

Spots and stripes in NLS-type equations with nearly one-dimensional potentials

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Abstract. We consider the existence of spots and stripes for a class of NLS-type equations in the presence of nearly one-dimensional localized potentials. Under suitable assumptions on the potential, we construct various types of waves which are localized in the direction of the potential and have single- or multihump, or periodic profile in the perpendicular direction. The analysis relies upon a spatial dynamics formulation of the existence problem, together with a center manifold reduction. This reduction procedure allows these waves to be realized as uni- or multipulse homoclinic orbits, or periodic orbits in a reduced system of ordinary differential equations.

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1. INTRODUCTION

Consider a system of NLS-type equations in the plane

$$i\partial_t U + \Delta U + V(x, y)U + F(U) = 0, \quad (1.1)$$

in which U is complex vector-valued, $U(t, x, y) \in \mathbb{C}^N$, V a given potential, and F a smooth nonlinearity. We are interested in solutions of the form

$$U(t, x, y) = e^{i\omega t} Q(x, y),$$

with complex-valued profiles Q satisfying the steady equation

$$\Delta Q - \omega Q + V(x, y)Q + F(Q) = 0. \quad (1.2)$$

In this paper we focus on the existence problem for both the single component system ($N = 1$), and the two-component system ($N = 2$). We suppose that the nonlinearity F is of NLS-type, more precisely that

$$F(U) = Uf(|U|^2),$$

in the first case, and of the form

$$F(U_1, U_2) = (U_1 f_1(|U_1|^2, |U_2|^2), U_2 f_2(|U_1|^2, |U_2|^2)),$$

when $U = (U_1, U_2)$ in the latter case. In both cases, the potential V is assumed to be nearly one-dimensional

$$V(x, y) = V_1(x) + \varepsilon V_2(x, y), \quad (1.3)$$

with V_1 and V_2 typically localized on \mathbb{R} and \mathbb{R}^2 , respectively, and ε a small parameter. We exclude the case of periodic potentials, but as explained in [Section 5](#) the type of approach used in this paper can be extended to such potentials in a straightforward manner. We restrict to mainly two types of solutions: those which are localized in y , namely *spots*, and those which have a periodic profile in y , i.e., *stripes*.

One motivation for investigating such systems is the question of wave formation in Bose-Einstein condensates in traps which are quasi-one dimensional. The governing equations for an N -component Bose-Einstein condensate are given by

$$i\partial_t U_j + \Delta U_j - \omega_j U_j + \sum_{k=1}^N a_{jk} |U_k|^2 U_j = V(x, y) U_j, \quad j = 1, \dots, N, \quad (1.4)$$

where $U_j \in \mathbb{C}$ is the mean-field wave-function of species j , $\mathbf{A} = (a_{jk}) \in \mathbb{R}^{N \times N}$ is symmetric, $\omega_j \in \mathbb{R}$ are free parameters and represent the chemical potential, and $V : \mathbb{R}^2 \mapsto \mathbb{R}$ represents the trapping potential (see [\[2–6, 13, 15, 17, 18, 22\]](#) and the references therein for further details). Clearly these equations are of the form [\(1.1\)](#), and we shall discuss the implications of our results for these equations in [Section 4](#). In order to obtain the existence of spots in the case of a truly one-dimensional trapping potential, i.e., when $V(x, y) = V(x)$, it must be assumed in [\(1.4\)](#) that $a_{jk} \in \mathbb{R}^+$. In other words, we must assume that the intra-species and inter-species interactions are attractive. If the interactions are repulsive, then it is necessary that we have a weak trapping potential in y in order to localize the solutions; otherwise, the solutions can only be stripes.

For the analysis of the existence problem we rely upon a *spatial dynamics* approach which consists in formulating the steady system [\(1.2\)](#) as an infinite dimensional dynamical system in which the evolutionary variable is the spatial variable y . This idea goes back to Kirchgässner [\[20\]](#) and has been extensively used for a wide range problems in extended domains (see e.g. [\[7, 9, 10, 27\]](#) and the references therein). The resulting dynamical system is typically ill-posed but the question of interest is that of existence of bounded solutions. One method which turned out to be particularly useful for answering this question is the center manifold reduction which shows that, under certain assumptions, the small bounded solutions of the infinite dimensional dynamical system lie on a *finite dimensional center manifold*. Consequently, these solutions can be found by solving an ordinary differential equation. In our case, the assumptions concern the potential V

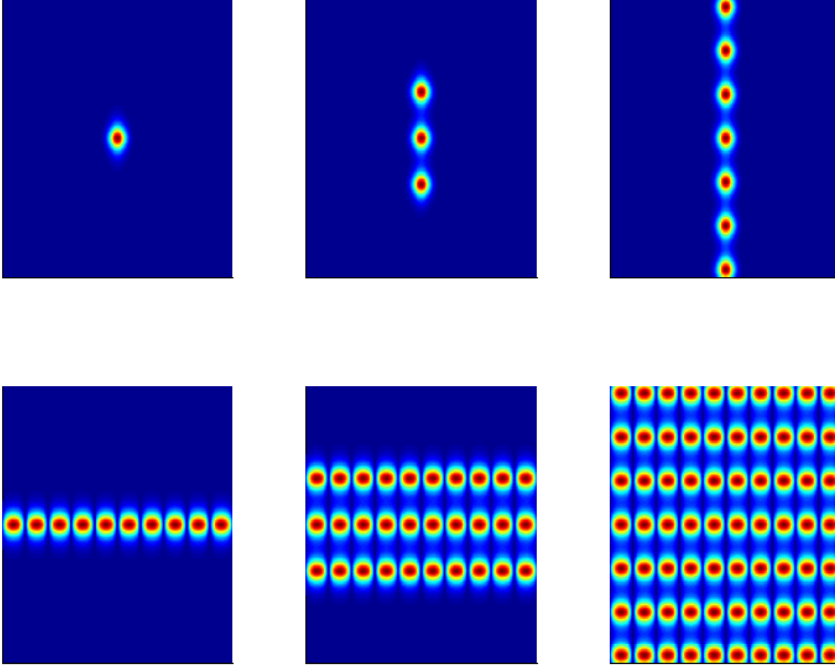


Figure 1: (color online) Plot of six different types of possible solutions. The top three figures represent possible solutions in a single-well channel: the left is a single spot, the middle is a multi-spot, and the right is a stripe. The bottom three figures represent possible solutions in an infinite-well, i.e., periodic channel. The left is a single spot, the middle is a multi-spot, and the right is a stripe.

and the ODE describes the profile of Q in y . In particular, spots and stripes are found as homoclinic orbits to zero and periodic orbits, respectively, of this ODE.

One of the many appealing aspects to using spatial dynamics is that it allows one to reduce the spatial dimension of the steady-state problem from two to one (or more generally, from $n \geq 2$ to one). Regarding (1.4) in the case that $N = 1$ there has been a great deal of formal work, and some recent rigorous work, in which this dimensional reduction is performed for the full PDE (see [1] and the references therein). In [1] it is shown that the dynamics of the full PDE are well-approximated by the one space dimension PDE under an assumption similar to that of (1.3) over long, but finite, time intervals. One of the underlying assumptions is that which is discussed in Section 2.2. The authors in [1] make the intriguing statement that if one uses the assumption discussed in Section 2.3, then the analysis is “more complicated and might involve coupling terms”. As is seen in Section 2.3, with respect to the steady-state problem these coupling terms lead to the existence of waves which are generically not localized, and which instead are asymptotically periodic waves with an exponentially small amplitude.

