(C(L)) = n(L) \).

On the other hand, by Lemma 2.10, for any eigenvalue \( \lambda \) of \( \mathcal{J}L \) with \( \text{Re} \lambda \geq 0 \) there is a linear subspace \( H_{\lambda} \subset I_{\lambda} \cap C(L) \) with dimension

\[ \dim[H_{\lambda}] = \begin{cases} k_+(\lambda), & \text{Re} \lambda > 0, \text{Im} \lambda = 0 \\ k_c(\lambda), & \text{Re} \lambda > 0, \text{Im} \lambda \neq 0 \\ k_-(\lambda), & \text{Re} \lambda = 0. \end{cases} \]

By Lemma 2.7, the direct sum of these subspaces is contained in \( C(L) \), so that \( \dim(C(L)) \geq k_+ + k_c + k_- \). Finally, by arguing as above, using the fact that the set of eigenvectors and generalized eigenvectors of \( \mathcal{J}L \) are a basis, and Lemma 2.7, again, we conclude that \( \dim(C(L)) = k_+ + k_c + k_- \). Together with the equality \( \dim(C(L)) = n(L) \), this completes the proof. \( \Box \)

Remark 2.14.

(a) It would be an interesting question to prove Theorem 2.13 by combining the results of [18, Appendix A] on Pontrjagin spaces and the SCS Basis Lemma. In particular, upon noticing that the operator \( i\mathcal{J}L \) is symmetric on \( H \) equipped with the indefinite, sesquilinear, Hermitian scalar product

\[ \{ u, v \} := \langle u, L v \rangle, \]

one is within the framework presented therein. A major hurdle to overcome in order to apply these ideas seems to be that completeness of the eigenfunctions of \( \mathcal{J}L \) in \( H \) does not necessarily imply completeness in the space supplied with the inner product given above.

(b) The result of Theorem 2.13 is in the same vein as that presented in [20, 21] for localized waves. The primary difference is that here \( L \) is assumed to be invertible with compact inverse. If desired, the assumption that \( L \) is nonsingular can be removed, and a result which takes into account that \( L \) has a nontrivial null-space can be formulated. However, since in our applications \( L \) is typically invertible, we restrict to this case, only. The fact that \( L \) has compact inverse allows us to give a self-contained proof which no longer relies upon the results of [14, 15].

(c) Recently [6, 27, 30] have initiated a program in which they prove, for localized waves, a result similar to that given in Theorem 2.13 without using any of the results in [14, 15]. In doing so, however, they typically assume that all of the eigenvalues of \( \mathcal{J}L \) are semi-simple [27], and that the operators \( \mathcal{J} \) and \( L \) have the special form

\[
\mathcal{J} = \begin{pmatrix} 0 & \text{id} \\ -\text{id} & 0 \end{pmatrix}, \quad L = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}.
\]

In [14, 15] equation (2.9) was also assumed; however, in [21] it was shown how this assumption could be circumvented.
2.1. Symmetries

The next goal is to refine the result of Theorem 2.13 under the assumption of some underlying symmetries. We will consider two different scenarios. In both cases we find the usual quadruplets \( \{ \pm \lambda, \pm \bar{\lambda} \} \). The second symmetry will be considered in more detail in the next subsection.

**Assumption 2.15.**

(a) Assume that \( \text{Im} \mathcal{J} = 0 \) and that there exists a bounded unitary operator \( \mathcal{U}_1 \) such that

\[
\mathcal{U}_1 \mathcal{J} = \mathcal{J} \mathcal{U}_1 \quad \text{and} \quad \mathcal{U}_1 \mathcal{L} = \mathcal{L} \mathcal{U}_1.
\]

(b) Assume that there exists a bounded unitary operator \( \mathcal{U}_2 \) such that

\[
\mathcal{U}_2 \mathcal{J} = -\mathcal{J} \mathcal{U}_2 \quad \text{and} \quad \mathcal{U}_2 \mathcal{L} = \mathcal{L} \mathcal{U}_2.
\]

**Proposition 2.16.** Under Assumption 2.15(a), if \( u \in H \) is a (generalized) eigenfunction of \( \mathcal{J} \mathcal{L} \) associated with the eigenvalue \( \lambda \), then \( \mathcal{U}_1 \bar{u} \) is a (generalized) eigenfunction of \( \mathcal{J} \mathcal{L} \) associated with the eigenvalue \( \bar{\lambda} \). Under Assumption 2.15(b), if \( u \in H \) is a (generalized) eigenfunction of \( \mathcal{J} \mathcal{L} \) associated with the eigenvalue \( \lambda \), then \( \mathcal{U}_2 u \) is a (generalized) eigenfunction of \( \mathcal{J} \mathcal{L} \) associated with the eigenvalue \( -\lambda \).

**Proof:** Consider the implication of Assumption 2.15(a). (The proof of Assumption 2.15(b) is similar to that of (a) and is left for the interested reader.) Suppose that \( u \) is an eigenfunction with associated eigenvalue \( \lambda \), so that \( \mathcal{J} \mathcal{L} u = \lambda u \). Upon taking the complex-conjugate of both sides one has that \( \mathcal{J} \mathcal{L} \bar{u} = \bar{\lambda} \bar{u} \). Applying \( \mathcal{U}_1 \) to both sides then yields the desired result. The case when \( u \) is a generalized eigenfunction is proved in the same way. \(\square\)

As a consequence of Proposition 2.16 one has that the spectrum of \( \mathcal{J} \mathcal{L} \) is also symmetric with respect to the real axis. First assume the symmetry of Assumption 2.15(a). For each invariant subspace \( I_\beta \) with \( \text{Im} \lambda \neq 0 \) there is an associated invariant subspace \( I_{\bar{\beta}} \) such that \( \dim(I_\beta) = \dim(I_{\bar{\beta}}) \). Furthermore, the subspaces have the same Jordan block structure. Now, if \( \Re \lambda \neq 0 \), then the result of Lemma 2.10 immediately shows that \( k_+ \) is symmetric with respect to both the imaginary and the real axes. Now suppose that \( \Re \lambda = 0 \), and let \( \{ u_1, \ldots, u_k \} \) be a basis for \( I_\beta \). Following the proof of Lemma 2.10 one has that

\[
(L(\lambda))_{ij} = \langle \mathcal{U}_1 u_i, u_j \rangle. \tag{2.10}
\]

As a consequence of Proposition 2.16 one has that \( \{ \mathcal{U}_1 u_1, \ldots, \mathcal{U}_1 u_k \} \) is a basis for \( I_{\bar{\lambda}} \). Upon using Assumption 2.15(a) one has that

\[
(L(\bar{\lambda}))_{ij} = \langle \mathcal{U}_1 \mathcal{L} u_i, \mathcal{U}_1 u_j \rangle = \langle \mathcal{U}_1 \mathcal{L} u_i, u_j \rangle = \langle \mathcal{L} u_i, u_j \rangle = \langle \mathcal{L} u_j, u_i \rangle = \langle \mathcal{L} u_i, u_j \rangle.
\]

Upon comparing with equation (2.10) one then sees that \( L(\lambda) = L(\bar{\lambda})^* \), which in particular implies that \( k_+^{-1} (\lambda) = k_+^{-1} (\bar{\lambda}) \). A similar argument holds when considering Assumption 2.15(b). The primary difference is that for \( \beta \in \mathbb{R} \), \( L(\beta) = L(-\beta) \). The result of Theorem 2.13 can now be modified in the following way.

**Theorem 2.17.** Consider the eigenvalue problem (2.1) under Assumption 2.1 and either Assumption 2.15(a) or Assumption 2.15(b). Then the spectrum of \( \mathcal{J} \mathcal{L} \) is symmetric with respect to both the imaginary and the real axis, and

\[ k_+ + k_- = n(L). \]

with \( k_+ \) and \( k_- \) even numbers.

2.2. Canonical form

We discuss in this section the particular case of the canonical form given in equation (2.9), when \( H = H \times H \), and

\[
\mathcal{J} = \begin{pmatrix} 0 & \text{id} \\ -\text{id} & 0 \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}. \tag{2.11}
\]
with $L_z$ symmetric operators on $\tilde{H}$ such that $\mathcal{J}$ and $L$ satisfy Assumption 2.1. One recovers the result of Theorem 2.17, since $\mathcal{J}L$ satisfies Assumption 2.15(b) with
$$
U_2 = \begin{pmatrix} \text{id} & 0 \\ 0 & -\text{id} \end{pmatrix}.
$$

(2.12)

The goal is to refine this result and state it in terms of the individual operators $L_z$ (see also [14, 15, 20, 27]). In order to do so, we review the basis result in Lemma 2.3, the orthogonality result in Lemma 2.7, and the structure of the matrix $L(\bar{\lambda})$ in Lemma 2.10 in this case.

**Notation 2.18.** For an operator $\mathcal{A}$ we set
$$
\sigma^r(\mathcal{A}) := \{ \bar{\lambda} \in \sigma(\mathcal{A}) : \text{Im}\,\bar{\lambda} > 0 \} \cup \{ \bar{\lambda} \in \sigma(\mathcal{A}) : \text{Im}\,\bar{\lambda} = 0 \text{ and Re}\,\bar{\lambda} > 0 \}.
$$

**Proposition 2.19.** Consider the eigenvalue problem of equation (2.1) under the Assumption 2.1 with $\mathcal{J}$ and $L$ given by equation (2.11). Suppose that $\bar{\lambda}$ is an eigenvalue of $\mathcal{J}L$ with $\bar{\lambda} \in \sigma^r(\mathcal{J}L)$. Let $\{u_1^\beta, \ldots, u_k^\beta\}$ be a basis for $E^\beta$ consisting of generalized eigenvectors of $\mathcal{J}L$, with $u_j^\beta = (u_j^\beta, v_j^\beta)^T$. Then the following properties hold:

(a) The set $\{U_2 u_1^\beta, \ldots, U_2 u_k^\beta\}$ is a basis for $E_{-\bar{\lambda}}$. Furthermore, $U_2 u_j^\beta = (u_j^\beta, -v_j^\beta)^T$.

(b) The set $\{u_1^\beta, U_2 u_j^\beta : j = 1, \ldots, k^\beta, \bar{\lambda} \in \sigma^r(\mathcal{J}L)\}$ is a basis for $H = \tilde{H} \times \tilde{H}$.

(c) The sets $\{u_j^\beta : j = 1, \ldots, k^\beta, \bar{\lambda} \in \sigma^r(\mathcal{J}L)\}$ and $\{v_j^\beta : j = 1, \ldots, k^\beta, \bar{\lambda} \in \sigma^r(\mathcal{J}L)\}$ are basis for $\tilde{H}$.

**Proof:** The property (a) is an immediate consequence of the Assumption 2.15(b) in Section 2.1 and of equation (2.12). Part (b) follows from (a) and the basis result in Lemma 2.3(c), and (c) is a consequence of (a) and (b).

The following result is a refinement of Lemma 2.7.

**Lemma 2.20.** Let $u_{\bar{\lambda}} := (u_{\bar{\lambda}}, v_{\bar{\lambda}})^T \in E_\bar{\lambda}$ and $u_\alpha := (u_\alpha, v_\alpha)^T \in E_\alpha$. If $\bar{\lambda} \pm \alpha \neq 0$, then
$$
\langle L_+ u_{\bar{\lambda}}, u_\alpha \rangle = \langle L_- v_{\bar{\lambda}}, v_\alpha \rangle = 0.
$$

**Proof:** Suppose that $\bar{\lambda} \pm \alpha \neq 0$, and further suppose that the eigenvectors $u_{\bar{\lambda}}, u_\alpha$ are not generalized eigenvectors. From equation (2.11) one has that
$$
L_+ u_{\bar{\lambda}} = -\bar{\lambda} u_{\bar{\lambda}}, \quad L_- v_{\bar{\lambda}} = \bar{\lambda} v_{\bar{\lambda}}; \quad L_+ u_\alpha = -\alpha u_\alpha, \quad L_- v_\alpha = \alpha v_\alpha,
$$

which upon using the fact that the operators $L_z$ are symmetric yields
$$
\langle L_+ u_{\bar{\lambda}}, u_\alpha \rangle = -\bar{\lambda}\langle v_{\bar{\lambda}}, u_\alpha \rangle = -\alpha \langle u_{\bar{\lambda}}, v_\alpha \rangle,
$$
$$
\langle L_- v_{\bar{\lambda}}, v_\alpha \rangle = \bar{\lambda}\langle u_{\bar{\lambda}}, v_\alpha \rangle = \alpha \langle v_{\bar{\lambda}}, u_\alpha \rangle.
$$

By Lemma 2.7 one has that
$$
\langle L_+ u_{\bar{\lambda}}, u_\alpha \rangle + \langle L_- v_{\bar{\lambda}}, v_\alpha \rangle = 0,
$$

so that
$$
(\bar{\lambda} - \alpha)(\langle u_{\bar{\lambda}}, v_\alpha \rangle) = (\bar{\lambda} - \alpha)(\langle v_{\bar{\lambda}}, u_\alpha \rangle) = 0.
$$

If $\bar{\lambda} - \alpha \neq 0$, then one gets the desired result. The proof for $u_{\bar{\lambda}}, u_\alpha$ being generalized eigenvectors follows that presented in Lemma 2.7 and is left for the interested reader.

Finally, we come back to the structure of the matrix $L(\bar{\lambda})$ in Lemma 2.10 in this case. We distinguish between $\text{Im}\,\bar{\lambda} = 0$ and $\text{Im}\,\bar{\lambda} \neq 0$.

**Lemma 2.21.** Suppose that $\bar{\lambda}$ is an eigenvalue of $\mathcal{J}L$ with $\text{Im}\,\bar{\lambda} = 0$. Let $\{u_1, \ldots, u_k\}$ be a basis for $E_{\bar{\lambda}}$ consisting of generalized eigenvectors of $\mathcal{J}L$, with $u_j = (u_j, v_j)^T$. Then the following properties hold:
(a) The Hermitian matrix \( L(\lambda) \) associated with the quadratic form \( \langle L|_{t_j}, \cdot \rangle \) on \( I_{\beta} = E_{\beta} \oplus E_{-\beta} \) is of the form

\[
L(\lambda) = L^1(\lambda) + L^2(\lambda),
\]

with

\[
L^1(\lambda) = L^1(\lambda) \left( \begin{array}{cc} \text{id} & \text{id} \\ \text{id} & \text{id} \end{array} \right), \quad L^2(\lambda) = L^2(\lambda) \left( \begin{array}{cc} \text{id} & -\text{id} \\ -\text{id} & \text{id} \end{array} \right),
\]

where

\[
(L^1(\lambda))_{ij} := \langle L^1 u_i, u_j \rangle, \quad (L^2(\lambda))_{ij} := \langle L^2 v_i, v_j \rangle.
\]

(b) We have that \( L^1(\lambda) = -L^2(\lambda) \).