When considering the application of the theoretical results to the Gross-Pitaevski equation, i.e., (1.4) with $N = 1$, it is natural to enquire as to how the results presented herein relate to [13–16]. In these works a Lyapunov-Schmidt reduction was used in order to construct weakly nonlinear solutions. The hypotheses for the applicability of the reduction required that ϵ in (1.3) satisfied $\epsilon \geq \epsilon_0$ for a given $\epsilon_0 > 0$. Thus, in this paper we are looking at the case for which the Lyapunov-Schmidt reduction does not hold. We will see that this is reflected in the solution structure in a profound manner: for $\epsilon > \epsilon_0$ and $V_2(x, y) = y^2$ one can show that to leading order the solutions of interest are for fixed x like $O(e^{-\epsilon y^2/2})$; however, for $\epsilon < \epsilon_0$ it will be seen that instead that the localized solutions for fixed x will be like $O(e^{-\epsilon \alpha |y|})$ for some $\alpha > 0$ independent of ϵ . Thus, very weak trapping potentials lead to waves which are localized, but not as strongly localized as for stronger potentials.

The paper is organized as follows. In Section 2 we consider the solution structure to (1.2) when $N = 1$. In particular, we prove the existence of waves which are localized in y for fixed x , i.e., spots. These solutions are graphically depicted in the left panels of Figure 1, where in the top panel $V(x)$ is a single-well potential, and in the bottom panel $V(x)$ is a periodic potential, e.g., $V(x) \propto \cos(2x)$. We also prove the existence of waves which are periodic in y for fixed x , i.e., stripes. These solutions are graphically depicted in the right panels of Figure 1. In Section 3 we consider the solution structure to (1.2) when $N = 2$. One of the intriguing results associated with this study is the existence of waves with a transverse multipulse configuration. In other

words, the possibility is now open to having a solution with more than one spot in the y -direction. These solutions are characterized in the middle panels of [Figure 1](#). In [Section 4](#) we apply the theoretical results of the previous two sections to a few physically relevant examples. Finally, in [Section 5](#) we conclude with a brief discussion of possible extensions of the present theory, in particular to the case of periodic potentials.

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2. ONE-COMPONENT SYSTEM

In this section we consider the complex-valued equation

$$\Delta Q - \omega Q + V(x, y)Q + Qf(|Q|^2) = 0, \quad Q(x, y) \in \mathbb{C}. \quad (2.1)$$

We start by describing the assumptions on the nonlinearity f and the potential V .

Hypothesis 2.1. Assume that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function with $f(0) = 0$, and

$$f(u) = c_0 u + O(u^2), \quad \text{as } u \rightarrow 0,$$

with $c_0 \neq 0$.

Hypothesis 2.2. Assume that $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a smooth function,

$$V(x, y) = V_1(x) + \varepsilon V_2(x, y)$$

in which $V_1 : \mathbb{R} \rightarrow \mathbb{R}$ and $V_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ have the following properties:

(a) The one-dimensional operator

$$\mathcal{L} := \partial_{xx} + V_1(x),$$

acting in $L^2(\mathbb{R}; \mathbb{C})$ has the spectrum

$$\text{spec}(\mathcal{L}) = \text{spec}_h(\mathcal{L}) \cup \{\gamma_n, \dots, \gamma_1\}, \quad \text{spec}_h(\mathcal{L}) \subset \{\lambda \in \mathbb{C} : \text{Re } \lambda \leq \gamma_*\},$$

for some $n \geq 1$, where $\gamma_* < \gamma_n < \dots < \gamma_1$, and γ_j , $j = 1, \dots, n$ are simple eigenvalues with associated normalized eigenfunctions g_j in the norm of $L^2(\mathbb{R}; \mathbb{C})$.

(b) The product $(u, v) \mapsto uv$ defines a continuous bilinear map on the domain H^1 of the square root operator $(\text{id} + \gamma_1 - \mathcal{L})^{1/2}$.

(c) The two-dimensional perturbation $V_2(\cdot, y) \in H^1$, for any $y \in \mathbb{R}$, and the map $y \mapsto V_2(\cdot, y)$ is smooth, bounded and even, $V_2(\cdot, y) = V_2(\cdot, -y)$.

Remark 2.3. For a bounded potential V_1 the space H^1 in the above hypothesis is simply $H^1(\mathbb{R}; \mathbb{C})$. Nevertheless, one may allow for unbounded potentials when H^1 is a suitably chosen weighted subspace of $H^1(\mathbb{R}; \mathbb{C})$. This will be important when considering the application to the Gross-Pitaevski equation, for which $V_1(x) = x^2$. In the following, we denote by $\langle \cdot, \cdot \rangle_{H^1}$ the scalar product in H^1 , and by $H_r^1 \subset H^1$ the Hilbert subspace consisting of real-valued functions.

Remark 2.4. The equation (2.1) possesses several symmetries which are essential in our analysis. There are two discrete symmetries

- reflection in y : $Q(\cdot, y) \mapsto Q(\cdot, -y)$,
- conjugation invariance: $Q \mapsto \bar{Q}$.

and the continuous symmetry

- phase invariance: $Q \mapsto e^{i\varphi} Q$, $\varphi \in (0, 2\pi)$,

In addition, if $\varepsilon = 0$, i.e., the potential is one-dimensional there is an additional continuous symmetry

- translation invariance in y : $Q \mapsto Q(\cdot, \cdot + \alpha)$, $\alpha \in \mathbb{R}$.

2.1. Spatial dynamics

The first step of the spatial dynamics approach consists in rewriting the equation (2.1) as a first order system in the spatial variable y ,

$$\begin{aligned} Q_y &= R \\ R_y &= (\omega - \mathcal{L})Q - \varepsilon V_2(x, y)Q - Qf(Q\bar{Q}), \end{aligned}$$

to which we add the complex conjugated equations

$$\begin{aligned} \bar{Q}_y &= \bar{R} \\ \bar{R}_y &= (\omega - \mathcal{L})\bar{Q} - \varepsilon V_2(x, y)\bar{Q} - \bar{Q}f(Q\bar{Q}). \end{aligned}$$

We regard this system as an infinite-dimensional non-autonomous dynamical system

$$\mathbf{Q}_y = \mathcal{A}_\omega \mathbf{Q} + \mathcal{F}(\mathbf{Q}, y; \varepsilon), \quad (2.2)$$

in the *real* Hilbert space

$$\mathcal{X} = \{\mathbf{Q} = (Q, R, \bar{Q}, \bar{R}) ; Q \in H^1, R \in L^2(\mathbb{R}; \mathbb{C})\},$$

equipped with the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{X}} = \operatorname{Re} \langle \cdot, \cdot \rangle_{H^1} + \operatorname{Re} \langle \cdot, \cdot \rangle$, in which $\langle \cdot, \cdot \rangle_{H^1}$ is the scalar product in H^1 and $\langle \cdot, \cdot \rangle$ the usual scalar product in $L^2(\mathbb{R}; \mathbb{C})$. In (2.2) we have

$$\mathcal{A}_\omega = \begin{pmatrix} 0 & \operatorname{id} & 0 & 0 \\ \omega - \mathcal{L} & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{id} \\ 0 & 0 & \omega - \mathcal{L} & 0 \end{pmatrix}, \quad \mathcal{F}(\mathbf{Q}, y; \varepsilon) = \begin{pmatrix} 0 \\ -\varepsilon V_2(\cdot, y)Q - Qf(Q\bar{Q}) \\ 0 \\ -\varepsilon V_2(\cdot, y)\bar{Q} - \bar{Q}f(Q\bar{Q}) \end{pmatrix}.$$

Two-dimensional waves to (2.1) are bounded orbits to this dynamical system, and we restrict to orbits close to the origin. Upon taking ω as bifurcation parameter, we find the bifurcation points from the spectrum of the linear operator \mathcal{A}_ω . We compute this spectrum from Hypothesis 2.2, which describes the spectral properties of \mathcal{L} , and find that there is an increasing sequence of bifurcation points at

$$\omega = \gamma_1, \quad \omega = \gamma_2, \quad \dots, \quad \omega = \gamma_n.$$

At each bifurcation point $\omega = \gamma_k$ the spectrum of \mathcal{A}_ω consists of

- one eigenvalue in zero, geometrically double and algebraically quadruple;
- $k - 1$ pairs of double semisimple complex conjugated eigenvalues

$$\pm i\sqrt{\gamma_j - \gamma_k}, \quad j = 1, \dots, k - 1;$$

and

- the rest of the spectrum lies at a distance $\sqrt{\gamma_k - \gamma_{k+1}} > 0$ from the imaginary axis.