**Proof:** The property (a) is the same as Proposition 2.19(a). Next, using Proposition 2.19(a) and equation (2.11) a straightforward calculation gives the equalities in part (b). Finally, part (c) follows from the proof of Lemma 2.7 which shows that the terms on the diagonal of \( L(\lambda) \) vanish, so that \( L^1(\lambda) + L^2(\lambda) = 0 \). \( \square \)

**Lemma 2.22.** Suppose that \( \lambda \) is an eigenvalue of \( J\mathcal{L} \) with \( \text{Im} \lambda \neq 0 \). Let \( \{u_1, \ldots, u_k\} \) be a basis for \( E_{\beta} \) consisting of generalized eigenvectors of \( J\mathcal{L} \) with \( u_j = (u_j, v_j)^T \). Then the Hermitian matrix \( L(\lambda) \) associated with the quadratic form \( \langle L|_{t_j}, \cdot \rangle \) on \( I_{\beta} \) has the following properties:

(a) If \( \text{Re} \lambda = 0 \), then \( I_{\beta} = E_{\beta} \) and

\[
L(\lambda) = L^1(\lambda) + L^2(\lambda),
\]

with

\[
L^1(\lambda) = L^1(\lambda) \left( \begin{array}{cc} \text{id} & \text{id} \\ \text{id} & \text{id} \end{array} \right), \quad L^2(\lambda) = L^2(\lambda) \left( \begin{array}{cc} \text{id} & -\text{id} \\ -\text{id} & \text{id} \end{array} \right).
\]

(b) If \( \text{Re} \lambda \neq 0 \), let \( \{x_1, \ldots, x_k\} \) be a basis for \( E_{-\beta} \) with \( x_j = (x_j, y_j)^T \). Then

\[
L(\lambda) = L^1(\lambda) + L^2(\lambda),
\]

with

\[
L^1(\lambda) = L^1(\lambda) \left( \begin{array}{cc} 0 & \text{id} \\ \text{id} & 0 \end{array} \right), \quad L^2(\lambda) := \langle L^1 u_i, x_j \rangle = \langle L^2 v_i, y_j \rangle.
\]

**Proof:** In both cases the equality \( L(\lambda) = L^1(\lambda) + L^2(\lambda) \) is an immediate consequence of equation (2.11).

(a) Suppose that \( \text{Re} \lambda = 0 \), so that \( -\lambda = \lambda \) and \( I_{\beta} = E_{\beta} \). Let

\[
\{(u_{j,k}, v_{j,k})^T : j = 1, \ldots, m, \ k = 1, \ldots, a_j\}, \quad \sum_{j=1}^m a_j = m(\lambda),
\]

be a basis that represents a Jordan block decomposition of \( E_{\beta} \) consisting of eigenvectors and generalized eigenvectors of \( J\mathcal{L} \) such that

\[
\langle L^1 u_{j,k}, u_{j,k} \rangle = -\lambda u_{j,k} - v_{j,k-1}, \quad v_{j,0} = 0
\]

\[
\langle L^2 v_{j,k}, u_{j,k} \rangle = \lambda u_{j,k} + v_{j,k-1}, \quad u_{j,0} = 0.
\]

First note that upon using equation (2.14) and the fact that \( L^1 \) is symmetric,

\[
\langle L^1 u_{j,k}, u_{j,k} \rangle = -\lambda (u_{j,k}, u_{j,k}) - (v_{j,k-1}, u_{j,k}'),
\]

\[
= \lambda (u_{j,k}, v_{j,k}') - (u_{j,k}, v_{j,k-1}).
\]

As a consequence of the last equality in equation (2.15) one has that

\[
\langle L^1 (u_{j,k}, v_{j,k}') \rangle = (u_{j,k}, v_{j,k}') - (v_{j,k-1}, u_{j,k}).
\]

Similarly one sees that

\[
\langle L^2 v_{j,k}, u_{j,k} \rangle = \lambda (u_{j,k}, v_{j,k}) + (u_{j,k-1}, v_{j,k}'),
\]

\[
= -\lambda (v_{j,k}, v_{j,k}') + (u_{j,k}, u_{j,k-1}).
\]
which implies that
\[ \mathcal{J}(u_{j,k}, u_{j,k'}) + (v_{j,k}, u_{j,k'}) = (u_{j,k}, u_{j,k'-1}) - (u_{j,k-1}, u_{j,k'}). \]  
(2.18)

Combining equation (2.16) and equation (2.18) yields the identity
\[ 2\mathcal{J}(u_{j,k}, u_{j,k'}) + (v_{j,k}, u_{j,k'}) = (u_{j,k}, u_{j,k'-1}) + (u_{j,k}, u_{j,k'-1}) - [(u_{j,k-1}, u_{j,k'}) + (u_{j,k-1}, u_{j,k'})]. \]  
(2.19)

Our aim is to prove that
\[ \langle \mathcal{L}_s u_{j,k}, u_{j,k'} \rangle = \langle \mathcal{L}_s' v_{j,k}, v_{j,k'} \rangle, \]
which in view of equation (2.15) and equation (2.17) is equivalent to proving that
\[ \mathcal{J}(u_{j,k}, u_{j,k'}) + (v_{j,k}, u_{j,k'}) = (u_{j,k}, u_{j,k'-1}) + (u_{j,k}, u_{j,k'-1}). \]

We claim that
\[ \langle u_{j,k}, u_{j,k'} \rangle + \langle v_{j,k}, u_{j,k'} \rangle = 0, \]  
(2.20)
for any \( j, j', k, \) and \( k' \), which then gives the required equality. Indeed, the equality clearly holds when either \( k = 0 \) or \( k' = 0 \), i.e.,
\[ \langle u_{j,0}, u_{j,k'} \rangle + \langle v_{j,0}, u_{j,k'} \rangle = 0, \quad k' = 0, \ldots, a, \]
and
\[ \langle u_{j,k}, u_{j,0} \rangle + \langle v_{j,k}, u_{j,0} \rangle = 0, \quad k = 0, \ldots, a. \]

Then using the equality (2.19), an induction argument for \( k = 0, \ldots, a \) and \( k' = 0, \ldots, a' \), gives equation (2.20).

(b) Now suppose that \( \text{Re} \mathcal{J} \neq 0 \); in particular, \( \mathcal{J} \neq \bar{\mathcal{J}} \neq 0 \). In addition to the basis already given for \( E_{\mathcal{J}} \), following the discussion leading to equation (2.14), let
\[ \{(x_{j,k}, y_{j,k})^T : j = 1, \ldots, m, k = 1, \ldots, a\} \]
be a basis for \( E_{\bar{\mathcal{J}}} \) such that
\[ \mathcal{L}_s x_{j,k} = \bar{\mathcal{J}} y_{j,k} - y_{j,k-1}, \quad y_{j,0} = 0 \]
\[ \mathcal{L}_s y_{j,k} = -\bar{\mathcal{J}} x_{j,k} + x_{j,k-1}, \quad x_{j,0} = 0. \]  
(2.21)

First consider the restriction \( \mathcal{L}_s (\mathcal{J})|_{E_{\mathcal{J}}} \). The analogue to equation (2.15) is
\[ \langle \mathcal{L}_s u_{j,k}, u_{j,k'} \rangle = -\mathcal{J}(u_{j,k}, u_{j,k'}) - (u_{j,k-1}, u_{j,k'}) \]
\[ = -\bar{\mathcal{J}}(u_{j,k}, u_{j,k'}) - (u_{j,k}, u_{j,k'-1}), \]  
(2.22)
and the analogue to equation (2.17) is
\[ \langle \mathcal{L}_s' v_{j,k}, v_{j,k'} \rangle = \mathcal{J}(u_{j,k}, u_{j,k'}) + (u_{j,k-1}, u_{j,k'}) \]
\[ = \bar{\mathcal{J}}(u_{j,k}, u_{j,k'}) + (u_{j,k}, u_{j,k'-1}). \]  
(2.23)

As a consequence of Lemma 2.7 one has that
\[ \langle \mathcal{L}_s u_{j,k}, u_{j,k'} \rangle + \langle \mathcal{L}_s v_{j,k}, v_{j,k'} \rangle = 0, \]
which by equation (2.22) and equation (2.23) implies that
\[ (\mathcal{J} - \bar{\mathcal{J}})(u_{j,k}, v_{j,k'}) = (u_{j,k}, u_{j,k'-1}) - (u_{j,k-1}, v_{j,k'}), \]  
(2.24)
where \( \mathcal{J} - \bar{\mathcal{J}} \neq 0 \). Now
\[ \langle u_{j,0}, v_{j,k'} \rangle = \langle u_{j,k}, v_{j,0} \rangle = 0, \quad k = 0, \ldots, a, \quad k' = 0, \ldots, a'. \]
so that by an induction argument from equation (2.24) we conclude that for all \(j, j', k,\) and \(k'\),

\[
\langle u_{j,k}, u_{j',k'} \rangle = 0.
\]

As a consequence of equation (2.22) and equation (2.23) this yields \(L_\pm(\bar{\mu})_{|E_\pm} = \theta\). Similarly, one has that \(L_\pm(\bar{\mu})_{|E_0} = \theta\).

Finally, consider

\[
\langle L_+, u_{j,k}, x_{j',k'} \rangle = -\bar{J}(u_{j,k}, x_{j',k'}) - \langle u_{j,k-1}, x_{j',k'} \rangle = \bar{J}(u_{j,k}, y_{j',k'}) - \langle u_{j,k}, y_{j',k'-1} \rangle.
\]

As a consequence of equation (2.25) one necessarily has that

\[
\bar{J} \{ u_{j,k}, y_{j',k'} \} = \langle u_{j,k} - u_{j',k'-1} \rangle - \langle u_{j,k-1}, x_{j',k'} \rangle.
\]

Similarly one sees that

\[
\langle L_-, u_{j,k}, x_{j',k'} \rangle = \bar{J}(u_{j,k}, y_{j',k'}) + \langle u_{j,k-1}, y_{j',k'} \rangle = -\bar{J}(u_{j,k}, x_{j',k'}) + \langle u_{j,k-1}, y_{j',k'-1} \rangle,
\]

which implies that

\[
\bar{J} \{ u_{j,k}, y_{j',k'} \} = \langle u_{j,k} - u_{j',k'-1} \rangle - \langle u_{j,k-1}, y_{j',k'} \rangle.
\]

Note the similarity of equation (2.26) with equation (2.16), as well as equation (2.28) with equation (2.18). We can then follow the induction argument leading to equation (2.20) to conclude that

\[
\langle u_{j,k}, y_{j',k'} \rangle = \langle L_+ u_{j,k}, x_{j',k'} \rangle = \langle L_- u_{j,k}, y_{j',k'} \rangle = 0,
\]

for all \(j, j', k,\) and \(k'\). Together with equation (2.25) and equation (2.27) this gives

\[
\langle L_+ u_{j,k}, x_{j',k'} \rangle = \langle L_- u_{j,k}, y_{j',k'} \rangle,
\]

which completes the proof. \(\square\)

**Remark 2.23.** In [30] the unfolding of a purely imaginary eigenvalue with \(m_4(\bar{\mu}) = 1\) and \(m_4(\bar{\mu}) = k\) was considered. Therein they showed that \(2n(L_+(\bar{\mu})) \in [k - 1, k + 1]\) if \(k\) is odd, and \(2n(L_+(\bar{\mu})) = k\) if \(k\) is even.

**Notation 2.24.** For a Hermitian matrix \(S \in \mathbb{C}^{\infty \times \infty}\) define the cone \(C(S)\) by

\[
C(S) := \{ c \in \mathbb{C}^n : (S, c) < 0 \} \cup \{ \theta \}.
\]

For self-adjoint operators \(S\) define the cone \(C(S)\) by

\[
C(S) := \{ u \in H : (Su, u) < 0 \} \cup \{ \theta \}.
\]

We can now refine the result in Theorem 2.17 in the following way. The result concerning \(k_c\) strengthens that presented in [15, Theorem 1.3] (also see [19] and [14, Theorem 2.6]). The result concerning \(k_c + k_c^{-}\) appears to be new.

**Theorem 2.25.** Consider the eigenvalue problem of equation (2.1) under the Assumption 2.1 with \(J\) and \(L\) given by equation (2.11). Then the spectrum of \(JL\) is symmetric with respect to both the imaginary and the real axis and

\[
k_c + k_c^{-} = n(L_+) + n(L_-),
\]

with \(k_c\) and \(k_c^{-}\) even numbers.

Furthermore, let \(\{ u_j, U_j \}_{j \in \mathbb{Z}}\) be a basis consisting of generalized eigenvectors for \(H = \tilde{H} \times \tilde{H}\), as in Proposition 2.19(b), with \(u_j = (u_j, v_j)^T\), and consider the Hermitian matrices \(L_k \in \mathbb{C}^{\infty \times \infty}\) defined as

\[
(L_+)_j := \langle L_+ u_j, u_j \rangle, \quad \langle L_- \rangle_j := \langle L_- u_j, u_j \rangle.
\]

Then

\[
k_c + k_c^{-} = 2 \dim[C(L_+) \cap C(L_-)],
\]

and

\[
k_c = [n(L_+) - n(L_-)] + 2 (\min[n(L_+), n(L_-)] - \dim[C(L_+) \cap C(L_-)]) \geq |n(L_+) - n(L_-)|.
\]
**Proof:** The first part follows immediately from equation (2.11) and Theorem 2.17. Now consider the second part, and first notice that by Lemma 2.20
\[
L_k = \bigoplus_{j \in \sigma^t(JL)} L_j(\bar{j}),
\]
where \(L_j(\bar{j})\) are the matrices defined in Lemma 2.21 and Lemma 2.22.

If \(\Im \bar{j} > 0\), then one has as a consequence of Lemma 2.22 that
\[
C(L_+(\bar{j})) = C(L_-(\bar{j})).
\]
In addition, if \(\Re \bar{j} \neq 0\) one has that
\[
k_\epsilon(\bar{j}) = n(L_+(\bar{j})) = \dim[C(L_+(\bar{j}))] = \dim[C(L_+(\bar{j})) \cap C(L_-(\bar{j}))],
\]
and similarly, if \(\Re \bar{j} = 0\)
\[
k_\epsilon^-(\bar{j}) = \dim[C(L_+(\bar{j})) \cap C(L_-(\bar{j}))].
\]
Next, if \(\Im \bar{j} = 0\), then by Lemma 2.21(c) one can conclude that
\[
C(L_-(\bar{j})) \cap C(L_+(\bar{j})) = \emptyset.
\]
Thus, upon summing over the eigenvalues \(\bar{j} \in \sigma^t(JL)\) and following the argument at the end of the proof of Theorem 2.13 one can conclude that
\[
\dim[C(L_+) \cap C(L_-)] = \sum_{\bar{j} \in \sigma \cap \sigma^t(JL)} k_\epsilon(\bar{j}) + \sum_{\bar{j} \in \sigma \cap \sigma^t(JL)} k_\epsilon^-(\bar{j}).
\]
Consequently, upon using the equalities \(k_\epsilon(\bar{j}) = k_\epsilon(\bar{j})\) and \(k_\epsilon^-(\bar{j}) = k_\epsilon^-(\bar{j})\), one has that
\[
k_\epsilon + k_\epsilon^- = 2 \dim[C(L_+) \cap C(L_-)].
\]
For the last property, assume without loss of generality that \(n(L_+) \geq n(L_-)\). Upon using the above result one has that
\[
k_\epsilon = n(L_+) + n(L_-) - (k_\epsilon + k_\epsilon^-)
= n(L_+) - n(L_-) + 2 (n(L_-) - \dim[C(L_+) \cap C(L_-)]).
\]
Furthermore, due to the basis result in Proposition 2.19(c) one can conclude that
\[
\dim[C(L_-)] = n(L_-).
\]
and since \(\dim[C(L_+) \cap C(L_-)] \leq \dim[C(L_-)]\) one gets the desired inequality. \(\square\)