This spectral picture suggests that the dynamical system (2.1) possesses a finite-dimensional local center manifold of dimension $4k$ (the sum of the algebraic multiplicities of the purely imaginary eigenvalues), which contains the set of (sufficiently) small bounded orbits, for ω close to γ_k . As we shall see later, the properties of \mathcal{L} and \mathcal{F} described above allow to use a non-autonomous version of the center manifold reduction [24] and construct this manifold. The dynamics on this manifold is described by a $4k$ -dimensional non-autonomous system of ODEs, and the bounded orbits of this system correspond to two-dimensional waves to (2.1). In particular, spots and stripes of (2.1) are obtained from homoclinic orbits to zero and periodic orbits, respectively, of the reduced system.

2.2. First bifurcation at $\omega = \gamma_1$

The simplest bifurcation is the first bifurcation at $\omega = \gamma_1$, when zero is the only purely imaginary eigenvalue of \mathcal{A}_ω .

2.2.1. Reduced system

Set $\omega = \gamma_1 + \mu$ and rewrite the dynamical system as

$$\mathbf{Q}_y = \mathcal{A}_1 \mathbf{Q} + \mu \mathcal{B}_1 \mathbf{Q} + \mathcal{F}(\mathbf{Q}, y; \varepsilon), \quad (2.3)$$

where $\mathcal{A}_1 = \mathcal{A}_{\gamma_1}$ and $\mu \mathcal{B}_1 = \mathcal{A}_{\gamma_1 + \mu} - \mathcal{A}_{\gamma_1}$. The spectrum of \mathcal{A}_1 satisfies

$$\text{spec}(\mathcal{A}_1) = \{0\} \cup \text{spec}_1(\mathcal{A}_1), \quad \text{spec}_1(\mathcal{A}_1) \subset \{\nu \in \mathbb{C}; |\text{Re } \nu| > \delta\}, \quad (2.4)$$

for some positive constant δ , and 0 is a quadruple eigenvalue with geometric multiplicity two. Following the general reduction procedure, we consider the kernel and generalized kernel of \mathcal{A}_1 ,

$$\ker(\mathcal{A}_1) = \text{span}(\mathbf{e}_1, \mathbf{f}_1), \quad \text{gker}(\mathcal{A}_1) = \text{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{f}_1, \mathbf{f}_2),$$

with

$$\mathbf{e}_1 = \begin{pmatrix} g_1 \\ 0 \\ g_1 \\ 0 \end{pmatrix}, \quad \mathbf{f}_1 = \begin{pmatrix} ig_1 \\ 0 \\ -ig_1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ g_1 \\ 0 \\ g_1 \end{pmatrix}, \quad \mathbf{f}_2 = \begin{pmatrix} 0 \\ ig_1 \\ 0 \\ -ig_1 \end{pmatrix},$$

where g_1 is the normalized vector in the kernel of $\mathcal{L} - \gamma_1$. The spectral projection onto the generalized kernel of \mathcal{A}_1 , $P : \mathcal{X} \rightarrow \text{gker } \mathcal{A}_1$, can be computed with the help of the $(L^2)^4$ -adjoint of \mathcal{A}_1 ,

$$\mathcal{A}_1^{\text{ad}} = \begin{pmatrix} 0 & \gamma_1 - \mathcal{L} & 0 & 0 \\ \text{id} & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_1 - \mathcal{L} \\ 0 & 0 & \text{id} & 0 \end{pmatrix}.$$

The kernel and generalized kernel of $\mathcal{A}_1^{\text{ad}}$ are spanned by

$$\ker(\mathcal{A}_1^{\text{ad}}) = \text{span}(\mathbf{e}_2, \mathbf{f}_2), \quad \text{gker}(\mathcal{A}_1^{\text{ad}}) = \text{span}(\mathbf{e}_2, \mathbf{e}_1, \mathbf{f}_2, \mathbf{f}_1),$$

and then

$$P\mathbf{Q} = \langle \mathbf{Q}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{Q}, \mathbf{e}_2 \rangle \mathbf{e}_2 + \langle \mathbf{Q}, \mathbf{f}_1 \rangle \mathbf{f}_1 + \langle \mathbf{Q}, \mathbf{f}_2 \rangle \mathbf{f}_2,$$

in which $\langle \cdot, \cdot \rangle$ denotes the $(L^2)^4$ -scalar product, and we have used the fact that the L^2 -norm of g_1 is equal to one.

According to the spectral decomposition (2.4), we make the Ansatz

$$\mathbf{Q}(y) = a_1(y)\mathbf{e}_1 + a_2(y)\mathbf{e}_2 + b_1(y)\mathbf{f}_1 + b_2(y)\mathbf{f}_2 + \mathbf{R}(y), \quad (2.5)$$

in which a_1 , a_2 , b_1 , and b_2 are real-valued functions depending upon y and

$$\mathbf{R}(y) \in (\text{id} - P)\mathcal{X}, \quad P\mathbf{R}(y) = 0, \quad \forall y \in \mathbb{R}.$$

We can now apply the center manifold reduction theorem mentioned above which gives us, for any (arbitrary but fixed) $k \geq 1$, a reduction map $\Psi : U \times \mathbb{R} \rightarrow (\text{id} - P)\mathcal{X}$ of class C^k , with U a neighborhood of the origin in \mathbb{R}^6 , such that for μ and ε close to zero, any sufficiently small bounded solution \mathbf{Q} to (2.3) is of the form (2.5) with

$$\mathbf{R}(y) = \Psi(a_1(y), a_2(y), b_1(y), b_2(y), y; \mu, \varepsilon), \quad \forall y \in \mathbb{R}. \quad (2.6)$$

Furthermore, the real-valued functions a_1 , a_2 , b_1 , and b_2 are solutions to the *reduced system*

$$\begin{aligned} a_{1y} &= a_2 + \varphi_1(a_1, a_2, b_1, b_2, y; \mu, \varepsilon) \\ a_{2y} &= \varphi_2(a_1, a_2, b_1, b_2, y; \mu, \varepsilon) \\ b_{1y} &= b_2 + \psi_1(a_1, a_2, b_1, b_2, y; \mu, \varepsilon) \\ b_{2y} &= \psi_2(a_1, a_2, b_1, b_2, y; \mu, \varepsilon) \end{aligned} \quad (2.7)$$

in which φ_j and ψ_j can be computed with the help of the formulae

$$\varphi_j = \mu \langle \mathcal{B}_1 \mathbf{Q}, \mathbf{e}_j \rangle + \langle \mathcal{F}(\mathbf{Q}, \varepsilon), \mathbf{e}_j \rangle, \quad \psi_j = \mu \langle \mathcal{B}_1 \mathbf{Q}, \mathbf{f}_j \rangle + \langle \mathcal{F}(\mathbf{Q}, \varepsilon), \mathbf{f}_j \rangle,$$

after substituting \mathbf{Q} from (2.5) and (2.6). In addition, if the vector field in (2.3) commutes (resp. anti-commutes) with an isometry S on \mathcal{X} , the reduction procedure insures that Ψ commutes with the induced isometry \tilde{S} on \mathbb{R}^4 , and that the reduced system (2.7) commutes (resp. anti-commutes) with \tilde{S} .