### 2.3. Refinements

The previous results can be refined so that they are not dependent upon the basis formed by the generalized eigenvectors. As a consequence of equation (2.14) one can write
\[
u_{j,k} = \bar{j}L_+^{-1}(u_{j,k} + \frac{1}{\bar{j}}u_{j,k-1}), \quad u_{j,0} = 0;
\]
hence,
\[
\langle L_-u_{j,k}, u_{j',k'} \rangle = |\bar{j}|^2 \langle L_+^{-1}(u_{j,k} + \frac{1}{\bar{j}}u_{j,k-1}), (u_{j',k'} + \frac{1}{\bar{j}}u_{j',k'-1}) \rangle.
\]
In particular, this implies that if \(m_\epsilon(\bar{j}) = m_\epsilon(\bar{j})\) for some \(\bar{j} \in \sigma(JL)\), then \(L_-(\bar{j}) = |\bar{j}|^2L_+^{-1}(\bar{j})\), where the latter matrix is formed in the obvious manner. The above results then are valid with \(L_+\) being replaced by \(L_-\). Furthermore, since a cone is independent of the basis, one can replace \(C(L_+) \cap C(L_-)\) with \(C(L_+) \cap C(L_-)\). This observation leads to the following result. The condition that \(\bar{j} \in \sigma(JL) \cap \mathbb{R}\) satisfy \(m_\epsilon(\bar{j}) = m_\epsilon(\bar{j})\) appears to be necessary. This issue will be further discussed after the proof.
Corollary 2.26. Consider the eigenvalue problem of equation (2.1) under the Assumption 2.1 with \( J \) and \( L \) given by equation (2.11). If all \( j \in \sigma(J, L) \cap \mathbb{R}^+ \) satisfy \( m_a(j) = m_b(j) \), then the result of Theorem 2.25 can be reformulated as

\[
k_\varepsilon + k_\varepsilon^* = 2 \dim[C(L_+) \cap C(L_-^1)],
\]

and

\[
k_\varepsilon = [n(L_+) - n(L_-)] + 2 \left( \min[n(L_+), n(L_-)] - \dim[C(L_+) \cap C(L_-^1)] \right) \geq [n(L_+) - n(L_-)].
\]

Proof: Suppose that \( \text{Im } j > 0 \). Suppose that each Jordan block has length \( a_j \) for \( j = 1, \ldots, m_b(j) \). Recall equation (2.29) and equation (2.30) for \( j = 1, \ldots, m_b(j) \) and \( k = 1, \ldots, a_j \). Now order the eigenfunctions as

\[
\{ u_1, 1, \ldots, u_1, a_1, u_2, 1, \ldots, u_2, a_2, \ldots, u_{m_b(j)}, 1, \ldots, u_{m_b(j)}, a_{m_b(j)} \},
\]

and define the nilpotent matrices \( N \in C^{\text{tot}} \) via

\[
(N_{ij})_{ij} = \begin{cases} 1, & i = 1, \ldots, t - 1, j = i + 1 \\ 0, & \text{otherwise}. \end{cases}
\]

If one sets

\[
\begin{array}{c}
S(j) = \text{diag} \left( S_{1}(j), \ldots, S_{m_b(j)}(j) \right), \\
S_{j}(j) = \text{id} + \frac{1}{j} N_{ij}.
\end{array}
\]

then upon substituting the expression of equation (2.29) into the quadratic form \( \langle L_-(j)c, c \rangle \) yields that if \( \text{Re } j = 0 \), then

\[
L_-(j) = j^2 S^*(j) L_-^1(j) S(j).
\]

(2.31)

where the notation consistent with Lemma 2.22(a) yields

\[
(L_-^1(j))_{ij} = \langle L_-^1 u_i, u_j \rangle.
\]

Note that \( S(j) = \text{id} + N \), where \( N \) is a nilpotent matrix; hence, \( S \) is nonsingular. On the other hand, if \( \text{Re } j > 0 \), then the matrix \( L_+(j) \) given in Lemma 2.22 is given by

\[
L_+(j) = j^2 S^*(j) L_-^1(j) S(j)
\]

where the notation consistent with Lemma 2.22(b) yields

\[
(L_-^1(j))_{ij} = \langle L_-^1 u_i, x_j \rangle.
\]

First suppose that \( j \in \mathbb{R}^+ \) with \( n(L_+(j)) = 0 \). The codimension of such eigenvalues is \( n(L_+) + n(L_-) \). As a consequence of the above discussion and Lemma 2.22 one has that

\[
C(L_+(j)) = C(L_-(j)) = \{ 0 \};
\]

consequently, by equation (2.31) one has that \( C(L_-^1(j)) = \{ 0 \} \). Now consider an eigenvalue for which \( n(L_+(j)) \geq 1 \), and initially assume that all such eigenvalues are semisimple, so that \( S(j) = \text{id} \) for each such eigenvalue. If \( \text{Im } j > 0 \), then as a consequence of Lemma 2.22 one has

\[
C(L_+(j)) = C(L_-^1(j)).
\]

In addition, if \( \text{Re } j \neq 0 \) one has that

\[
k_\varepsilon(j) = n(L_+(j)) = \dim[C(L_+(j))] = \dim[C(L_+(j)) \cap C(L_-^1(j))],
\]

and similarly, if \( \text{Re } j = 0 \)

\[
k_\varepsilon^*(j) = \dim[C(L_+(j)) \cap C(L_-^1(j))].
\]

Next, if \( \text{Im } j = 0 \), then by Lemma 2.21(c) one can conclude that

\[
C(L_+(j)) \cap C(L_-^1(j)) = \{ 0 \}.
\]
Thus, upon summing over the eigenvalues \( \lambda \in \sigma(\mathcal{L}) \) and following the argument at the end of the proof of Theorem 2.13 one can conclude that
\[
\dim(C(L_+) \cap C(L_-)) = \sum_{\lambda \in \sigma(\mathcal{L})} k_+(\lambda) + \sum_{\lambda \in \sigma(\mathcal{L})} k_-(\lambda).
\]
Consequently, upon using the equalities \( k_+(\lambda) = k_-(\lambda) \) and \( k_+(\lambda) = k_-(\lambda) \), one has that
\[
k_+ + k_- = 2 \dim(C(L_+) \cap C(L_-)).
\]
Recall that the eigenvectors and generalized eigenvectors form a basis. Since cones are independent of a given basis, and since \( C(L_+) \) corresponds to taking a cone with a particular basis, one finally has that
\[
k_+ + k_- = 2 \dim(C(L_+) \cap C(L_-)).
\]
All that is left to do is to remove the assumption that the finite number of eigenvalues with \( \text{Im} \lambda \in \mathbb{R}^+ \) having \( n(L_+(\lambda)) \geq 1 \) are semisimple. It is important to keep in mind that there are only a finite number of such eigenvalues. Eigenvalues having a nontrivial Jordan block are nongeneric; hence, a generic perturbation will cause these eigenvalues to unfold to semisimple eigenvalues. Let \( \mathcal{L}_k^+ \) correspond to a relatively compact symmetric perturbation of the operators \( \mathcal{L}_k \). If the eigenvalue is originally purely imaginary, then it will unfold into eigenvalues which are either purely imaginary or have nonzero real part (e.g., see \([30]\)). If the eigenvalue is originally complex with nonzero real part, then it will unfold into eigenvalues having the same property. In either case, for the operators \( \mathcal{L}_k^+ \) one recovers the desired result since the eigenvalues are now all semisimple. Since cones are open sets which depend continuously upon \( \epsilon \), one can conclude that the result holds in the limit. □

**Remark 2.27.** If the assumption that all \( \lambda \in \sigma(\mathcal{L}) \cap \mathbb{R}^+ \) satisfy \( m_+(\lambda) = m_-(\lambda) \) in Corollary 2.26 is dropped, then all that can be said is that
\[
k_+ + k_- \leq 2 \dim(C(L_+) \cap C(L_-)).
\]

The above perturbation argument fails in the case of initially real-valued eigenvalues, for in this case the perturbed eigenvalues will either be purely real (no cone intersection) or complex with nonzero imaginary part (cones coincide). Hence, the limiting behavior is unclear. It is natural to wonder if a different argument will allow one to remove the condition that all \( \lambda \in \sigma(\mathcal{L}) \cap \mathbb{R}^+ \) satisfy \( m_+(\lambda) = m_-(\lambda) \).

Consider the following example, which suggests that it is indeed necessary. Let \( \mathcal{L} \in \mathbb{R}^{4 \times 4} \) with
\[
\mathcal{L}_+ = \begin{pmatrix} 2a & a + b \\ a + b & 2b \end{pmatrix}, \quad \mathcal{L}_- = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

It is not difficult to check that \( \sigma(\mathcal{L}_+) = \{ \pm \sqrt{b - a} \} \), and that each eigenvalue \( \lambda \) satisfies \( m_+(\lambda) = 1 \) and \( m_-(\lambda) = 2 \). Upon noting that \( \mathcal{L}_+^{-1} = \mathcal{L}_- \) one has that
\[
C(L_+) = \{(x, y) : ax^2 + (a + b)xy + by^2 < 0\}, \quad C(L_-^{-1}) = \{(x, y) : x^2 - y^2 < 0\}.
\]

Note that
\[
\partial C(L_+) = \{(x, y) \in \mathbb{R}^2 : y = -x \text{ and } y = -\frac{a}{b}x \}, \quad \partial C(L_-^{-1}) = \{(x, y) \in \mathbb{R}^2 : y = \pm x \}.
\]

Assuming that \( b > a \), so that \( \sigma(\mathcal{L}_+) \subset \mathbb{R} \), it is not difficult to check that if \( a, b \in \mathbb{R}^+ \), then \( \dim(C(L_+) \cap C(L_-^{-1})) = 0 \); otherwise, \( \dim(C(L_+) \cap C(L_-^{-1})) = 1 \). Thus, it is not possible to use the dimension of the intersection of the cones in order to say anything definitive in this case. On the other hand, if \( b < a \), so that \( \sigma(\mathcal{L}_+) \subset \mathbb{iR} \), then one has \( \dim(C(L_+) \cap C(L_-^{-1})) = 1 \).

In practice it is difficult to precisely calculate \( \dim(C(L_+) \cap C(L_-^{-1})) \). However, one can derive a computable lower bound for this quantity. Let \( N_+ (N_-) \subset C(L_+) (C(L_-^{-1})) \) be maximal subspaces, and suppose that an orthonormal basis for each is given by \( \{ \phi_1^+, \ldots, \phi_{n_+}^+ \} \), where \( n_+ = n(L_+) \). It is clear that
\[
\dim(N_+ \cap N_-) \leq \dim(C(L_+) \cap C(L_-^{-1})).
\]
If \( u \in N_+ \cap N_- \), then there exists vectors \( c^+ \in \mathbb{C}^n \) such that
\[
  u = \sum_{j=1}^n c_j^+ \phi_j^+ = \sum_{j=1}^n c_j^- \phi_j^-.
\]

Upon using the orthonormality of the basis functions one sees that the above is equivalent to
\[
  c^+ = Nc^-, \quad c^- = N^* c^+; \quad N_g := \langle \phi_i^+, \phi_i^- \rangle.
\]

The above is equivalent to the system
\[
  c^+ = NN^* c^+, \quad c^- = N^* Nc^-.
\]

Thus,
\[
  \dim(N_+ \cap N_-) = \dim(\ker(NN^* - \text{id})) = \dim(\ker(N^* N - \text{id})].
\]

which yields
\[
  \dim(\ker(NN^* - \text{id})) \leq \dim(C(\mathcal{L}_+) \cap C(\mathcal{L}_-)).
\]

As a consequence of equation (2.33) one now has the following result.

Corollary 2.28. Under the nondegeneracy assumption given in Corollary 2.26 one has that
\[
  k_+ + k_- \geq 2 \dim(\ker(NN^* - \text{id})],
\]

where \( N \) is defined in equation (2.32).

For a final theoretical result, we are now in position to refine the result of Theorem 2.13. As is seen in [21], the eigenvalue problem of equation (2.1) is equivalent to that of equation (2.11) upon setting
\[
  \mathcal{L}_+ \coloneqq \mathcal{L}, \quad \mathcal{L}_- \coloneqq -\mathcal{J} \mathcal{L} \mathcal{J}.
\]

The primary effect of looking at the original system in this fashion is that for equation (2.34) there is now an artificial symmetry with respect to the real axis. In particular, if \((\bar{\lambda}; u_{\bar{\lambda}})\) is an eigenvalue/eigenfunction pair for equation (2.1), then for equation (2.34) one has the pairs \((\bar{\lambda}; (u_{\bar{\lambda}}, \mathcal{J}^{-1} u_{\bar{\lambda}}))\) and \((-\bar{\lambda}; (u_{\bar{\lambda}}, -\mathcal{J}^{-1} u_{\bar{\lambda}}))\). The major implication of introducing this artificial symmetry is that each of the quantities \( k_+ \) and \( k_- \) in Theorem 2.13 is now doubled for equation (2.34). The Jordan block structure of the eigenvalues are not affected, however.

Taking into account the doubling of \( k_+ \) and \( k_- \), under the appropriate nondegeneracy condition one can now use Corollary 2.26 to state that for equation (2.1),
\[
  k_+ + k_- = \dim(C(\mathcal{L}) \cap C(-\mathcal{J}^{-1} \mathcal{L}^{-1} \mathcal{J}^{-1})).
\]

Let us now derive a lower bound similar to that given in Corollary 2.28. Let \( N \subset C(\mathcal{L}) \) be a maximal subspace, and suppose that an orthonormal basis for \( N \) is given by \{\( \phi_1, \ldots, \phi_n \)\}, where \( n = n(\mathcal{L}) \). It is a straightforward calculation to show that an orthonormal basis for the maximal subspace of \( C(-\mathcal{J}^{-1} \mathcal{L}^{-1} \mathcal{J}^{-1}) \) is given by \{\( \mathcal{J} \phi_1, \ldots, \mathcal{J} \phi_n \)\}. Consequently, the matrix \( N \) derived in equation (2.32) satisfies \( N_g = \langle \psi, \mathcal{J} \psi \rangle \).

Since \( N \) is skew-symmetric, one has that \( NN^* = -N^2 \), so that
\[
  \dim(\ker(NN^* - \text{id})) = \dim(\ker(iN - \text{id})] + \dim(\ker(iN + \text{id})].
\]

Upon applying the above discussion to Corollary 2.28, we have now proved the following.

Corollary 2.29. Suppose that for the eigenvalue problem of equation (2.1) that all \( \bar{\lambda} \in o(\mathcal{J} \mathcal{L}) \cap \mathbb{R}^+ \) satisfy \( m_+(\bar{\lambda}) = m_-(\bar{\lambda}) \). Then one can refine Theorem 2.13 by further stating that
\[
  k_+ + k_- = \dim(C(\mathcal{L}) \cap C(-\mathcal{J}^{-1} \mathcal{L}^{-1} \mathcal{J}^{-1})) \geq \dim(\ker(iN - \text{id})] + \dim(\ker(iN + \text{id})].
\]

where \( N \in C(\mathcal{L})^{n(\mathcal{L})} \) is skew-symmetric and is given by
\[
  N_g = \langle \psi, \mathcal{J} \psi \rangle.
3. Eigenvalue problems of NLS-type

In this section we apply the general results of Section 2 to the eigenvalue problem
\[ \mathcal{J}L u = \lambda u, \]  
(3.1)
where \( \mathcal{J} = J \in \mathbb{R}^{n \times n} \) is a skew-symmetric, invertible matrix, and
\[ L := -D_2 \partial_x^2 + D_1 \partial_x + Q(x). \]  
(3.2)
Here \( D_2 \in \mathbb{R}^{n \times n} \) is a positive definite, symmetric matrix, \( D_1 \in \mathbb{R}^{n \times n} \) a skew-symmetric matrix, and \( Q : \mathbb{R} \to \mathbb{R}^{n \times n} \) a smooth map with \( Q(x) \) symmetric for all \( x \in \mathbb{R} \). The form of the operator is motivated by the intended application, i.e., the study of spectra of periodic waves in equations of NLS type (e.g., see [2–4, 11]) and the references therein). Therefore, unlike in [20], we will assume here that the map \( Q \) is 2L-periodic for some \( L > 0 \), i.e., \( Q(x + 2L) = Q(x) \).

The operators \( L \) and \( \mathcal{J}L \) are considered in \( L^2(\mathbb{R}; \mathbb{R}^n) \). Consequently, all of the spectra are continuous, and there are no isolated point eigenvalues with finite multiplicity. However, upon using a Bloch-wave decomposition, i.e., Floquet theory, one has that solving equation (3.1) is equivalent to solving for each \( y \in [-\pi, \pi) \) the eigenvalue problem
\[ \mathcal{J}L_y u = \lambda_y u, \quad u(-L) = u(L), \]  
(3.3)
where the operator \( L_y \) is formed by replacing
\[ \partial_x \mapsto \partial_x + \frac{y}{2L}, \]
so that
\[ L_y := -D_2 \left( \partial_x + \frac{y}{2L} \right)^2 + D_1 \left( \partial_x + \frac{y}{2L} \right) + Q(x). \]  
(3.4)
More precisely, one has that
\[ \sigma(\mathcal{J}L) = \bigcup_{y \in [-\pi, \pi)} \sigma(\mathcal{J}L_y). \]  
(3.5)
where the operator \( \mathcal{J}L_y \) is taken in the Hilbert space \( H := L^2_{\text{per}}([-L, L]; \mathbb{C}^n) \) equipped with the standard inner product (e.g., see [28]).

In contrast to \( \mathcal{J}L \) the operator \( \mathcal{J}L_y \) has compact resolvent, since its domain \( H^2_{\text{per}}([-L, L]; \mathbb{C}^n) \) is compactly embedded in \( H \), and it satisfies Assumption 2.1 provided \( 0 \not\in \sigma(L_y) \). Indeed, Assumption 2.1(a) is clearly satisfied, and Assumption 2.1(b) holds since \( L_y \) is self-adjoint, with compact resolvent, and invertible. Moreover, \( L_y \) is a relatively compact perturbation of the positive operator \( -D_2 \left( \partial_x + \frac{y}{2L} \right)^2 \), which implies that its spectrum is bounded from below, \( \sigma(L_y) \subset [-M, \infty) \), for some \( M > 0 \). Together with the fact that \( \sigma(L_y) \) is a discrete set, this shows that \( L_y \) has only a finite number of negative eigenvalues, which implies Assumption 2.1(c). Next, \( \mathcal{J}L_y \) is a relatively compact perturbation of the operator with constant coefficients
\[ \mathcal{A}_y^0 := -\mathcal{J}D_2 \left( \partial_x + \frac{y}{2L} \right)^2 + \mathcal{J}D_1 \left( \partial_x + \frac{y}{2L} \right), \]
and the difference
\[ \mathcal{B} := \mathcal{J}L_y - \mathcal{A}_y^0 = Q(x). \]
is bounded. Consequently, equation (2.2) holds with \( r = 0 \). A standard Fourier analysis shows that the spectrum of \( \mathcal{A}_y^0 \) consists of eigenvalues \( \lambda_{kj} = i\omega_{kj}, j = 1, \ldots, n, k \in \mathbb{Z} \), with \( (\lambda_{kj})_{k=1}^{\infty} \) eigenvalues of the skew-symmetric matrix
\[ \mathcal{J}D_2 \left( k + \frac{y}{2L} \right)^2 + i\mathcal{J}D_1 \left( k + \frac{y}{2L} \right), \quad k \in \mathbb{Z}. \]
Analyzing the location of these eigenvalues on the imaginary axis, one finds that \( \lambda_{kj} = O(k^2) \), as \( k \to \infty \), and conclude that Assumption 2.1(d)(ii)-(iii) holds for \( s = 1 \) and \( r' = 1/2 \). This shows that \( \mathcal{J} \) and \( L_y \) satisfy Assumption 2.1.
We point out that as a consequence of equation (3.4), one has that

$$\overline{\mathcal{L}_y} = \mathcal{L}_{-y}. \tag{3.6}$$

so that if \( \tilde{\eta} \in \sigma(\mathcal{J}\mathcal{L}_y) \) with associated eigenfunction \( v \), then \( \tilde{\eta} \in \sigma(\mathcal{J}\mathcal{L}_{-y}) \) with associated eigenfunction \( \overline{v} \). Thus, one can conclude that for equation (3.1) one has the standard Hamiltonian quartet \( \{\pm \tilde{\eta}, \pm \overline{\eta}\} \in \sigma(\mathcal{J}) \). However, this quartet is recovered only by considering \( \sigma(\mathcal{J}\mathcal{L}_y) \cup \sigma(\mathcal{J}\mathcal{L}_{-y}) \) for \( 0 \leq y \leq \pi \).

We can now apply the results in Section 2, more precisely Theorem 2.13, which together with equation (3.6) give the following.

**Theorem 3.1.** Consider the eigenvalue problem (3.3). Let \(-\pi \leq y < \pi \) be such that \( \dim[\ker(\mathcal{L}_y)] = 0 \). One has that

$$k(y) + k_y(y) + k_y(-y) = n(\mathcal{L}_y).$$

Furthermore, \( \sigma(\mathcal{J}\mathcal{L}_y) = \overline{\sigma(\mathcal{J}\mathcal{L}_{-y})} \), and \( k(y) = k(y), k_y(y) = k_y(-y) \), \( k_y(y) = k_y(-y) \).

**Remark 3.2.**

(a) As a consequence of the last part of Theorem 3.1, we may restrict from now on to \( y \in [0, \pi] \).

(b) In stability problems, \( \mathcal{J}\mathcal{L} \) represents the linearization about the periodic wave and if the problem possesses a translation invariance in space (which is often the case) then the derivative of the periodic wave belongs to the kernel of \( \mathcal{J}\mathcal{L}_0 \), so that the condition \( \dim[\ker(\mathcal{L}_y)] = 0 \) does not hold for \( y = 0 \). However, for \( y \neq 0 \) this zero eigenvalue of \( \mathcal{J}\mathcal{L}_0 \) typically moves off the origin, so that this condition is satisfied. One expects in such problems that this assumption will only be violated for a finite number of values of \( y \). As we shall see, in our examples here the condition \( \dim[\ker(\mathcal{L}_y)] = 0 \) is only violated for \( y = 0 \). Notice, that since the full spectrum of \( \mathcal{J}\mathcal{L} \) is continuous, this restriction does not qualitatively effect the final result.

We consider now the particular case \( D_1 = \theta \) and \( Q(-x) = Q(x) \). Set

$$\mathcal{U}_1[u(x)] := u(-x), \tag{3.7}$$

so that \( \mathcal{U}_1 \) is a unitary operator on \( H \). Then it is easy to verify that \( \mathcal{U}_1\mathcal{L} = \mathcal{L}\mathcal{U}_1 \) and \( \mathcal{U}_1\mathcal{J} = \mathcal{J}\mathcal{U}_1 \). For the Bloch operator \( \mathcal{J}\mathcal{L}_y \) it is not difficult to check that \( \mathcal{U}_1\mathcal{L}_{-y} = \mathcal{L}_{-y}\mathcal{U}_1 \), and then

$$\mathcal{U}_1\overline{\mathcal{L}_y} = \mathcal{L}_y\mathcal{U}_1.$$ Consequentely, \( \mathcal{U}_1, \mathcal{J}, \) and \( \mathcal{L}_y \) satisfy Assumption 2.15(a). Also note that in this case \( \sigma(\mathcal{J}\mathcal{L}_y) = \sigma(\mathcal{J}\mathcal{L}_{-y}) \). By applying the results of Theorem 2.17 we obtain the following.

**Theorem 3.3.** Under the assumptions in Theorem 3.1, further suppose that \( D_1 = \theta \) and \( Q(-x) = Q(x) \). Then the spectrum of \( \mathcal{J}\mathcal{L}_y \) is symmetric with respect to both the imaginary and the real axis and the result in Theorem 3.1 holds with \( k(y) \) and \( k_y(-y) \) even numbers.

Finally, consider the canonical form

$$\mathcal{J} = \begin{pmatrix} \theta & \text{id} \\ -\text{id} & \theta \end{pmatrix}, \quad \mathcal{L} = \text{diag}(\mathcal{L}_+, \mathcal{L}_-). \tag{3.8}$$

which is achieved by taking

$$D_2 = \text{diag}(D_2^+, D_2^-), \quad D_1 = \theta, \quad Q(x) = \text{diag}(Q_+(x), Q_-(x)).$$

Then in (3.4) one finds

$$\mathcal{L}_y = \text{diag}(\mathcal{L}_+(y), \mathcal{L}_-(y)).$$

As in the previous case, one has \( \sigma(\mathcal{J}\mathcal{L}_y) = \sigma(\mathcal{J}\mathcal{L}_{-y}) \), and the result of Theorem 2.25 gives:

**Theorem 3.4.** Assume that \( \mathcal{J} \) and \( \mathcal{L} \) are as in equation 3.8. Let \(-\pi \leq y < \pi \) be such that \( \dim[\ker(\mathcal{L}_y(y))] = 0 \). Then the result in Theorem 2.25 holds for \( \mathcal{J} \) and \( \mathcal{L}_y \).
3.1. NLS equation with periodic potential

Consider the nonlinear Schrödinger equation
\[ iq_t + q_{xx} + \omega q + f(x, |q|^2)q = 0, \]
where \( f \) is a smooth, real-valued map, and \( f(x + 2L, \cdot) = f(x, \cdot) \). Here \( q(x, t) \in \mathbb{C} \) and \( (x, t) \in \mathbb{R} \times \mathbb{R}^+ \). A physically relevant example is
\[ f(x, |q|^2) = g|q|^2 + V_0 \text{sn}^2(x, k), \]
where \( g = \pm 1 \) and \( \text{sn}(x, k) \) is the Jacobian elliptic sine function with \( 0 \leq k < 1 \) [e.g., see \( 3, 4 \)]. In this case, if \( g = +1 \) then equation (3.9) is a model equation for an attractive Bose-Einstein condensate (BEC), whereas if \( g = -1 \) equation (3.9) is a model equation for a repulsive BEC. The form of the potential allows one to explicitly solve the steady-state problem associated with equation (3.9) via the use of elliptic functions. We discuss this particular case in the next section.

Suppose that \( Q(x) \) is a real-valued, periodic steady solution to equation (3.9) that satisfies \( Q(x + 2L) = \pm Q(x) \). Note that this implies that \( Q^2(x) \) is \( 2L \)-periodic. Upon linearizing about this solution one finds the eigenvalue problem given by equation (3.1) with \( J \) and \( L \) as in (3.8), where
\[ L_+ := -\partial_x^2 + Q_+(x), \quad Q_+(x + 2L) = Q(x), \]
with
\[ Q_+(x) := -f(x, \nabla^2(x)) - 2D_2f(x, \nabla^2(x))\nabla^2(x) - \omega, \]
\[ Q_-(x) := -f(x, \nabla^2(x)) - \omega. \]

Note that \( L_\pm \) are operators of Hill type (see \( 24 \)). Their spectra in \( L^2(\mathbb{R}) \) consist of unions of bands with nontrivial interior \([\mu_0, \mu_1] \cup [\mu_2, \mu_1] \cup [\mu_2, \mu_3] \cup \cdots \), where the solution of the eigenvalue problem corresponding to the band edge \( \mu_j \) is periodic with period \( 2L \) (the periodic eigenvalue), and the solution corresponding to the band edge \( \mu'_j \) is periodic with period \( 4L \) (the antiperiodic eigenvalue). In terms of Floquet theory the eigenvalue \( \mu_j \) has a Floquet exponent of \( \gamma = 0 \), while the eigenvalue \( \mu'_j \) has a Floquet exponent of \( \gamma = \pi \). In terms of the associated Bloch operators \( L_\pm(y) \), the band edge \( \mu_j \) corresponds to an eigenvalue for \( y = 0 \), while the band edge \( \mu'_j \) corresponds to an eigenvalue for \( y = \pi \). If \( \mu_{j-1}' = \mu_{2j} \) or \( \mu_{2j-1}' = \mu_{2j} \), then this corresponds to a closed gap, and such a point is referred to as a double point. For each \( y \in (0, \pi) \) there is precisely one value of \( \mu(y) \) in the interior of each band such that \( y \) is the Floquet exponent for that value of \( \mu(y) \); furthermore, \( \mu(y) \) depends continuously upon \( y \) and \( \mu(y) = \mu(-y) \). At the band edge, in addition to the periodic or antiperiodic solution there is a solution which grows linearly, unless the band edge is a double point, in which case there are two linearly independent periodic or antiperiodic solutions.

We are now in position to present another proof of the instability result in [2, Theorem 3.3], as well as its natural generalization.

**Theorem 3.5.** The periodic wave \( Q(x) \) is spectrally unstable in each of the following two cases:

(a) if \( \mu = 0 \) is in the interior of a band of \( \sigma(L_+) \);

(b) if \( L_+ \) and \( L_- \) have a different number of bands contained in \( \mathbb{R}^+ \cup \{0\} \).

In both cases, \( k_0(y) \geq 1 \), for some \( y \in (0, \pi) \).

**Proof:** We use the results of Theorem 3.4. Since \( L_- Q = 0 \), and \( Q(x) \) satisfies \( Q(x + 2L) = \pm Q(x) \), one has that \( \mu = 0 \) is a band edge for \( L_- \). The set \( \sigma(L_-) \cap (\mathbb{R}^+ \cup \{0\}) \) will then contain a finite number of bands, so that for each \( y \in (0, \pi) \) one will have \( n(L_-(y)) = k \) for some fixed \( k \in \mathbb{N}_0 \), which is independent of \( y \). Now consider the operator \( L_+ \) under the assumption that \( \mu = 0 \) is in the interior of a band, say (without loss of generality) \([\mu_j, \mu_{j+1}]\), for some \( j \in \mathbb{N}_0 \). Let \( y_k \in (0, \pi) \) be such that \( \mu(y_k) = 0 \). One then has that
\[ n(L_+(y)) = \begin{cases} j + 1, & 0 < y < y_k \\ j, & y_k < y < \pi. \end{cases} \]
Together with $n(\mathcal{L}_-(\gamma)) = k$, upon applying Theorem 3.4, this gives
\[
k_0(\gamma) \geq \begin{cases} 
|k - (j + 1)|, & 0 < \gamma < \gamma_c \\
|k - j|, & \gamma_c < \gamma < \pi.
\end{cases}
\]
Consequently, one has that $k_0(\gamma) \geq 1$ for either $\gamma \in (0, \gamma_c)$ or $\gamma \in (\gamma_c, \pi)$ (or both, if $k \not\in \{j, j+1\}$).

The second part is proved in the same way, since the fact that $\mathcal{L}_+$ and $\mathcal{L}_-$ have different number of bands in $\mathbb{R}^- \cup \{0\}$ implies that $n(\mathcal{L}_+(\gamma)) - n(\mathcal{L}_-(\gamma)) \neq 0$, for certain $\gamma \in (0, \pi)$. \hfill \Box

**Remark 3.6.** If $\mu = 0$ is not in the interior of a band of $\sigma(\mathcal{L}_+)$, or if the number of bands is equal, then it may be the case that the underlying wave still has associated unstable spectra; however, it is no longer necessarily true that some of this unstable spectra is purely real.

### 3.1.1. Example: Bose-Einstein condensates

We consider now the particular case of the nonlinearity given in equation (3.10). Following the work presented in [3] one can explicitly find three real-valued steady-state periodic solutions $Q(x)$ to equation (3.9):

(a) $Q(x) = \sqrt{-g(V_0 + 2k^2)} \, \text{sn}(x, k)$, \hspace{1em} $\omega = 1 + k^2$;

(b) $Q(x) = \sqrt{g(V_0 + 2k^2)} \, \text{cn}(x, k)$, \hspace{1em} $\omega = 1 - 2k^2 - V_0$;

(c) $Q(x) = \sqrt{g(V_0 + 2k^2)/k} \, \text{dn}(x, k)$, \hspace{1em} $\omega = -2 + 2k^2 - V_0/k^2$;

for $0 \leq k < 1$. Regarding the operators $\mathcal{L}_+$ in equation (3.11) one has that
\[
Q_+(x) = -\omega^+_{\text{sn, cn, dn}} + (2V_0 + 6k^2) \, \text{sn}^2(x, k), \hspace{1em} Q_-(x) = -\omega^-_{\text{sn, cn, dn}} + 2k^2 \, \text{sn}^2(x, k),
\]

where
\[
\omega^+_{\text{sn, cn}} = 1 + k^2, \hspace{1em} \omega^+_{\text{dn}} = 1 + 4k^2 + 2V_0, \hspace{1em} \omega^-_{\text{sn, cn, dn}} = 4 + k^2 + \frac{2V_0}{k^2}.
\]

and
\[
\omega^-_{\text{sn, cn}} = 1 + k^2, \hspace{1em} \omega^-_{\text{dn}} = 1.
\]

Note that $Q_+(x + 2K(k)) = Q_-(x)$, where $K(k)$ is the elliptic integral of the first kind. Further note that $Q_-(x)$ are even in $x$; hence, the result of Theorem 3.4 is applicable here, as well as the result of Theorem 3.5.