In order to determine the dynamics of this reduced system near the origin, we compute a Taylor expansion for the vector field in (2.7). First, since the first and third components of $\mathcal{B}_1 \mathbf{Q}$ and \mathcal{F} in (2.3) and the second and fourth components of \mathbf{e}_1 and \mathbf{f}_1 all vanish, the scalar products in the formulae for φ_1 and ψ_1 are zero so that $\varphi_1 = \psi_1 = 0$. Next, we introduce the complex-valued variables $A_j = a_j + ib_j$ so that the reduced system becomes

$$\begin{aligned} A_{1y} &= A_2 \\ A_{2y} &= \Phi_2(A_1, A_2, \bar{A}_1, \bar{A}_2, y; \mu, \varepsilon) \end{aligned} \quad (2.8)$$

together with the complex conjugated equations. The reduced system possesses a number of symmetries which are induced by the symmetries of the original equation in Remark 2.4. The conjugation and the phase invariances imply that the reduced vector field in (2.8) commutes with the induced symmetries

$$\mathcal{S}_c(A_1, A_2) = (\bar{A}_1, \bar{A}_2), \quad \mathcal{S}_\varphi(A_1, A_2) = (e^{i\varphi} A_1, e^{i\varphi} A_2),$$

and the reflection in y implies that (2.8) is reversible, that is the vector field anti-commutes with the symmetry

$$\mathcal{R}(A_1, A_2) = (A_1, -A_2).$$

Notice also that real-valued and purely imaginary solutions of (2.1) lie in the invariant subspaces

$$\{(A_1, A_2), \operatorname{Im} A_1 = \operatorname{Im} A_2 = 0\} \quad \text{and} \quad \{(A_1, A_2), \operatorname{Re} A_1 = \operatorname{Re} A_2 = 0\},$$

respectively. Finally, we can use normal form theory and transform (2.8) into a system of the form

$$\begin{aligned} A_{1y} &= A_2 \\ A_{2y} &= A_1 Q(|A_1|^2, \frac{i}{2}(A_1 \bar{A}_2 - \bar{A}_1 A_2), y; \mu, \varepsilon) + O((|A_1| + |A_2|)^{2m+2}) \end{aligned} \quad (2.9)$$

with Q polynomial of degree m in its first two arguments with coefficients depending upon y , μ , and ε . A direct calculation gives us the lowest order terms ($m = 1$) in this system

$$\begin{aligned} A_{1y} &= A_2 \\ A_{2y} &= \mu A_1 - \varepsilon h(y) A_1 - c_1 A_1 |A_1|^2, \end{aligned} \quad (2.10)$$

in which

$$h(y) = \int_{\mathbb{R}} V_2(x, y) g_1^2(x) dx, \quad c_1 = c_0 \int_{\mathbb{R}} g_1^4(x) dx.$$

2.2.2. One-dimensional potentials

We start the analysis of the reduced system with the case $\varepsilon = 0$, when the potential V is one-dimensional, $V(x, y) = V_1(x)$. Then the truncated reduced system (2.10) is equivalent to the steady cubic NLS equation

$$A_{1yy} = \mu A_1 - c_1 A_1 |A_1|^2, \quad (2.11)$$

in which the sign of c_1 , which is the same as the sign of c_0 , accounts for either the focusing or the defocusing NLS. The set of bounded solutions of this steady NLS equation (which is also the steady Ginzburg-Landau equation) is well-known. Up to translations $A_1(\cdot + \alpha)$, $\alpha \in \mathbb{R}$, and phase rotations $e^{i\varphi} A_1$, $\varphi \in \mathbb{R}$, we have the following classification of bounded solutions:

- if $\mu < 0$ and $c_1 < 0$, the equation possesses periodic solutions, quasi-periodic solutions, and a one-parameter family of homoclinic solutions to nonzero periodic solutions;
- if $\mu > 0$ and $c_1 < 0$, there are no nontrivial bounded solutions;
- if $\mu < 0$ and $c_1 > 0$, the equation possesses periodic and quasi-periodic solutions;
- if $\mu > 0$ and $c_1 > 0$, the equation possesses periodic solutions, quasi-periodic solutions, and one real-valued, reversible homoclinic solution to zero, given by the formula

$$a_1^*(y) = \frac{\sqrt{2\mu}}{\sqrt{c_1}} \operatorname{sech}(\sqrt{\mu}y). \quad (2.12)$$

Actually, this picture persists for a general class of reversible four-dimensional vector fields [11], and in particular for our system. Each of these bounded solutions gives a two-dimensional solution of (2.1) with profile in y determined by the shape of A_1 . As we are interested in spots and stripes, we restrict to orbits homoclinic to zero and to periodic orbits.

Homoclinic orbits exist only in the last case, when $\mu > 0$ and $c_1 > 0$. In contrast to the general situation in [11], our system possesses the additional symmetries \mathcal{S}_c and \mathcal{S}_φ which allow to show that the homoclinic solution for $\mu > 0$ and $c_1 > 0$ persists as a *real-valued*, reversible homoclinic solution. Indeed, the subspace of real-valued solutions is invariant under the flow of (2.8), and we can restrict to the second order autonomous ODE

$$a_{1yy} = \Phi_2(a_1, a_{1y}, a_1, a_{1y}, 0; \mu, 0) = \mu a_1 - c_1 a_1^3 + \mathcal{R}(a_1, a_{1y}; \mu), \quad (2.13)$$

where in view of the normal form (2.9) we have $\mathcal{R}(a_1, a_{1y}; \mu) = O(|\mu|^2|a_1| + |\mu||a_1|^3 + (|a_1| + |a_{1y}|)^5)$. The homoclinic orbit given by (2.12) satisfies this equation at lowest order ($\mathcal{R} \equiv 0$), and, as it is well-known, it persists under reversible perturbations in two dimensions (see e.g. [19, Proposition 5.1]). In particular, in this case we find an even homoclinic solution for (2.13) which decays in y with exponential rate $e^{-\eta\sqrt{\mu}y}$, for some $\eta \in (0, 1)$, and sufficiently small μ .

Periodic orbits exist when $\mu < 0$, or when $c_1 > 0$ and $\mu > 0$. In all cases, there are two types of periodic orbits, up to translations in y and phase rotations. First, there are periodic orbits with constant modulus and linear phase, of the form

$$A_1(y) = \alpha e^{i\kappa y}, \quad c_1 \alpha^2 = \mu + \kappa^2.$$

Using polar coordinates and the implicit function theorem, it is not difficult to prove that these solutions persist as periodic solutions with constant modulus and linear phase of (2.9), with $\varepsilon = 0$. Next, in each case there is a one-parameter family of real-valued periodic waves of size $O(|\mu|^{1/2})$, which persist as real-valued solutions of the full normal form (2.9), for the same reasons as the homoclinic orbit above.

Finally, going backwards the steps in the reduction procedure we obtain the following result.

Theorem 2.5. *Assume that Hypotheses 2.1 and 2.2 hold. Then for any $\omega = \gamma_1 + \mu$, with μ sufficiently small, the equation (2.1) possesses the following solutions, up to translations in y and phase rotations.*

- (a) *If $\mu > 0$ and $c_0 > 0$, there is a spot solution*

$$Q(x, y) = \frac{\sqrt{2\mu}}{\sqrt{c_1}} \operatorname{sech}(\sqrt{\mu}y) g_1(x) + O(\mu),$$

which is real-valued and even in y .

- (b) *If $\mu < 0$, or if $\mu > 0$ and $c_0 > 0$, there are two one-parameter families of stripes, one consisting of solutions of the form*

$$Q(x, y) = a\sqrt{\mu} e^{ik\sqrt{\mu}y} g_1(x) + O(\mu),$$

with $k^2 = -1 + c_1 a^2 + O(\mu)$, and the other one consisting of solutions of the form

$$Q(x, y) = \sqrt{\mu} a_1(\sqrt{\mu}y) g_1(x) + O(\mu),$$

which are real-valued, and periodic and even in y .

Remark 2.6. In fact, the center manifold reduction shows that the solutions in Theorem 2.5 are of class C^k , and that the spot solution decays exponentially in y with decay rate $e^{-\eta\sqrt{\mu}|y|}$, for some $\eta \in (0, 1)$.

2.2.3. Nearly one-dimensional potentials

We consider now the general case $\varepsilon \neq 0$, and focus on the existence of spots. Since we expect homoclinics to zero to be real-valued, just as in the case $\varepsilon = 0$, we restrict to *real-valued* solutions. Then we have the second order ODE

$$a_{1yy} = \mu a_1 - \varepsilon h(y) a_1 - c_1 a_1^3 + \mathcal{R}(a_1, a_{1y}, y; \mu, \varepsilon), \quad (2.14)$$

in which $\mathcal{R}(a_1, a_{1y}; \mu) = O(|a_1|(|\mu|^2 + |\varepsilon|^2) + |a_1|^3(|\mu| + |\varepsilon|) + (|a_1| + |a_{1y}|)^5)$.

First, notice that for sufficiently small $\varepsilon = o(|\mu|)$, the additional non-autonomous terms in the reduced system are of higher order. It is then easy to check that the arguments showing the persistence of the homoclinic in (2.12) remain valid, and that the result in Theorem 2.5 holds for $\varepsilon = o(|\mu|)$, as well.

Next, we consider the parameter regime $\mu = O(|\varepsilon|)$. We set $\mu = |\varepsilon|\nu$ and consider the rescaled variables

$$z = |\varepsilon|^{\frac{1}{2}} y, \quad a_1 = |\varepsilon|^{\frac{1}{2}} \tilde{a}_1,$$

so that the equation (2.14) becomes

$$\tilde{a}_{1yy} = \nu \tilde{a}_1 - \text{sign}(\varepsilon) h(y) \tilde{a}_1 - c_1 \tilde{a}_1^3 + O(|\varepsilon \tilde{a}_1|). \quad (2.15)$$

We make the following additional assumption on V_2 .

Hypothesis 2.7. The Sturm-Liouville operator $L_\varepsilon = \partial_{yy} + \text{sign}(\varepsilon)h(y)$ acting in $L^2(\mathbb{R})$ with domain $H^2(\mathbb{R})$, in which h is the reduced potential

$$h(y) = \int_{\mathbb{R}} V_2(x, y) g_1^2(x) dx,$$

has the spectrum

$$\text{spec}(\partial_{yy} + \text{sign}(\varepsilon)h(y)) = (-\infty, \delta_*] \cup \{\delta_m, \dots, \delta_1\},$$

where $\delta_* < \delta_m < \dots < \delta_1$, and δ_j , $j = 1, \dots, m$ are simple eigenvalues with associated normalized eigenfunctions h_j .

Under this hypothesis, for the rescaled equation (2.15) we find a sequence of bifurcation points at

$$\nu_k = \delta_k, \quad k = 1, \dots, m.$$

At each bifurcation point a simple eigenvalue crosses the origin, so that we have a classical bifurcation from a simple eigenvalue. The bifurcating solutions can be analyzed with the help of a standard Liapunov-Schmidt reduction. With the Ansatz

$$\nu = \nu_k + \tilde{\nu}, \quad \tilde{a}_1(y) = \alpha h_k(y) + r_1(y),$$

where $\alpha \in \mathbb{R}$ and $\langle h_k, r_1 \rangle = 0$, we find the bifurcation equation

$$\tilde{\nu} = d_1 \alpha^2 + O(|\varepsilon| + |\alpha|^2(|\tilde{\nu}| + |\alpha|^2)), \quad d_1 = c_1 \int_{\mathbb{R}} h_k^4(y) dy,$$

which is invariant under the reflection $\alpha \mapsto -\alpha$, due to the phase rotation invariance \mathcal{S}_φ of the reduced system (2.8). We conclude that there is a value $\tilde{\nu}_\varepsilon = O(|\varepsilon|)$, such that there is a pair of solutions for $\tilde{\nu} > \tilde{\nu}_\varepsilon$ (resp. $\tilde{\nu} < \tilde{\nu}_\varepsilon$), when $d_1 > 0$ (resp. $d_1 < 0$),

Going back to the original equation (2.1) we obtain the following result.

Theorem 2.8. *Assume that Hypotheses 2.1, 2.2, and 2.7 hold, for some sufficiently small ε . Then there exist $\tilde{\nu}_\varepsilon = O(|\varepsilon|)$ such that for any $\omega = \gamma_1 + |\varepsilon|(\delta_k + \tilde{\nu})$, with $\tilde{\nu}$ close to $\tilde{\nu}_\varepsilon$, $\tilde{\nu} > \tilde{\nu}_\varepsilon$ (resp. $\tilde{\nu} < \tilde{\nu}_\varepsilon$), if $d_1 > 0$ (resp. $d_1 < 0$), the equation (2.1) possesses, up to phase rotations, a real-valued spot solution satisfying*

$$Q(x, y) = \sqrt{\frac{|\varepsilon|(\tilde{\nu} - \tilde{\nu}_\varepsilon)}{d_1}} h_k(y) g_1(x) + O(|\varepsilon|^{1/2} |\tilde{\nu} - \tilde{\nu}_\varepsilon|).$$

Remark 2.9. The same type of arguments can be used to show the existence of stripe solutions. For $\varepsilon = o(|\mu|)$, one can argue with the implicit function theorem in a space of periodic functions, whereas in the case $\mu = O(|\varepsilon|)$ it is enough to replace the Hypothesis 2.7 by a periodicity condition on the potential the reduced potential h . Then the Liapunov-Schmidt reduction can be used in suitable chosen spaces of periodic functions.

Now that we have an example worked out for which $\varepsilon > 0$, it will henceforth be assumed that $\varepsilon = 0$. This is done for the sake of simplicity; otherwise, an analysis similar to that leading to Theorem 2.8 can be performed for the normal form equation at hand.

2.3. Second bifurcation

We consider now the second bifurcation at $\omega = \gamma_2$, in the case $\varepsilon = 0$, and also restrict to real-valued solutions. Then in the dynamical system (2.2) it is enough to keep the first two equations and by setting $\omega = \gamma_2 + \mu$ we find a system

$$\mathbf{Q}_y = \mathcal{A}_2 \mathbf{Q} + \mu \mathcal{B}_2 \mathbf{Q} + \mathcal{F}(\mathbf{Q}), \quad (2.16)$$

in the real Hilbert space $\mathcal{X}_r = \{\mathbf{Q} = (Q, R); Q \in H_r^1, R \in L^2(\mathbb{R}; \mathbb{R})\}$. Here $\mathcal{A}_2 = \mathcal{A}_{\gamma_2}$ and $\mu \mathcal{B}_2 = \mathcal{A}_{\gamma_2 + \mu} - \mathcal{A}_{\gamma_2}$, with

$$\mathcal{A}_\omega = \begin{pmatrix} 0 & \text{id} \\ \omega - \mathcal{L} & 0 \end{pmatrix}, \quad \mathcal{F}(\mathbf{Q}) = \begin{pmatrix} 0 \\ -Qf(Q^2) \end{pmatrix}.$$