First consider the operator $\mathcal{L}_-$. The operator
\[
\mathcal{H}_1 := -\partial_x^2 + 2k^2 \, \text{sn}^2(x, k)
\]
is the 1-gap Lame operator, and consequently its spectrum is known precisely (e.g., see [9]). The band edges are given by $\mu \in \{k^2, 1 + k^2\}$, and the associated periodic and antiperiodic eigenfunctions are given by $\{\text{dn}(x, k), \text{cn}(x, k), \text{sn}(x, k)\}$. Thus, for each $\gamma \in (0, \pi)$ one has that
\[
n(\mathcal{L}_-(\gamma)) \begin{cases} 
1, & Q(x) \propto \text{sn}(x, k) \\
1, & Q(x) \propto \text{cn}(x, k) \\
0, & Q(x) \propto \text{dn}(x, k).
\end{cases}
\]

(3.12)

For the operator $\mathcal{L}_+$, notice that if $V_0 = -2k^2$ then $\mathcal{L}_+ = \mathcal{L}_-$.

First suppose that $Q(x) \propto \text{dn}(x, k)$. Since $n(\mathcal{L}_-(\gamma)) = 0$, by Theorem 3.4 it follows that
\[
k_0(\gamma) = n(\mathcal{L}_+(\gamma)), \hspace{1em} k_0^-(\gamma) = k_0(\gamma) = 0.
\]

In addition, if $V_0 = -2k^2$, then $n(\mathcal{L}_+(\gamma)) = n(\mathcal{L}_-(\gamma)) = 0$, so that the wave $Q \equiv 0$ is spectrally stable. Next, a standard argument shows that the band edge $\mu_0(V_0)$ of $\mathcal{L}_+$, satisfies
\[
\frac{d}{dV_0} \mu_0 = -\frac{2}{k^2} \langle \text{dn}^2(x, k), u^2(x) \rangle < 0.
\]
In this case the result of Theorem 3.4 is inconclusive. However, for the eigenvalue
\( g(\gamma) = 0 \), for any \( \gamma \in (0, \pi) \), if \( g = -1 \), whereas \( n(L_{\gamma}(\gamma)) \geq 1 \), for some \( \gamma \in (0, \pi) \), if \( g = +1 \). Hence, in the case \( g = -1 \) the wave is spectrally stable, and in fact a ground state, whereas in the case \( g = +1 \) the wave is spectrally unstable (see \([3, 4]\) for a previous proof of this result).

Next suppose that \( Q(x) \propto \text{cn}(x, k) \). The operator
\[
\mathcal{H}_2 := -\partial_x^2 + 6k^2 \text{sn}^2(x, k)
\]
is the 2-gap Lame operator, and its spectrum is known precisely (e.g., see \([9]\)): in particular, the band edges are given by
\[
\mu \in \left\{ 2 \left( 1 + k^2 - \sqrt{k^4 - k^2 + 1} \right), 1 + k^2, 1 + 4k^2, 4 + k^2, 2 \left( 1 + k^2 + \sqrt{k^4 - k^2 + 1} \right) \right\}.
\]

Furthermore, the eigenfunction associated with \( \mu = 1 + k^2 \) is \( \partial_x \text{sn}(x, k) \), and the eigenfunction associated with \( \mu = 1 + 4k^2 \) is \( \partial_x \text{cn}(x, k) \).

![Figure 1](color online) The spectrum of \( L_{\gamma} \) associated with \( Q(x) \propto \text{cn}(x, k) \) in the case that \( g = +1 \).

First it suppose that \( g = +1 \). Then if \( V_0 = 0 \) we find \( L_+ = \mathcal{H}_2 - 1 - 4k^2 \), so that \( n(L_+(\gamma)) = 1 \), for \( \gamma \in (0, \pi) \). In this case the result of Theorem 3.4 is inconclusive. However, for the eigenvalue \( \mu_2'(V_0) \) which satisfies \( \mu_2'(0) = 0 \) one can show that
\[
\frac{d}{dV_0} \mu_2' = -2(\text{cn}^2(x, k), u^2(x)) < 0.
\]
Hence, \( \mu_2'(V_0) < 0 \) for \( V_0 > 0 \) (see Figure 1), which by Theorem 3.5 implies that the wave is unstable with \( k_0(\gamma) \geq 1 \), for \( V_0 > 0 \) and some values of \( \gamma \). (see \([2, \text{Section 5.2}]\). If \( V_0 \in (-2k^2, 0) \) then one can only conclude from Theorem 3.4 that
\[
k_0(\gamma) + k_0(\gamma) + k_0^{-1}(\gamma) = 2, \quad \gamma \in (0, \pi).
\]

We present in Figure 2 a numerical computation of the spectrum done with the package described in \([5, 7]\). This shows that the wave is spectrally unstable for \( k = 0.8 \) and \( V_0 = -0.1 \), whereas it is spectrally stable for \( k = 0.8 \) and \( V_0 = -0.8 \). Note, however, that in both cases \( k_0(\gamma) = 0 \).

Now suppose that \( g = -1 \). For the eigenvalue \( \mu_1'(V_0) \) which satisfies \( \mu_1'(-2k^2) = 0 \) one can show that
\[
\frac{d}{dV_0} \mu_1' = -2(\text{cn}^2(x, k), u^2(x)) < 0.
\]
Hence, \( \mu_1'(V_0) > 0 \) for \( V_0 < -2k^2 \), which by Theorem 3.5 implies that the wave is unstable with \( k_0(\gamma) \geq 1 \) for some values of nonzero \( \gamma \).

Finally suppose that \( Q(x) \propto \text{sn}(x, k) \). For \( V_0 = -2k^2 \) we have \( L_+ = L_- \), and for the eigenvalue \( \mu_2'(V_0) \) which satisfies \( \mu_2'(-2k^2) = 0 \) one can show that
\[
\frac{d}{dV_0} \mu_2' = 2(\text{cn}^2(x, k), u^2(x)) > 0.
\]
Figure 2: The spectrum associated with $Q(x) \propto \text{cn}(x, k)$ for $g = +1$ in the case that $k = 0.8$ and $V_0 = -0.8$ (left panel) and $k = 0.8$ and $V_0 = -0.1$ (right panel). In both cases $k_\gamma(y) + k_\gamma(y) + k_\gamma^{-}(y) = 2$.

If $g = +1$, then $\mu_1'(V_0) < 0$ for $V_0 < -2k^2$, which by Theorem 3.5 implies that the wave is unstable with $k_\gamma(y) \geq 1$ for some values of nonzero $\gamma$ (see also the discussion in [2, Section 5.1]).

Next, suppose that $g = -1$. Then $L_\pm = \mathcal{H}_2 - 1 - k^2$, for $V_0 = 0$, and for the eigenvalue $\mu_1'(V_0)$ which satisfies $\mu_1'(0) = 0$ one can show that

$$\frac{d}{dV_0} \mu_1' = 2\langle \text{sn}^2(x, k), u^2(x) \rangle > 0.$$ 

Hence, $\mu_1'(V_0) > 0$ for $V_0 > 0$ (see Figure 3), which by Theorem 3.5 implies again that the wave is unstable with $k_\gamma(y) \geq 1$ for some values of $\gamma$ and for $V_0 > 0$. If $V_0 \in (-2k^2, 0)$ then one can only conclude from Theorem 3.4 that

$$k_\gamma(y) + k_\gamma(y) + k_\gamma^{-}(y) = 2, \quad \gamma \in (0, \pi).$$

In Figure 4 it is seen that the wave is spectrally unstable for $k = 0.8$ and $V_0 = -0.8$, whereas it is spectrally stable in the integrable case of $V_0 = 0$. Note that $k_\gamma(y) = 0$ for all values of $\gamma$ in both cases.

### 3.2. Quasi-periodic solutions to the defocusing NLS equation

In this section we consider the classical defocusing NLS equation

$$i q_t + q_{xx} + q - |q|^2 q = 0.$$  \hspace{1cm} (3.13)

As it is well-known this equation possesses, up to continuous symmetries, a two-parameter family of quasi-periodic steady solutions [e.g., see [4, 8, 10]]. Any such quasi-periodic solution can be written as

$$q(x) = e^{i \phi x} \mathcal{Q}(2k \phi). \hspace{1cm} (3.14)$$
where \( p \in \mathbb{R} \), \( k \in \mathbb{R}^+ \), and \( Q : \mathbb{R} \to \mathbb{C} \) is \( 2\pi \)-periodic. Here \( k = \pi/T \), where \( T \in \mathbb{R}^+ \) is the minimal period of the modulus \(|q_s|\). Alternatively, one can parameterize this family by the “angular momentum” \( \Omega \) and the “energy” \( E \):

\[
\Omega := \text{Im}(\partial_s q_s) \in \mathbb{R}, \quad E := \frac{1}{2} |\partial_s q_s|^2 + \frac{1}{2} |q_s|^2 - \frac{1}{4} |q_s|^4 \in \mathbb{R},
\]

which are conserved quantities for the time-independent equation. Notice that if \( \Omega = 0 \), then the solution \( q_s \) is real-valued and periodic with period \( 2T \); otherwise, \( q_s \) is complex-valued and quasi-periodic (see [10] for further details).

The linearized eigenvalue problem associated with the quasi-periodic solution \( q_s \) to equation (3.13) turns out to be an eigenvalue problem of the form (3.1) with \( J \) being the canonical skew-symmetric matrix (see equation (3.8)).

\[
D_2 = 4k^2 \text{id}, \quad D_1 = 4pkJ, \quad Q(x) = \begin{pmatrix}
p^2 - 1 + 3R^2 + I^2 \\
2RI \\
p^2 - 1 + R^2 + 3I^2
\end{pmatrix},
\]

where \( Q(x) := R(x) + iI(x) \). Note that although the underlying solution \( q_s \) is quasi-periodic, the linear operator \( \mathcal{L} \) obtained in this way has periodic coefficients with \( L = \pi \). We can now introduce the associated Bloch operators \( J \mathcal{L}_y \) as explained in the first part of Section 3.

The stability of these quasi-periodic solutions has been studied in [10, 11] (see also the references therein). While the result in [10] shows that these waves are orbitally stable with respect to perturbations which preserve the period \( T \) of modulus and the Floquet exponent \( p \), the result in [11] proves the spectral stability of the waves with small amplitude, \(|q_s| \ll 1\), with respect to localized and bounded perturbations. More precisely, the latter result shows that the spectrum of \( J \mathcal{L} \) in \( L^2(\mathbb{R}) \times L^2(\mathbb{R}) \) entirely lies on the imaginary axis. The proof also relies upon a Bloch-wave decomposition, together with a detailed analysis of the spectra of \( J \mathcal{L}_y \). In particular, this result shows that

\[
k_c(y) = k_s(y) = 0,
\]

for any \( y \in (0, \pi) \), provided \(|q_s|\) is sufficiently small. Our purpose here is to apply the general Theorem 3.1 in order to obtain a spectral information for all waves \( q_s \). We prove the following:

**Theorem 3.7.** For any \( y \in (0, \pi) \) one has that \( \dim(\ker(\mathcal{L}_y)) = 0 \) and

\[
k_c(y) + k_c(y) + k'_c(y) = 2.
\]

**Proof:** The main part of the proof consists in proving that \( \dim(\ker(\mathcal{L}_y)) = 0 \). We use the idea presented in [8, Section 3]. Write the complex-valued solution \( Q = R + iI \) as the vector \( Q = (R, I)^T \). As a consequence of

\[\text{Figure 4: The spectrum associated with } Q(x) \propto \text{sn}(x, k) \text{ for } g = -1 \text{ in the case that } k = 0.8 \text{ and } V_0 = -0.8 \text{ (left panel)} \text{ and } k = 0.8 \text{ and } V_0 = 0 \text{ (right panel). In both cases } k_c(y) + k_s(y) + k'_c(y) = 2.\]
the translation invariance and the rotation invariance of equation (3.13), two linearly independent solutions to the linear problem \( L u = 0 \) are given by \( u_1 = Q_k \) and \( u_2 = iQ \). In [10, equation (3.7)] it is shown that two other linearly independent solutions are given by

\[
u_3 = Q_k + 2xQ_x, \quad \nu_4 = Q_p + ixQ.
\]

where \( Q_k \) and \( Q_p \) denote the derivatives of \( Q \) with respect to \( k \) and \( p \), respectively. Define \( \Psi(x) \in \mathbb{R}^{4 \times 4} \) by

\[
\Psi(x) := \begin{pmatrix}
  u_1 & u_2 & u_3 & u_4 \\
  u_{1,x} & u_{2,x} & u_{3,x} & u_{4,x}
\end{pmatrix},
\]

and set

\[
\Phi := \Psi(2\pi)\Psi(0)^{-1}.
\]

As is seen in [8, Section 3], \( \mu \) is an eigenvalue of the \( 4 \times 4 \)-matrix \( \Phi \) with \( |\mu| = 1 \) if and only if \( j = 0 \) is an eigenvalue of \( L_y \) for \( y = \arg(\mu) \).

We now compute the matrix \( \Phi \) and show that it has only one eigenvalue, \( \mu = 1 \). As constructed in [10, Section 2] the wave \( q \) satisfies the initial condition

\[
Q(0) = r_2 > 0, \quad Q_x(0) = i\frac{\Omega}{2k} \left( \frac{\Omega}{r_2} - pr_2 \right) \in i\mathbb{R},
\]

and in addition \( Q_{xx}(0) \in \mathbb{R} \). Then we conclude that

\[
\Psi(0) := \left[
\begin{array}{cccc}
0 & 0 & R_k(0) & R_p(0) \\
L_k(0) & R(0) & 0 & 0 \\
R_{xx}(0) & -I_k(0) & 0 & 0 \\
0 & 0 & I_{xx}(0) + 2I_k(0) & I_{px}(0) + R(0)
\end{array}
\right].
\]

In order to calculate \( \Psi(2\pi) \) we use the fact that \( G(\cdot) \) is \( 2\pi \)-periodic. Clearly, \( (u_j, u_{j,x})^T(2\pi) = (u_j, u_{j,x})^T(0) \) for \( j = 1, 2 \), while

\[
(u_3, u_{3,x})^T(2\pi) = (u_3, u_{3,x})^T(0) + 4\pi(0, I_k(0), R_{xx}(0), 0)^T
\]

\[
(u_4, u_{4,x})^T(2\pi) = (u_4, u_{4,x})^T(0) + 2\pi(0, R(0), -2I_k(0), 0)^T.
\]

A relatively straightforward calculation now shows that

\[
\Phi = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\alpha_1 & 1 & 0 & \alpha_3 \\
\alpha_2 & 0 & 1 & \alpha_4 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

where \( \alpha_j \in \mathbb{R} \) for \( j = 1, \ldots, 4 \), and that \( \mu = 1 \) is an eigenvalue of \( \Phi \) with algebraic multiplicity four.