The spectrum of \mathcal{A}_2 satisfies

$$\text{spec}(\mathcal{A}_2) = \{0, \pm i\sqrt{\gamma_2 - \gamma_1}\} \cup \text{spec}_1(\mathcal{A}_2), \quad \text{spec}_1(\mathcal{A}_2) \subset \{\nu \in \mathbb{C}; |\text{Re } \nu| > \delta\}, \quad (2.17)$$

with some positive constant δ . Here zero is a double eigenvalue with geometric multiplicity one, and $\pm i\sqrt{\gamma_2 - \gamma_1}$ are simple eigenvalues. The kernel and generalized kernel of \mathcal{A}_2 are spanned by

$$\ker(\mathcal{A}_2) = \text{span}(\mathbf{e}_1), \quad \mathbf{e}_1 = \begin{pmatrix} g_2 \\ 0 \end{pmatrix}, \quad \text{gker}(\mathcal{A}_2) = \text{span}(\mathbf{e}_1, \mathbf{e}_2), \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ g_2 \end{pmatrix},$$

where g_2 belongs to the kernel of $\mathcal{L} - \gamma_2$ and has L^2 -norm equal to one. In addition, we consider the kernels

$$\ker(\mathcal{A}_2 \mp i\sqrt{\gamma_2 - \gamma_1}) = \text{span}(\mathbf{f}_\pm), \quad \mathbf{f}_\pm = \begin{pmatrix} g_1 \\ \pm i\sqrt{\gamma_2 - \gamma_1} g_1 \end{pmatrix},$$

and the spectral projection corresponding to the spectral decomposition (2.17),

$$P\mathbf{Q} = \langle \mathbf{Q}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{Q}, \mathbf{e}_2 \rangle \mathbf{e}_2 + \langle \mathbf{Q}, \mathbf{f}_+^{\text{ad}} \rangle \mathbf{f}_+ + \langle \mathbf{Q}, \mathbf{f}_-^{\text{ad}} \rangle \mathbf{f}_-,$$

in which

$$\mathbf{f}_\pm^{\text{ad}} = \frac{1}{2} \begin{pmatrix} g_1 \\ \pm \frac{i}{\sqrt{\gamma_2 - \gamma_1}} g_1 \end{pmatrix}.$$

We now make the Ansatz

$$\mathbf{Q}(y) = a_1(y)\mathbf{e}_1 + a_2(y)\mathbf{e}_2 + B(y)\mathbf{f}_+ + \overline{B}(y)\mathbf{f}_- + \mathbf{R}(y), \quad (2.18)$$

in which a_1 and a_2 are real-valued functions depending upon y , B is complex-valued, and

$$\mathbf{R}(y) \in (\text{id} - P)\mathcal{X}, \quad P\mathbf{R}(y) = 0, \quad \forall y \in \mathbb{R}.$$

By arguing again with a center-manifold reduction theorem we obtain a four-dimensional reduced system for a_1 , a_2 , and B . This system possesses again a reversibility symmetry now given through

$$\mathcal{R}(a_1, a_2, B) = (a_1, -a_2, \overline{B}),$$

and as a reminiscence of the rotation invariance it is invariant under

$$\mathcal{S}(a_1, a_2, B) = (-a_1, -a_2, -B).$$

In order to study the solutions of this system we use normal form theory. According to the results in [21, Chapter 3] we find the normal form

$$\begin{aligned} a_{1y} &= a_2 \\ a_{2y} &= a_1 P(a_1^2, |B|^2, \mu) + O((|a_1| + |a_2| + |B|)^{2m+2}) \\ B_y &= iBQ(a_1^2, |B|^2, \mu) + O((|a_1| + |a_2| + |B|)^{2m+2}), \end{aligned} \quad (2.19)$$

in which P and Q are real valued polynomials of degree m satisfying

$$P(0, 0, 0) = 0, \quad Q(0, 0, \mu) = \sqrt{\gamma_2 - \gamma_1} + O(\mu).$$

At lowest order we obtain

$$\begin{aligned} a_{1y} &= a_2 \\ a_{2y} &= \mu a_1 - c_2 a_1^3 \\ B_y &= i\sqrt{\gamma_2 - \gamma_1} B, \end{aligned} \quad (2.20)$$

in which the coefficient c_2 is given by

$$c_2 = c_0 \int_{\mathbb{R}} g_2^4(x) dx.$$

For the truncated system (2.20), we find two types of periodic orbits, orbits for which a_1 is periodic in y and $B = 0$, and orbits for which $a_1 = 0$ and $B(y) = e^{i\sqrt{\gamma_2 - \gamma_1} y} B(0)$, just as in Section 2.2.1. The persistence of these orbits for the full normal form can be shown, for instance, using the implicit function theorem. The question of existence of homoclinic orbits turns out to be more complicated in this case. Indeed, for the truncated system (2.20) homoclinic orbits to zero exist in the case $c_2 > 0$ and $\mu > 0$, but in contrast to the first bifurcation these homoclinic orbits typically do not persist for the full system [21]. The origin is now surrounded by a one-parameter family of periodic orbits, the ones for which $a_1 = 0$ above, and instead of the homoclinic orbit to zero from the truncated normal form, for the full system one finds a family of homoclinic connections to small periodic orbits. The amplitude of these periodic orbits may be exponentially small but not zero. (We refer to [28] for further details on the existence of such orbits in systems of this type). In the original equation (2.1), such solutions correspond to waves which are decaying in x and have a localized profile with small periodic oscillations at infinity in y .

Remark 2.10. The next bifurcations can be analyzed in the same way. To leading order the equations will be of the form of (2.20). The equations for $\mathbf{a} = (a_1, a_2)$ will be exactly of the same form, and, for the n -th bifurcation, there will be $n - 1$ uncoupled equations for $\mathbf{B} = (B_1, \dots, B_{n-1})$ of the form of the B -equation in (2.20). Since the size of \mathbf{B} increases each time, we do not expect homoclinic orbits to zero to exist, but only homoclinic orbits to small periodic orbits, just as for the second bifurcation. There will be again two types of periodic solutions, associated with the \mathbf{a} -equations and associated with the \mathbf{B} -equations which are expected to persist for the full normal form.

3. TWO-COMPONENT SYSTEM

The analysis of systems can be performed in the same way. Our main interest is new phenomena generated by the coupling terms in the system, and for simplicity we shall restrict here to real-valued solutions and to one-dimensional potentials. We consider the two-component system

$$\begin{aligned} \Delta Q_1 - \omega_1 Q_1 + V_1(x)Q_1 + Q_1 f_1(|Q_1|^2, |Q_2|^2) &= 0 \\ \Delta Q_2 - \omega_2 Q_2 + V_2(x)Q_2 + Q_2 f_2(|Q_1|^2, |Q_2|^2) &= 0, \end{aligned} \quad (3.1)$$

describing solutions of the form $U(t, x, y) = (e^{i\omega_1 t} Q_1(x, y), e^{i\omega_2 t} Q_2(x, y))$ of (1.1), when $n = 2$. The assumptions on the potentials V_1, V_2 and nonlinear terms f_1 and f_2 are similar to those for the single equation.

Hypothesis 3.1. Assume that $f_1, f_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ are smooth functions with $f_1(0, 0) = f_2(0, 0) = 0$, and

$$f_1(u_1, u_2) = c_{11}u_1 + c_{12}u_2 + O(u_1^2 + u_2^2), \quad f_2(u_1, u_2) = c_{21}u_1 + c_{22}u_2 + O(u_1^2 + u_2^2).$$

Hypothesis 3.2. Assume that $V_1, V_2 : \mathbb{R} \rightarrow \mathbb{R}$ are smooth functions satisfying Hypothesis 2.2(a)-(b).

For $j = 1, 2$, we denote by $\gamma_{jn} < \dots < \gamma_{j1}$ the simple eigenvalues of the operator

$$\mathcal{L}_j := \partial_{xx} + V_j(x),$$

by g_{jn}, \dots, g_{j1} the associated eigenfunctions normalized in the norm of $L^2(\mathbb{R}, \mathbb{R})$, and by H_j^1 and $H_{j_r}^1$ the Hilbert spaces introduced in Hypothesis 2.2(b).

Remark 3.3. The system (3.1) possesses the same symmetries as the single equation, i.e., translation invariance in y , phase invariance, reflection in y , and conjugation invariance.

3.1. Bifurcation problem

We consider the existence of real-valued solutions to the system (3.1). Writing $\mathbf{Q} = (Q_1, R_1, Q_2, R_2)$ one has the dynamical system

$$\mathbf{Q}_y = \mathcal{A}(\omega_1, \omega_2)\mathbf{Q} + \mathcal{F}(\mathbf{Q}), \quad (3.2)$$

in the *real* Hilbert space

$$\mathcal{X}_r = \{\mathbf{Q} = (Q_1, R_1, Q_2, R_2); Q_1 \in H_{1r}^1, Q_2 \in H_{2r}^1, R_1, R_2 \in L^2(\mathbb{R}; \mathbb{R})\}.$$

In (3.2) we have

$$\mathcal{A}(\omega_1, \omega_2) = \begin{pmatrix} 0 & \text{id} & 0 & 0 \\ \omega_1 - \mathcal{L}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{id} \\ 0 & 0 & \omega_2 - \mathcal{L}_2 & 0 \end{pmatrix}, \quad \mathcal{F}(\mathbf{Q}) = \begin{pmatrix} 0 \\ -Q_1 f_1(Q_1^2, Q_2^2) \\ 0 \\ -Q_2 f_2(Q_1^2, Q_2^2) \end{pmatrix}.$$

The spectrum of the linear operator $\mathcal{A}(\omega_1, \omega_2)$ is easily obtained from the spectrum of \mathcal{L}_j , $j = 1, 2$. In the parameter plane (ω_1, ω_2) we now find two sequences of bifurcation curves given by the lines $\omega_1 = \gamma_{1k}$ and $\omega_2 = \gamma_{2k}$, $k = 1, \dots, n$. Any point on one of these lines is a codimension one bifurcation point with a double non-semisimple zero eigenvalue, except for the intersection points $(\gamma_{1k}, \gamma_{2\ell})$ which are codimension two bifurcation points with a quadruple zero eigenvalue having a 2×2 Jordan block. The simplest bifurcation is again the one related to the first eigenvalues γ_{j1} of \mathcal{L}_j . We focus on the first codimension two bifurcation at $(\gamma_{11}, \gamma_{21})$.

We set

$$\omega_1 = \gamma_{11} + \mu_1, \quad \omega_2 = \gamma_{21} + \mu_2,$$

and rewrite the dynamical system as

$$\mathbf{Q}_y = \mathcal{A}_1 \mathbf{Q} + \mu_1 \mathcal{B}_1 \mathbf{Q} + \mu_2 \mathcal{B}_2 \mathbf{Q} + \mathcal{F}(\mathbf{Q}), \quad (3.3)$$

where $\mathcal{A}_1 = \mathcal{A}(\gamma_{11}, \gamma_{21})$ and $\mu_1 \mathcal{B}_1 + \mu_2 \mathcal{B}_2 = \mathcal{A}(\gamma_{11} + \mu_1, \gamma_{21} + \mu_2)$. At $\mu_1 = \mu_2 = 0$ the operator \mathcal{A}_1 has precisely one eigenvalue on the imaginary axis, $\nu = 0$ of algebraic multiplicity four and geometric multiplicity two. We are in the presence of a codimension two $0^2 0^2$ bifurcation. The behavior of the four critical eigenvalues for small μ_1 and μ_2 is given in Figure 2.

For the analysis of this bifurcation we use again a center manifold reduction. We now have

$$\ker(\mathcal{A}_1) = \text{span}\{\mathbf{e}_1, \mathbf{e}_2\}, \quad \text{gker}(\mathcal{A}_1) = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\},$$

where

$$\mathbf{e}_1 = \begin{pmatrix} g_{11} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 0 \\ g_{21} \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ g_{11} \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ g_{21} \end{pmatrix},$$

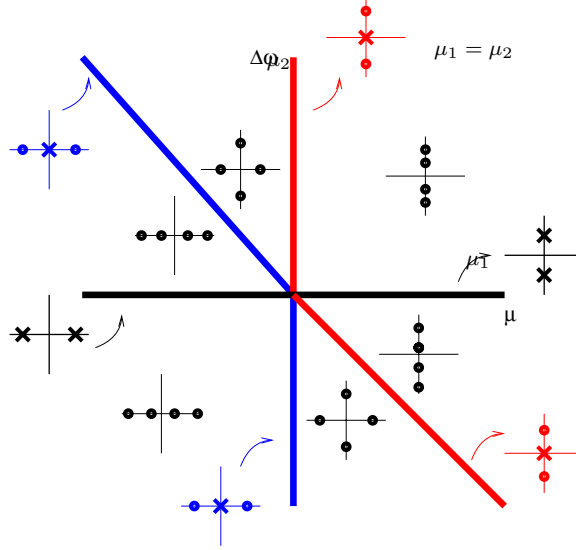


Figure 2: Critical eigenvalues in the $0^2 0^2$ -bifurcation in (3.3) for small μ_1 and μ_2 . Simple eigenvalues are represented by \bullet and double eigenvalues by \times . At $\mu_1 = \mu_2 = 0$ the four eigenvalues collide in the origin which is an algebraically quadruple and geometrically double eigenvalue.

where g_{11} and g_{21} are the normalized vectors in the kernel of $\mathcal{L}_1 - \gamma_{11}$ and $\mathcal{L}_2 - \gamma_{21}$, respectively. Proceeding as in Section 2.2.1, we make the Ansatz

$$\mathbf{Q}(y) = a_1(y)\mathbf{e}_1 + a_2(y)\mathbf{e}_2 + a_3(y)\mathbf{e}_3 + a_4(y)\mathbf{e}_4 + \mathbf{R}(y), \quad (3.4)$$

and apply the center-manifold reduction theorem. After straightforward calculations we find the lowest order terms in the reduced system,

$$\begin{aligned} a_{1y} &= a_3 \\ a_{2y} &= a_4 \\ a_{3y} &= \mu_1 a_1 - (d_{11}a_1^2 + d_{12}a_2^2)a_1 \\ a_{4y} &= \mu_2 a_2 - (d_{21}a_1^2 + d_{22}a_2^2)a_2 \end{aligned} \quad (3.5)$$

where

$$\begin{aligned} d_{11} &= c_{11} \int_{\mathbb{R}} g_{11}^4(x) dx, & d_{12} &= c_{12} \int_{\mathbb{R}} g_{11}^2(x) g_{21}^2(x) dx, \\ d_{21} &= c_{21} \int_{\mathbb{R}} g_{11}^2(x) g_{21}^2(x) dx, & d_{22} &= c_{22} \int_{\mathbb{R}} g_{21}^4(x) dx; \end{aligned}$$

(here c_{jk} are the coefficients in the expansions of f_j in Hypothesis 3.1).

Just as in the case of a single equation, this reduced system possesses several symmetries which are induced by the symmetries of the original system (3.1). Since we restrict here to real-valued solutions, the conjugation invariance of (3.1) becomes trivial, while the phase invariance reduces to $\mathcal{S}_\pi = -\text{id}$, and the reflection in y gives reversibility symmetry $\mathcal{R}(a_1, a_2, a_3, a_4) = (a_1, a_2, -a_3, -a_4)$. In addition, notice that the subspaces $(a_2, a_4) = (0, 0)$ and $(a_1, a_3) = (0, 0)$ are invariant, and solutions in these invariant spaces correspond to one-component solutions $Q_2 \equiv 0$ and $Q_1 \equiv 0$, respectively, in the system (3.1). In both subspaces, the flow is the same as the one of the reduced equation for real-valued solutions of the single equation in Section 2, and in particular, we find a pair of homoclinic orbits under the assumption that

$$\mu_1 > 0, \quad d_{11} > 0,$$

and

$$\mu_2 > 0, \quad d_{22} > 0,$$

respectively. If $\mu_1 = \mu_2$, then by setting

$$a_1 = \alpha a_2, \quad \alpha^2 := \frac{d_{22} - d_{12}}{d_{11} - d_{21}} > 0, \quad (3.6)$$

the system (3.5) collapses to

$$\begin{aligned} a_{2y} &= a_4 \\ a_{4y} &= -\mu_2 a_2 - (d_{22} + \alpha^2 d_{21}) a_2^3. \end{aligned} \quad (3.7)$$

There is yet another homoclinic solution to (3.7), which as a consequence of (3.6) is a vector-soliton solution to (3.5) (e.g., see [25] for an example).