As a consequence of the above argument, the number \( n(L_y) \) of negative eigenvalues of \( L_y \) is independent of \( p \) and \( k \). Thus, it is sufficient to compute \( n(L_y) \) for a particular value, e.g. \( \Omega = 0 \). This is the case (b), with \( g = -1 \) and \( V_0 = 0 \), presented in Section 3.1.1 in which \( n(L_y) = 2 \). One can now apply Theorem 3.1 and get the desired result.

**Remark 3.8.**

(a) The result in the above theorem allows to give an alternative proof for the spectral stability result obtained in [11] for small waves. The idea of proof in [11] is to show that the eigenvalues of \( JL_y \) are purely imaginary by locating successively (a) the spectrum for values of \( y \) bounded away from 0, (b) the eigenvalues outside a neighborhood of the origin for small \( y \), and finally (c) the eigenvalues close to the origin for small \( y \). The result of Theorem 3.7, more precisely the fact that \( k_0(y) + k_0(y) \) is closed, allows to treat in a different way the first two situations, (a) and (b). However, the proof of the case (c) remains the same. We present a spectral stability proof based on the result of Theorem 3.7 in the next section for the periodic waves of the generalized KdV equation.
(b) The orbital stability result in [10, Proposition 3.1], mentioned above, relies upon the general approach in [16, 17]. In particular, it shows that for γ = 0 the kernel of \( L_0 \) is two-dimensional, due to translation invariance and rotation invariance, and \( n(L_0) = 1 \). For this situation, one can adapt the results presented in [20, 21] to conclude that

\[
k_r(0) + k_c(0) + k_r^-(0) = 0.
\]

(3.15)

In addition, this orbital stability result shows that instabilities may arise as sideband instabilities, only. However, the result in [11, Theorem 2] shows that this is not the case for small waves, \( 0 < |q_s| \ll 1 \), and numerical evidence suggests that the waves are spectrally stable for all \( k \in \mathbb{R}^+ \) and all \( p \in \mathbb{R} \).

c) One can also construct quasi-periodic solutions to the focusing NLS equation

\[
iq_t + q_{xx} + \omega q + |q|^2 q = 0, \quad \omega \in \{-1, +1\}.
\]

The method of proof for Theorem 3.7 does not rely upon the form of the nonlinear term; instead, it uses the symmetries inherent in the problem. Consequently, the result of Theorem 3.7 holds in this case, as well. Nevertheless, in this case the small waves, \( 0 < |q_s| \ll 1 \), are spectrally unstable, and the result in [11, Section 5] allows to conclude that

\[
k_r(\gamma) = 0, \quad k_c(\gamma) = 1, \quad k_r^-(\gamma) = 1,
\]

for sufficiently small \( \gamma \neq 0 \) (see also [1, 2, 22, 26, 29] and the references therein for spectral stability results in NLS-type equations). In contrast, in the case \( \gamma = 0 \) the equation (3.15) holds for sufficiently small waves [11], just as in the defocusing case. Numerical evidence strongly suggests that equation (3.15) holds for the entire family of waves [10, Section 4].

4. Eigenvalue Problems of KdV-type

In this section we apply the general results of Section 2 to the eigenvalue problem

\[
\mathcal{J} \mathcal{L} u = \lambda u.
\]

(4.1)

where

\[
\mathcal{J} := \partial_x, \quad \mathcal{L} := -\partial_x^2 + Q(x).
\]

(4.2)

The form of the operator is motivated by the intended application, i.e., the study of the spectra of periodic waves to the generalized KdV equation. We will therefore assume that \( Q \) is \( 2L \)-periodic for some \( L > 0 \), i.e., \( Q(x + 2L) = Q(x) \), and consider the operators \( \mathcal{L} \) and \( \mathcal{J} \mathcal{L} \) in \( L^2(\mathbb{R}; \mathbb{R}) \). Again, all of the spectra will be continuous, and there will be no isolated point eigenvalues with finite multiplicity.

Upon using the Bloch-wave decomposition, just as in the previous section, one has that

\[
\sigma(\mathcal{J} \mathcal{L}) = \bigcup_{\gamma \in [-\pi, \pi]} \sigma(\mathcal{J}_\gamma \mathcal{L}_\gamma).
\]

(4.3)

in which the Bloch operators

\[
\mathcal{J}_\gamma := \partial_x + i \frac{\gamma}{2L}, \quad \mathcal{L}_\gamma := -\left( \partial_x + i \frac{\gamma}{2L} \right)^2 + Q(x).
\]

(4.4)

are acting in the Hilbert space \( H := L^2_{\text{per}}([-L, L]; \mathbb{C}) \). Note that \( \mathcal{J}_\gamma \) is invertible with a compact inverse for each \( \gamma \neq 0 \), and that \( \mathcal{L}_\gamma \) has compact resolvent. By arguing as in Section 3, one can then show that these operators satisfy Assumption 2.1 for any \( \gamma \neq 0 \), provided \( 0 \notin \sigma(\mathcal{L}_\gamma) \), so that we can apply the results given in Section 2.

As a consequence of equation (4.2) one has that

\[
\mathcal{J}_\gamma \mathcal{L}_\gamma = \mathcal{J}_{-\gamma} \mathcal{L}_{-\gamma}.
\]
Furthermore, one can conclude that for equation (4.1) one has the standard Hamiltonian quartet \((\pm J, \pm J^*)\) and \(J^\star = J\). Again, this quartet is recovered only by considering \(\sigma(\mathcal{F}_y L_y) \cup \sigma(\mathcal{F}_y L_{-y})\) for \(0 \leq y \leq \pi\). The following is an immediate application of Theorem 2.13.

**Theorem 4.1.** Consider the eigenvalue problem of equation (4.1). Let \(y \in [-\pi, \pi]\setminus\{0\}\) be such that \(\dim(\ker(L_y)) = 0\). One has that

\[
k_i(y) + k_k(y) + k_i^*(y) = n(L_y).
\]

Furthermore, \(\sigma(\mathcal{F}_y L_y) = \sigma(\mathcal{F}_y L_{-y})\), and \(k_i(y) = k_i(-y), k_k(y) = k_k(-y), k_i^*(y) = k_i^*(-y)\).

### 4.1. Periodic waves to the gKdV equation

Consider the generalized KdV equation with power nonlinearity, written in a frame of reference moving with (positive) speed \(c\):

\[
q_t + (q_{xx} - cq + q^{p+1})_x = 0, \quad p \geq 1, \quad c > 0.
\]

The scaling invariance of this equation allows us to set \(c = 1\), without loss of generality. The purpose of this section is to study the spectral stability of steady periodic waves of (4.5).

The steady-state solutions of (4.5) satisfy the ODE

\[
q_{xx} = q - q^{p+1} + b.
\]

for some constant \(b\). We first restrict to \(b = 0\), and recall below some well-known facts about the solutions of this second order ODE. (Note that for the KdV equation, \(p = 1\), one can always assume \(b = 0\), due to the Galilean invariance.) This equation has critical points at \((q, q_0) = (0, 0)\) and \((q, q_0) = (1, 0)\). In the \((q, q_0)\) phase plane the nonlinear center \((1, 0)\) is enclosed in a one-parameter family of periodic orbits which are bounded by an orbit which is homoclinic to the saddle point \((0, 0)\). These periodic orbits can be parameterized by \(0 < a < 1\); we express this dependence as \(q_a(x)\) with \((q_a, \partial_x q_a)(0) = (a, 0)\). The period \(T_a\) is smooth and satisfies

\[
\lim_{a \to 0^+} T_a = +\infty, \quad \lim_{a \to 1^-} T_a = \frac{2\pi}{\sqrt{p}}.
\]

Up to spatial translations, we then have a one-parameter family of steady periodic waves \(q_a\) to equation (4.5). While \(q_a \to 1\) as \(a \to 1\), in the limit \(a \to 0\) one finds the (well-known) solitary wave of the gKdV equation.

Upon linearizing about the periodic wave \(q_a\), we obtain a linear eigenvalue problem of the form equation (4.1) with

\[
\mathcal{F} = \partial_x, \quad L = -\partial_x^2 + Q_a(x; c),
\]

where

\[
Q_a(x; c) := 1 - (p + 1)q^p_a(x).
\]

Note that \(L\) is a Hill operator of the form discussed in Section 3.1 with \(L = T_a/2\). As a consequence of the spatial translation invariance associated with equation (4.5) one knows that \(L\partial_c q_a = 0\). Since \(\partial_c q_a\) is periodic with the same period as \(q_a\), and since this solution has precisely one sign change over its period, by using a Sturm-Liouville type argument and the properties presented in Section 3.1 one can conclude that there are precisely two bands of spectra for \(L\) contained in the closed left-half of the complex plane. Furthermore, \(J = 0\) corresponds to a band edge with \(y = 0\) for all \(a\). A direct application of Theorem 4.1 now gives:

**Theorem 4.2.** Consider the eigenvalue problem (4.1) with \(\mathcal{F}\) and \(L\) given by equation (4.7). Let \(y \in [-\pi, \pi]\setminus\{0\}\). One has that

\[
k_i(y) + k_k(y) + k_i^*(y) = 2.
\]

**Remark 4.3.** Suppose that \(p > 4\). As is seen in [12, Section 3], if \(0 < a \ll 1\), i.e., if the periodic wave \(q_a\) is sufficiently close to the solitary wave \(q_0\) in phase space, then the wave is spectrally unstable. The instability is due to the presence of a simple loop of unstable eigenvalues in a neighborhood of \(J_0\), where \(J_0 \in \mathbb{R}^+\) is an eigenvalue associated with the linearization about \(q_0(x)\). In this situation, one can then conclude that

\[
1 \leq k_i(y) + k_k(y) \leq 2, \quad 0 \leq k_i^*(y) \leq 1.
\]
Our goal is to refine the conclusion of Theorem 4.2 for the periodic waves which are close to the equilibrium $(1,0)$, i.e. for $0 < 1 - a < 1$, in order to obtain a spectral (in)stability result for these waves. We now consider the entire family of periodic waves, i.e. for $b 
eq 0$ also. For small $b$ the ODE equation (4.6) has the equilibrium $Q_b$ with expansion

$$Q_b = 1 + \frac{1}{p} b - \frac{p + 1}{2p^2} b^2 + O(|b|^3).$$

As is true for $b = 0$, this equilibrium is surrounded by a one-parameter family of periodic orbits which give the periodic solutions to equation (4.6)

$$q_{a,b}(x) = P_{a,b}(k_{a,b} x),$$

in which $P_{a,b}$ is a $2\pi$-periodic even solution of

$$k_{a,b}^2 v_{xz} - v + v^{p+1} = b.$$

A straightforward calculation gives the expansions

$$P_{a,b}(z) = Q_b + \cos(z) a - \frac{p + 1}{4} \alpha^2 + \frac{p + 1}{12} \cos(2z) \alpha^2 + O(|\alpha(\alpha^2 + b^2)|),$$

$$k_{a,b}^2 = p + (p + 1)b - \frac{p(p + 1)(p + 4)}{12} \alpha^2 - \frac{p + 1}{p} b^2 + O(|\alpha|^2 + |b|^3).$$

Together with the spatial translation invariance this family of solutions gives a three-parameter family of steady periodic waves to the gKdV equation (4.5). Notice that

$$J_{\mu} = \partial_\mu, \quad L_{\mu} = -k_{a,b}^2 \partial_{zz} + 1 - (p + 1)P_{a,b}^p.$$  

For the associated Bloch operators it is more convenient to set $\mu = \gamma/2\pi$, so that $-1/2 \leq \mu < 1/2$, and then we have

$$J_{\mu} = \partial_\mu + i \mu, \quad L_{\mu a,b} = -k_{a,b}^2(\partial_\mu + i \mu)^2 + 1 - (p + 1)P_{a,b}^p.$$  

These operators have $2\pi$-periodic coefficients so that we consider their action in the Hilbert space $H := L^2_{per}([-\pi, \pi]; \mathbb{C})$. Our main result is the following:

**Theorem 4.4.** Assume $p \geq 1$ and consider the periodic wave $q_{a,b}$ for $a$ and $b$ sufficiently small. If $p < 2$ one has that

$$k_1(\mu) = 0, \quad k_2(\mu) = 0, \quad k_2^-(\mu) = 2,$$

for any $\mu \in \left[-\frac{1}{2}, \frac{1}{2}\right] \setminus \{0\}$, whereas if $p > 2$ one has that

$$k_1(\mu) = 0, \quad k_2(\mu) = 1, \quad k_2^-(\mu) = 1,$$

for sufficiently small $\mu = o(|a|)$. Consequently, the periodic wave $q_{a,b}$ is spectrally stable if $p < 2$ and spectrally unstable if $p > 2$.

**Proof:** Without loss of generality we restrict to $0 \leq \mu \leq 1/2$ (see the last part of Theorem 4.1). The idea is to locate the spectrum of $J_{\mu} L_{\mu a,b}$, starting from the case $a = b = 0$, i.e., the constant solution, in order to determine those values of $\mu$ which could potentially lead to an instability upon perturbation. Using perturbation arguments and the result in Theorem 4.2 we show that for sufficiently small $a$ and $b$ the spectrum of $J_{\mu} L_{\mu a,b}$ splits into two parts: $\sigma_1(J_{\mu} L_{\mu a,b})$ consisting of purely imaginary eigenvalues for all $\mu \in [0, 1/2]$, and $\sigma_2(J_{\mu} L_{\mu a,b})$ consisting of three small eigenvalues which are purely imaginary for all values of $\mu \in [\mu_0, 1/2]$, for any $\mu_0 > 0$. Then, we locate these three eigenvalues for small values of $\mu$, and show that they are purely imaginary for $p < 2$, and that two of them move off the imaginary axis for $p > 2$. This gives the result in the theorem.

When $a = b = 0$ all operators have constant coefficients and a standard Fourier analysis gives the spectra

$$\sigma(L_{0,0}) = \{ \eta_j(\mu) := p((j + \mu)^2 - 1), j \in \mathbb{Z} \} \subset \mathbb{R},$$

Spectra of periodic waves
and
\[ \sigma(J_\mu L_{0,0,\mu}) = \{ \beta_j(\mu) := ip(j + \mu)((j + \mu)^2 - 1), \ j \in \mathbb{Z} \} \subset i\mathbb{R}. \]

In both cases the eigenvalues are semi-simple with associated eigenfunction \( u_j(x) := e^{ix} \) for \( j \in \mathbb{Z} \). Also note that there is a one-to-one correspondence between the eigenvalues of \( L_{0,0,\mu} \) and \( J_\mu L_{0,0,\mu} \) which preserves the eigenfunctions.

First consider \( \sigma(L_{0,0,\mu}) \). It is not difficult to show that for any \( \mu \in [0, 1/2] \) we have
\[ \eta_0(\mu) \geq \frac{5}{4}p, \quad ||v|| \geq 2. \]
and
\[ -p \leq \eta_0(\mu) \leq -\frac{3}{4}p, \quad 0 \leq \eta_1(\mu) \leq \frac{5}{4}p, \quad -\frac{3}{4}p \leq \eta_{-1}(\mu) \leq 0. \]

Similarly, for \( \sigma(J_\mu L_{0,0,\mu}) \) we find
\[ \text{Im} \ f_j(\mu) \in \left( -\infty, -\frac{15}{8}p \right] \cup [6p, +\infty), \quad ||v|| \geq 2 \]
and
\[ \text{Im} \ f_j(\mu) \in \left[ \frac{3}{8}p, \frac{15}{8}p \right], \quad j = -1, 0, +1 \]
(see Figure 5). In particular, this provides us with a spectral decomposition
\[ \sigma(J_\mu L_{0,0,\mu}) = \sigma_1(J_\mu L_{0,0,\mu}) \cup \sigma_2(J_\mu L_{0,0,\mu}). \]

where
\[ \sigma_1(J_\mu L_{0,0,\mu}) \subset \left( -\infty, -\frac{15}{8}p \right] \cup [6p, +\infty), \quad \sigma_2(J_\mu L_{0,0,\mu}) \subset \left[ \frac{3}{8}p, \frac{15}{8}p \right]. \]

with the property that for any \( v \) in the spectral subspace associated with \( \sigma_1(J_\mu L_{0,0,\mu}) \) we have
\[ \langle L_{0,0,\mu}v, v \rangle \geq \frac{5}{4}p ||v||^2. \]

Note that these hold uniformly for any \( \mu \in [0, 1/2] \). A standard perturbation argument then shows that similar properties hold for sufficiently small \( a \) and \( b \). More precisely, one has that
\[ \sigma(J_\mu L_{a,b,\mu}) = \sigma_1(J_\mu L_{a,b,\mu}) \cup \sigma_2(J_\mu L_{a,b,\mu}), \]

where
\[ \sigma_1(J_\mu L_{a,b,\mu}) \subset (\infty, -p] \cup [5p, +\infty), \quad \sigma_2(J_\mu L_{a,b,\mu}) \subset \left[ \frac{2}{3}p, 3p \right]. \]

and for any \( v \) in the spectral subspace associated with \( \sigma_1(J_\mu L_{a,b,\mu}) \) we have
\[ \langle L_{a,b,\mu}v, v \rangle \geq p ||v||^2. \]

for any \( \mu \in [0, 1/2] \) and sufficiently small \( a \) and \( b \). The latter inequality together with the result in Lemma 2.10 now implies that the eigenvalues in \( \sigma_1(J_\mu L_{a,b,\mu}) \) are all purely imaginary.

It is left then to determine the placement of the eigenvalues in \( \sigma_2(J_\mu L_{a,b,\mu}) \). These are the smooth continuation for small \( a \) and \( b \) of the eigenvalues \( \beta_{L_1}(\mu) \) and \( \beta_0(\mu) \) of \( J_\mu L_{0,0,\mu} \). First notice that two such eigenvalues coincide only if \( \mu = 0 \), when \( \beta_{L_1}(0) = \beta_0(0) = 0 \). Furthermore, for any \( \mu_0 > 0 \) there exists a positive constant \( C_0 \) such that
\[ |\beta_j(\mu) - \beta_k(\mu)| \geq C_0 p, \quad j, k \in \{-1, 0, 1\}, \quad \mu \in [\mu_0, 1/2]. \]

Consequently, these eigenvalues will remain simple and distinct under small perturbations (for sufficiently small \( a \) and \( b \)) for all \( \mu \in [\mu_0, 1/2] \). Together with the symmetry of the spectrum with respect to the imaginary axis this shows that these three eigenvalues are purely imaginary, so that for such values of \( \mu \) there are no unstable eigenvalues of \( J_\mu L_{a,b,\mu} \). Using the result from Theorem 4.2 we conclude that
\[ k_4(\mu) = 0, \quad k_5(\mu) = 0, \quad k_7(\mu) = 2, \quad \mu \in [\mu_0, 1/2]. \]
Figure 5: (color online) The union of the spectra of the operators $L_{0,0,\mu}/p$ (top panel) and $J_{\mu}L_{0,0,\mu}/p$ (bottom panel) for $0 \leq \mu \leq 1/2$. Together with the spectra for $-1/2 \leq \mu < 0$ this gives the spectrum in $L^2(\mathbb{R})$ of the operators $L = L_{0,0}$ and $JL = J_{\mu}L_{0,0}$. The labeling via the index $j$ corresponds to the eigenvalues $\eta(\mu)$ (top panel) and $\lambda(\mu)$ (bottom panel).

for sufficiently small $a$ and $b$.

It remains to locate now the three eigenvalues $\bar{\eta}_{\pm 1}(\mu)$ and $\bar{\eta}_0(\mu)$ for small $\mu$. We proceed as in [11, Proposition 4.8], by computing the $3 \times 3$ matrix representing the action of $J_{\mu}L_{a,b,\mu}$ on the spectral space associated to these three eigenvalues. We denote this matrix $M_{a,b,\mu}$.

First, at $a = b = 0$ the operator has constant coefficients, and by choosing the real basis

$$
\xi_0 = \cos(z), \quad \xi_1 = \sin(z), \quad \xi_2 = 1,
$$

a direct calculation gives the matrix

$$
M_{0,0,\mu} = p \begin{pmatrix}
2i\mu + i\mu^3 & 3\mu^2 & 0 \\
-3\mu^2 & 2i\mu + i\mu^3 & 0 \\
0 & 0 & -i\mu + i\mu^3
\end{pmatrix}.
$$

Next, at $\mu = 0$ we have that $J_0L_{a,0,0}(\partial_z P_{a,b}) = 0$, due to translation invariance, so that the kernel of $J_0L_{a,0,0}$ is at least one dimensional. Furthermore, differentiating the equation satisfied by $P_{a,b}$ with respect to $a$ we find

$$
J_0L_{a,0,0}(\partial_a P_{a,b}) = \partial_a(k_{a,b}^2) \partial_z^2 P_{a,b},
$$

and a similar equality is obtained by differentiating with respect to $b$. Consequently,

$$
J_0L_{a,0,0}(\partial_b(k_{a,b}^2)\partial_a P_{a,b} - \partial_a(k_{a,b}^2)\partial_b P_{a,b}) = 0,
$$

which provides a second vector in the kernel of $J_0L_{a,0,0}$. Finally, notice that

$$
J_0L_{a,0,0}(P_{a,b}) = p k_{a,b}^2 \partial_z^3 P_{a,b} - p \partial_z P_{a,b},
$$

and then

$$
J_0L_{a,0,0}(\partial_b(k_{a,b}^2)P_{a,b} - p k_{a,b}^2 \partial_b P_{a,b}) = -p \partial_b(k_{a,b}^2) \partial_z P_{a,b},
$$

which gives a principal vector in the generalized kernel of $J_0L_{a,0,0}$. In particular, this shows that the three eigenvalues $\bar{\eta}_{\pm 1}(\mu)$ and $\bar{\eta}_0(\mu)$ vanish at $\mu = 0$, for all $a$ and $b$, when 0 is a geometrically double and algebraically triple eigenvalue of $J_0L_{a,0,0}$. Furthermore, we have found three vectors in the generalized
kernel which allow to choose a basis, which is compatible to the one found at $\alpha = b = 0$,

\[
\xi_0 = \frac{1}{p + 1} \left( \partial_b(k_{a,b}^2) \partial_a P_{a,b} - \partial_a(k_{a,b}^2) \partial_b P_{a,b} \right) \\
= \cos(z) \left( \frac{2p - 1}{6} \alpha + \frac{p + 1}{6} \cos(2z) \alpha - \frac{2}{p} \cos(z) b + O(a^2 + b^2) \right), \\
\xi_1 = -\frac{1}{a} \partial_a P_{a,b} = \sin(z) + \frac{p + 1}{6} \sin(2z) \alpha + O(a^2 + b^2), \\
\xi_2 = \partial_b(k_{a,b}^2) P_{a,b} - p\, k_{a,b}^2 \partial_b P_{a,b} = 1 + (p + 1) \cos(z) \alpha - \frac{p + 1}{p} b + O(a^2 + b^2).
\]

In this basis we obtain

\[
M_{a,b,0} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & p \alpha \, \partial_b(k_{a,b}^2) \\
0 & 0 & 0
\end{pmatrix}, \\
p \alpha \, \partial_b(k_{a,b}^2) = p(p + 1) \alpha - 2(p + 1) \alpha b + O(|\alpha|(a^2 + b^2)).
\]

The two basis above can be extended to a basis for small $a$, $b$, and $\mu$, in which we then have

\[
M_{a,b,\mu} = M_{0,0,\mu} + M_{a,b,0} + a \mu M_1 + b \mu M_2 + O(|\mu|(a^2 + b^2)) + O(|\mu (|a| + |b|)|).
\]

By computing the terms of order $O(|\mu|)$ and $O(|\mu|)$ in the expansions of $J_\mu \mathcal{L}_{a,b,\mu}$ we find

\[
M_1 = \begin{pmatrix}
0 & 0 & 2i p(p + 1) \\
0 & 0 & 0 \\
i p(p - 2)/2 & 0 & 0
\end{pmatrix}, \\
M_2 = \begin{pmatrix}
2 i(p + 1) & 0 & 0 \\
0 & 2 i(p + 1) & 0 \\
0 & 0 & 2 i(p + 1)
\end{pmatrix}.
\]

In order to determine the location of the three eigenvalues of $M_{a,b,\mu}$ we consider the characteristic polynomial

\[
P(\jmath) = \jmath^3 + c_2 \jmath^2 + c_1 \jmath + c_0,
\]

which has coefficients $c_j$ depending upon $a$, $b$, and $\mu$. Since the spectrum of $J_\mu \mathcal{L}_{a,b,\mu}$ is symmetric with respect to the imaginary axis it is not difficult to conclude that the coefficients $c_2$ and $c_0$ are purely imaginary, whereas $c_1$ is real. Next, recall that $P_{a,b}(z + \pi) = P_{a,b}(z)$ and $k_{a,b} = k_{-a,-b}$, so that the spectra of $J_\mu \mathcal{L}_{a,b,\mu}$ and $J_\mu \mathcal{L}_{-a,-b,\mu}$ are the same. Consequently, the coefficients $c_j$ of the polynomial $P$ are even functions in $\alpha$.

Furthermore, since $\sigma(J_\mu \mathcal{L}_{a,-b,\mu}) = \sigma(J_\mu \mathcal{L}_{-a,b,\mu})$, we also have that $c_2$ and $c_4$ are odd in $\mu$, while $c_1$ is even in $\mu$. Together with the expansion of $M_{a,b,\mu}$ above, these properties imply that the eigenvalues of $M_{a,b,\mu}$ are of the form $\jmath = ip\mu X$ with $X$ root of the polynomial

\[
Q(X) = X^3 + d_2 X^2 + d_1 X + d_0,
\]

with real coefficients $d_i$ which are all even in $\alpha$ and $\mu$. The location of the roots of this polynomial is determined by its discriminant,

\[
\Delta_{a,b,\mu} = 18d_2 d_1 d_0 + d_2^2 d_1^2 - 4d_2 d_1^2 d_0 - 4d_2^2 d_0 - 27d_0^2,
\]

that we may write as

\[
\Delta_{a,b,\mu} = \Delta_{0,b,\mu} + a \alpha^2 + O(a^2(|b| + \mu^2)).
\]

Since at $\alpha = 0$ the operator $J_\mu \mathcal{L}_{a,0,\mu}$ has constant coefficients, the three eigenvalues can be computed explicitly,

\[
\jmath_0(0, b, \mu) = i k_{0,b}^2 \mu (\mu^2 - 1), \\
\jmath_{+1}(0, b, \mu) = i k_{0,b}^2 \mu (\mu^2 + 3 \mu + 2),
\]

from which we find

\[
\Delta_{0,b,\mu} = 2916 \frac{k_{0,b}^2}{p^2} \mu^2 (\mu^2 - 1)^2 \geq 0.
\]

From the explicit expansion of $M_{a,0,\mu}$, using Maple we also compute,

\[
\alpha = -162(p + 1)(p - 2).
\]
Then if \( p < 2 \), the discriminant \( \Delta_{a,b,\mu} \) is positive for sufficiently small \( a, b, \) and \( \mu \), so that the three eigenvalues are purely imaginary. If \( p > 2 \), then \( a < 0 \), and the discriminant is negative for sufficiently small \( \mu = o(a) \). Consequently the polynomial \( Q \) has two complex roots, which then proves that \( M_{a,b,\mu} \) has a pair of complex eigenvalues \( (k_c(\mu) = 1) \). This completes the proof. \( \square \)

We now close with the results of a few numerical simulations. In this discussion let \( k = k_{a,b} \). It will be henceforth assumed that \( b = 0 \). First suppose that \( p < 2 \). As a consequence of Theorem 4.2 the wave is spectrally stable for \( 0 < p - k^2 \ll 1 \). There is strong numerical evidence that the waves remain spectrally stable for all values of \( 0 < k^2_{a,b} < p \). In the case that \( p = 2 \), for which the result of Theorem 4.2 is inconclusive, one again has numerical evidence that the wave is spectrally stable for all values of \( 0 < k^2 < 2 \).

Now suppose that \( p > 2 \), so that the wave is spectrally unstable for \( 0 < p - k^2 \ll 1 \). First suppose that \( p = 3 \). It can be seen numerically that there is a \( k^*_3 > 0 \) such that the wave is spectrally unstable for \( k^*_3 < k^2 < 3 \), while it is spectrally stable for \( k^2 < k^*_3 \). The spectra is illustrated in Figure 6. Next, suppose that \( p = 5 \). It can now be seen numerically that the wave is spectrally unstable for all values of \( 0 < k^2 < 5 \). However, unlike the case of \( p = 3 \), the curve of unstable spectra does not necessarily intersect the point \( \lambda = 0 \) for all values of \( k^2 \) (see Figure 7). The primary difference between the cases of \( p = 3 \) and \( p = 5 \) is that in the former case the limiting solitary wave is spectrally stable, whereas in the latter case the limiting solitary wave is spectrally unstable. Hence, as already discussed in Remark 4.3, in the latter case for \( 0 < k^2 \ll 1 \) one has a band of unstable spectra which is disjoint from \( \lambda = 0 \) surrounding the unstable eigenvalue associated with the solitary wave.

**Figure 6:** The spectrum associated with the operator given in equation (4.7) in the case that \( p = 3 \). The left panel corresponds to \( k^2 \sim 2.73 \), the middle panel corresponds to \( k^2 \sim 1.53 \), and the right panel corresponds to \( k^2 \sim 0.12 \). In all cases one has that \( k_r(\gamma) + k_c(\gamma) + k_*^c(\gamma) = 2 \) for \( \gamma \neq 0 \).

**Figure 7:** The spectrum associated with the operator given in equation (4.7) in the case that \( p = 5 \). The left panel corresponds to \( k^2 \sim 3.68 \), the middle panel corresponds to \( k^2 \sim 0.68 \), and the right panel corresponds to \( k^2 \sim 0.18 \). In all cases one has that \( k_r(\gamma) + k_c(\gamma) + k_*^c(\gamma) = 2 \) for \( \gamma \neq 0 \). Furthermore, in the latter two cases one has that \( k_r(\gamma) \geq 1 \) for some values of \( \gamma \).
APPENDIX: PROOF OF THE SCS BASIS LEMMA

Part (a) is an immediate consequence of Assumption 2.1(a)-(b), and part (b) follows from part (a). In order to prove the parts (c) and (d) we first use the Keldysh completeness theorem (see [13, Theorem X.4.1]) to show that the set of eigenvectors and generalized eigenvectors of $\mathcal{J}L$ is complete in $H$. Set $\mathcal{A} := \mathcal{J}L$ and write

$$\mathcal{A} = (\mathcal{A}_0 - i\omega) + (i\omega + \mathcal{B}),$$

for some $i\omega, \in i\mathbb{R}$ which belongs to the resolvent set of $\mathcal{A}_0$ (since the spectrum of $\mathcal{A}_0$ is a countable set such an $i\omega$ exists). Now by Assumption 2.1(d) the operator $(\mathcal{A}_0 - i\omega)^{-1}$ is compact and skew-adjoint. Since $\mathcal{A}$ is invertible, one has that

$$\mathcal{A}^{-1} = (\mathcal{A}_0 - i\omega)^{-1}(\text{id} + \mathcal{S})^{-1},$$

in which

$$\mathcal{S} := (i\omega + \mathcal{B})(\mathcal{A}_0 - i\omega)^{-1},$$

is compact, and $(\text{id} + \mathcal{S})^{-1}$ is bounded. Consequently, we may write

$$(\text{id} + \mathcal{S})^{-1} = \text{id} + \tilde{\mathcal{S}}, \quad \tilde{\mathcal{S}} := -\mathcal{S}(\text{id} + \mathcal{S})^{-1},$$

so that $\tilde{\mathcal{S}}$ is compact. Summarizing, we have

$$i\mathcal{A}^{-1} = K(\text{id} + \tilde{\mathcal{S}}),$$

in which $K := i(\mathcal{A}_0 - i\omega)^{-1}$ is self-adjoint and $\tilde{\mathcal{S}}$ is compact. Together with the Assumption 2.1(d)(ii), by arguing with the Keldysh completeness theorem [13, Theorem X.4.1], we conclude that the set of eigenvectors and generalized eigenvectors of $i\mathcal{A}^{-1}$ is complete in $H$. Furthermore, with the possible exception of a finite number,

$$\sigma(i\mathcal{A}^{-1}) \subset \{\rho e^{i\theta} : \rho \in \mathbb{R}^+, |\theta| < \epsilon \text{ or } |\theta - \pi| < \epsilon\},$$

for a given $\epsilon > 0$. This implies that the set of eigenvectors and generalized eigenvectors of $\mathcal{A}$ is also complete in $H$ and that (d) holds.

In order to prove that it forms a basis for $H$ we use the necessary and sufficient condition in [25, Lemma 6.1, p. 26]. We construct an increasing sequence of closed sets $(F_k)_{k \geq k_0}$ in the complex plane, for $k_0$ sufficiently large, such that $\bigcup_{k \geq k_0} F_k = \mathbb{C}$, and consider the spectral subspaces $(H_k)_{k \geq k_0}$ of $H$ spanned by the eigenfunctions and generalized eigenfunctions of $\mathcal{A}$ associated to the eigenvalues lying inside $F_k$. These form an increasing sequence of subspaces of $H$ which is complete (since $\bigcup_{k \geq k_0} F_k = \mathbb{C}$), and then according to [25, Lemma 6.1, p. 26] it suffices to show that the corresponding spectral projectors $\Pi_k : H \to H_k$ satisfy

$$\sup_{k \geq k_0} \|\Pi_k\| < \infty.$$

For the construction of the projectors we rely upon the Dunford integral formula and a perturbation argument in which we regard $\mathcal{A}$ as a relatively bounded perturbation of $\mathcal{A}_0$.

We start by constructing the sets $F_k$. Consider the subsequence of eigenvalues $(i\omega_{n_k})_{k \in \mathbb{Z}}$ of $\mathcal{A}_0$ in Assumption 2.1(d)(iii). For each $k \geq 1$ consider the paths $\Gamma_{\pm k}$ defined by

$$\Gamma_{\pm k} = \Gamma_{\pm k}^1 \cup \Gamma_{\pm k}^2 \cup \Gamma_{\pm k}^3,$$

(see Figure 8) in which

$$\Gamma_{\pm k}^1 = [\beta = R_{\pm k} \pm it, \ t \in [0, R_{\pm k}/2]], \quad \Gamma_{\pm k}^2 = [\beta = t \pm R_{\pm k}/2, \ t \in [-R_{\pm k}, R_{\pm k}]],$$

$$\Gamma_{\pm k}^3 = [\beta = -R_{\pm k} \pm it, \ t \in [0, R_{\pm k}/2]],$$

and $R_{\pm k}$ are the positive constants given by

$$R_k = 2(\omega_{n_k} + r_k), \quad r_k = \frac{1}{3} (\omega_{n_k+1} - \omega_{n_k}) \geq \frac{C}{3} \omega_{n_k+1}.'$$

(A.1)
Spectra of periodic waves

and

\[ R_k = 2(|\omega_{n_k} + r_{-k}|, \quad r_{-k} = \frac{1}{3} (|\omega_{n_k} - \omega_{n_{k-1}}|) \geq \frac{c}{3} |\omega_{n_{k-1}}|^r. \]

Assume, without loss of generality, that \( R_{-k} < R_k \). We now construct a closed path oriented counterclockwise \( \Gamma_k \) surrounding the eigenvalues \( i\omega_{n_0}, \ldots, i\omega_0, \ldots, i\omega_{n_0} \) by closing \( \Gamma^{+}_k \cup \Gamma^{-}_k \) (after an appropriate change of orientation of \( \Gamma^j_k \)) with the appropriately oriented paths

\[ \Gamma^0_k = \{ \bar{\eta} = \pm t, \ t \in [R_{-k}, R_k] \}. \]

Then any \( \bar{\eta} \in \Gamma_k \) belongs to the resolvent set of \( A_0 \) and using the fact that the set of eigenvectors of \( A_0 \) forms an orthonormal basis of \( H \) we obtain

\[ \|(\bar{\eta} - A_0)^{-1}\| = \frac{1}{\text{dist}(\bar{\eta}, \sigma(A_0))}. \] (A.2)

The set \( F_k \) is the closed bounded set inside \( \Gamma_k \).

We claim that for \( k \geq k_0 \), with \( k_0 \) sufficiently large, any \( \bar{\eta} \in \Gamma_k \) also belongs to the resolvent set of \( \mathcal{A} \). In order to see this, we start by writing

\[ \bar{\eta} - \mathcal{A} = (\text{id} - \mathcal{B}(\bar{\eta} - A_0)^{-1})(\bar{\eta} - A_0), \]

and estimate now the norm of \( \mathcal{B}(\bar{\eta} - A_0)^{-1} \). According to the inequality (2.2) we find

\[ \|\mathcal{B}(\bar{\eta} - A_0)^{-1}\| \leq a\|(\bar{\eta} - A_0)^{-1}\| + b\|A_0\|\|(\bar{\eta} - A_0)^{-1}\|. \] (A.3)

The norm of \( (\bar{\eta} - A_0)^{-1} \) is given in (A.2) and for the norm of \( |A_0\|\|(\bar{\eta} - A_0)^{-1}\| \) we have

\[ \|\mathcal{B}(\bar{\eta} - A_0)^{-1}\| \leq \sup_{n \in \mathbb{Z}} \frac{|\omega_n|^r}{|\bar{\eta} - i\omega_n|}, \]

(see also Remark 2.2(c)). We consider successively \( \bar{\eta} \in \Gamma^1_{+k}, \Gamma^2_{+k}, \Gamma^0_{+k} \), the other paths can be treated in the same way.

First, for \( \bar{\eta} \in \Gamma^1_{+k} \) we have \( \bar{\eta} = R_k + it, \ t \in [0, R_k/2] \), so that

\[ \|(\bar{\eta} - A_0)^{-1}\| \leq \frac{1}{R_k}. \] (A.4)

and we claim that there exists a positive constant \( C_r \) such that

\[ \|\mathcal{B}(\bar{\eta} - A_0)^{-1}\| \leq \frac{C_r}{R_k^{1+r}}. \] (A.5)
Indeed,
\[ |||\mathcal{A}_0||| (\lambda - \mathcal{A}_0)^{-1}||| \leq \sup_{n \in \mathbb{Z}} \frac{\sqrt{2} |\omega_n|^{r'}}{R_k + |\omega_n - t|}, \]
so that for \(0 \leq n \leq n_k\) we find
\[ |||\mathcal{A}_0||| (\lambda - \mathcal{A}_0)^{-1}||| \leq \frac{\sqrt{2} |\omega_n|^{r'}}{R_k} \leq \frac{\sqrt{2}}{R_k^{1-r'}}. \]
For \(n \geq n_k + 1\) we obtain
\[ |||\mathcal{A}_0||| (\lambda - \mathcal{A}_0)^{-1}||| \leq \left( \frac{r}{1-r} \right)^r \frac{\sqrt{2}}{(R_k - t)^{1-r'}} \leq \left( \frac{r}{1-r} \right)^r \frac{\sqrt{2}}{(R_k/2)^{1-r'}}. \]
and for \(n \leq 0\),
\[ |||\mathcal{A}_0||| (\lambda - \mathcal{A}_0)^{-1}||| \leq \left( \frac{r}{1-r} \right)^r \frac{\sqrt{2}}{(R_k + t)^{1-r'}} \leq \left( \frac{r}{1-r} \right)^r \frac{\sqrt{2}}{(R_k)^{1-r'}}. \]
Next, consider \(\tilde{\lambda} = iR_k/2 + t \in \Gamma^2_{z_k}\), \(t \in [-R_k, R_k]\), when
\[ \|(\lambda - \mathcal{A}_0)^{-1}\| = \frac{1}{|\tilde{\lambda} - i\omega_n|} = \frac{1}{|ir_k + t|} = \frac{1}{\sqrt{r_k^2 + t^2}}. \]
and
\[ |||\mathcal{A}_0||| (\lambda - \mathcal{A}_0)^{-1}||| \leq \sup_{n \in \mathbb{Z}} \frac{|\omega_n|^{r'}}{|\omega_n - \omega_n - r_k|}. \]
For \(0 \leq n \leq n_k\), using the inequality (2.3), we find
\[ |||\mathcal{A}_0||| (\lambda - \mathcal{A}_0)^{-1}||| \leq \frac{|\omega_n|^{r'}}{r_k} \leq \frac{|\omega_n + 1 - \omega_n| |\omega_n|^{r'-r'}}{cr_k} = \frac{3}{c|\omega_n|^{r'-r'}}. \]
for \(n \geq n_k + 1\),
\[ |||\mathcal{A}_0||| (\lambda - \mathcal{A}_0)^{-1}||| \leq \frac{\omega_n^{r'} + 1}{\omega_n + 1 - \omega_n - r_k} = \frac{\omega_n^{r'} + 1}{2r_k} \leq \frac{3}{2c|\omega_n|^{r'-r'}}. \]
and for \(n \leq 0\),
\[ |||\mathcal{A}_0||| (\lambda - \mathcal{A}_0)^{-1}||| \leq \left( \frac{r}{1-r} \right)^r \frac{1}{(\omega_n + r_k)^{1-r'}} \leq \left( \frac{r}{1-r} \right)^r \frac{1}{|\omega_n|^{1-r'}}. \]
Summarizing, for \(\tilde{\lambda} \in \Gamma^2_{z_k}\) we have
\[ |||\mathcal{A}_0||| (\lambda - \mathcal{A}_0)^{-1}||| \leq \frac{C}{\omega_n^y}. \]
where
\[ C = \max \left( \frac{3}{c}, \left( \frac{r}{1-r} \right)^r \right) > 0, \quad y = \min (r' - r, 1 - r) > 0. \]
Finally, for \(\tilde{\lambda} = t \in \Gamma^0_{-z_k}, t \in [R_k, R_k]\), we find
\[ \|(\lambda - \mathcal{A}_0)^{-1}\| \leq \frac{1}{t}. \]
and
\[ |||\mathcal{A}_0||| (\lambda - \mathcal{A}_0)^{-1}||| \leq \sup_{n \in \mathbb{Z}} \frac{\sqrt{2} |\omega_n|^{r'}}{|\omega_n + t| + t} \leq \frac{c'}{t^{1-r'}}. \]
for some \(c' > 0\). One obtains similar estimates for \(\Gamma^0_{-z_k}, \Gamma^1_{-z_k}, \Gamma^2_{-z_k}\), and \(\Gamma^3_{z_k}\).
Together with the inequality (A.3) the estimates (A.4)-(A.9) imply that for \(k \geq k_0, k_0\) sufficiently large, we have
\[ \|B(\lambda - \mathcal{A}_0)^{-1}\| \leq \frac{1}{2}. \]
Consequently, $\text{id} - B(\lambda - A_0)^{-1}$ is invertible, so that $\lambda$ belongs to the resolvent set of $\mathcal{A}$ and

$$(\lambda - \mathcal{A})^{-1} = (\lambda - A_0)^{-1}(\text{id} - B(\lambda - A_0)^{-1})^{-1}.$$  

We consider now the spectral projectors $\Pi_k$ and $\Pi_k^0$ of $\mathcal{A}$ and $\mathcal{A}_0$, respectively, associated with the closed set $F_k$. They can be computed by the Dunford integral formula,

$$
\Pi_k = \frac{1}{2\pi i} \int_{\partial F_k} (\lambda - \mathcal{A})^{-1} d\lambda, \quad \Pi_k^0 = \frac{1}{2\pi i} \int_{\partial F_k} (\lambda - \mathcal{A}_0)^{-1} d\lambda.
$$

According to [25, Lemma 6.1, p. 26] we need to show that

$$
\sup_{k \geq k_0} \|\Pi_k\| < \infty.
$$

Since the set of eigenvectors of $\mathcal{A}_0$ forms a basis of $H$ we have that

$$
\sup_{k \geq k_0} \|\Pi_k^0\| < \infty,
$$

so that it is enough to check that

$$
\sup_{k \geq k_0} \|\Pi_k - \Pi_k^0\| < \infty.
$$

We write

$$
\Pi_k - \Pi_k^0 = \frac{1}{2\pi i} \int_{\Gamma_k} \left( (\lambda - \mathcal{A})^{-1} - (\lambda - \mathcal{A}_0)^{-1} \right) d\lambda,
$$

and estimate successively the norm of the last integral term along the curves $\Gamma_{\pm k}$. Again, we give the calculations for $\Gamma_{\pm k}^1$, $\Gamma_{\pm k}^2$, $\Gamma_{\pm k}^0$, the other paths can be treated in the same way.

First for $\Gamma_{\pm k}^1$, from (A.4) and (A.10) we find

$$
\left\| \frac{1}{2\pi i} \int_{\Gamma_k} \left( (\lambda - \mathcal{A})^{-1} - (\lambda - \mathcal{A}_0)^{-1} \right) d\lambda \right\| \leq \frac{1}{2\pi} \int_0^{\omega_k + r_k} \frac{1}{R_k} \, dt = \frac{1}{2\pi} \frac{\omega_k + r_k}{R_k} = \frac{1}{4\pi}.
$$

For $\Gamma_{\pm k}^2$, using successively (A.6), (A.3), and (A.7), we obtain

$$
\left\| \frac{1}{2\pi i} \int_{\Gamma_k} \left( (\lambda - \mathcal{A})^{-1} - (\lambda - \mathcal{A}_0)^{-1} \right) d\lambda \right\| \leq \frac{1}{n} \int_{-r_k}^{r_k} \frac{1}{\sqrt{r_k^2 + t^2}} \left( \frac{a}{\sqrt{r_k^2 + t^2}} + \frac{bC}{\omega_m^\lambda} \right) \, dt
$$

$$
= \frac{2a}{n} \frac{\arctan(R_k/r_k)}{r_k} + \frac{2bC}{n\omega_m^\lambda} \ln \left( \frac{R_k + \sqrt{R_k^2 + r_k^2}}{r_k} \right)
$$

$$
\leq \frac{2a}{n} \frac{1}{r_k} + \frac{2bC}{n\omega_m^\lambda} \ln \left( \frac{R_k + \sqrt{R_k^2 + r_k^2}}{r_k} \right)
$$

for some $M_1 > 0$ and any $k \geq k_0$; here the last inequality is a consequence of (A.1). Finally, for $\Gamma_{\pm k}^0$, from (A.8), (A.9), and the fact that $R_{k+} \to \infty$, as $k \to \infty$, we find

$$
\left\| \frac{1}{2\pi i} \int_{\Gamma_k} \left( (\lambda - \mathcal{A})^{-1} - (\lambda - \mathcal{A}_0)^{-1} \right) d\lambda \right\| \leq \frac{1}{n} \int_{R_k}^{r_k} \left( \frac{a}{t^2} + \frac{bC}{\omega_m^\lambda} \right) \, dt \leq M_0,
$$

for some $M_0 > 0$ and any $k \geq k_0$. These show that $\sup_{k \geq k_0} \|\Pi_k - \Pi_k^0\| < \infty$ for $k \geq k_0$, and thus ends the proof of the proposition.
References