While we do not attempt here a full description of this bifurcation, we shall focus in the next sections on two particular cases. First, in Section 3.2.1 we construct two-component homoclinic orbits to (3.5) by analyzing the bifurcation problems at the first two homoclinic solutions above. Next, in Section 3.2.2 we use the third homoclinic solution to construct multipulse homoclinic solutions to (3.5) by applying the theory presented in [9]. These multipulse homoclinic solutions correspond to solutions of the system (3.1) which consist of a finite number of spots in the y direction.

3.2. Localized waves

3.2.1. Two-component homoclinic orbits

We focus on the bifurcation problem for the homoclinic orbit in the invariant subspace $a_2 = a_4 = 0$, the other homoclinic can be treated in the same way. We take

$$\mu_1 > 0, \quad d_{11} > 0,$$

and choose the positive homoclinic orbit to

$$a_{1yy} = \mu_1 a_1 - d_{11} a_1^3 + O((|a_1| + |a_{1y}|)^5),$$

given by

$$a_1^*(y) = \sqrt{\frac{2\mu_1}{d_{11}}} \operatorname{sech}(\sqrt{\mu_1} y) + O(\mu_1).$$

We seek solutions (a_1, a_2) close to $(a_1^*, 0)$ for the system

$$\begin{aligned} a_{1yy} &= \mu_1 a_1 - (d_{11} a_1^2 + d_{12} a_2^2) a_1 + h.o.t. \\ a_{2yy} &= \mu_2 a_2 - (d_{21} a_1^2 + d_{22} a_2^2) a_2 + h.o.t.. \end{aligned}$$

Here μ_1 is a fixed (small) parameter and μ_2 will be used as bifurcation parameter. Upon rescaling

$$z = \sqrt{\mu_1} y, \quad a_1(y) = \sqrt{\mu_1} U_1(z), \quad a_2(y) = \sqrt{\mu_1} U_2(z), \quad \mu_2 = \mu_1 \varepsilon,$$

we obtain the system

$$\begin{aligned} U_{1zz} &= U_1 - (d_{11} U_1^2 + d_{12} U_2^2) U_1 + O(\mu_1) \\ U_{2zz} &= \varepsilon U_2 - (d_{21} U_1^2 + d_{22} U_2^2) U_2 + O(\mu_1). \end{aligned} \quad (3.8)$$

We set

$$(U_1, U_2) = (U_1^* + V_1, V_2), \quad U_1^*(z) = \sqrt{\frac{2}{d_{11}}} \operatorname{sech}(z),$$

and find the equation for $\mathbf{V} = (V_1, V_2)$,

$$\mathcal{L}_\varepsilon \mathbf{V} + \mathcal{F}(\mathbf{V}; \mu_1, \varepsilon) = 0, \quad (3.9)$$

with linear part at $\mu_1 = 0$, $\mathcal{L}_\varepsilon \mathbf{V} = (L_0 V_1, L_\varepsilon V_2)$, where

$$\begin{aligned} L_0 V_1 &= -V_{1zz} + V_1 - 3d_{11}U_1^{*2}V_1 = -V_{1zz} + V_1 - 6 \operatorname{sech}^2(z)V_1, \\ L_\varepsilon V_2 &= -V_{2zz} + (1 + \varepsilon)V_2 - d_{21}U_1^{*2}V_2 = -V_{2zz} + \varepsilon V_2 - \frac{2d_{21}}{d_{11}} \operatorname{sech}^2(z)V_2. \end{aligned}$$

The operators L_0 and L_ε are Sturm-Liouville operators and their spectra can be computed explicitly. We find for L_0 ,

$$\operatorname{spec} L_0 = \{-3, 0\} \cup [1, +\infty),$$

and for L_ε ,

$$\operatorname{spec} L_\varepsilon = \{\nu_1 + \varepsilon, \dots, \nu_N + \varepsilon\} \cup [\varepsilon, +\infty),$$

in which $-3, 0$ and $\nu_1 + \varepsilon, \dots, \nu_N + \varepsilon$ are simple eigenvalues. The number N and the values ν_n depend upon d_{21}/d_{11} . We have for

$$(N-1)N < \frac{2d_{21}}{d_{11}} \leq N(N+1), \quad N \geq 1, \quad (3.10)$$

that

$$\nu_n = - \left(\left(\frac{2d_{21}}{d_{11}} + \frac{1}{4} \right)^{1/2} - n + \frac{1}{2} \right)^2, \quad n = 1, \dots, N.$$

Consequently, under (3.10) for some $N \geq 1$, we find N bifurcation points at

$$\varepsilon_n = -\nu_n, \quad n = 1, \dots, N. \quad (3.11)$$

At each bifurcation point the operator L_ε has a one-dimensional kernel spanned by either an even or an odd function $h_n(z)$, which is exponentially decaying in z as $|z| \rightarrow \infty$ (actually, there are explicit formulae for these functions, see e.g. [8]).

Each of these bifurcations can be viewed as a bifurcation from a simple eigenvalue in a suitably chosen function space. At the bifurcation point, the operator \mathcal{L}_ε has a double zero eigenvalue with two associated eigenvectors, one in the kernel of L_0 and one in the kernel of L_ε . But the kernel of L_0 is always one-dimensional and spanned by the derivative U_{1y}^* , due to translation invariance. It therefore does not play an essential role in the bifurcation analysis. A convenient way of eliminating this one-dimensional kernel is by restricting to spaces of functions (V_1, V_2) with V_1 even function. Since U_{1y}^* is an odd function, in such a space L_0 is invertible, and its spectrum reduces to $\operatorname{spec} L_0 = \{-3\} \cup [1, +\infty)$. Of course, restricting to even first component V_1 requires some parity condition for the second component, as well. Going back to the full reduced system, and using its symmetries, we check that for the second component V_2 we may restrict to either even or odd functions. Choosing V_2 with the same parity as the eigenfunction in the kernel of L_ε at the bifurcation point ε_n , the kernel of \mathcal{L}_ε is one-dimensional, so that we are in the presence of a bifurcation from a simple eigenvalue.

We set $\varepsilon = -\nu_n + \tilde{\varepsilon}$ and $(V_1, V_2) = (W_1, \alpha h_n + W_2)$, where $\langle h_n, W_2 \rangle = 0$, and take (V_1, V_2) in $L_e^2(\mathbb{R}) \times L_e^2(\mathbb{R})$, if h_n is an even function, or $L_o^2(\mathbb{R}) \times L_o^2(\mathbb{R})$, if h_n is an odd function, where

$$L_e^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) ; f(z) = f(-z)\}, \quad L_o^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) ; f(z) = -f(-z)\}.$$

Then using the Liapunov-Schmidt reduction for the equation (3.9) we obtain the bifurcation equation $\tilde{\varepsilon} = \varphi_n(\alpha, \mu_1)$, with φ_n defined in a neighborhood of $(0, 0)$ such that $\varphi_n(0, 0) = 0$. For the system (3.1) this gives the following result.

Theorem 3.4. *Assume that Hypotheses 3.1 and 3.2 hold, and that $c_{11} > 0$ and $c_{21} > 0$. Take $N \geq 1$ such that (3.10) holds, and set $\omega_1 = \gamma_{11} + \mu_1$, $\omega_2 = \gamma_{21} + \mu_1 \varepsilon$. Then for $\mu_1 > 0$ sufficiently small, there exists a sequence of bifurcation points $\varepsilon_n = -\nu_n$, $n = 1, \dots, N$, with ν_n given by (3.11), and a sequence of functions $\varphi_n(\alpha, \mu_1)$ defined in a neighborhood of $(0, 0)$, $\varphi_n(0, 0) = 0$, such that for $\varepsilon = -\nu_n + \varphi_n(\alpha, \mu_1)$, the system (3.1) possesses a two-component localized solution (Q_1, Q_2) with Q_1, Q_2 real-valued functions such that*

$$Q_1(x, y) = \sqrt{\frac{2\mu_1}{d_{11}}} \operatorname{sech}(\sqrt{\mu_1} y) g_{11}(x) + O(\mu_1), \quad Q_2(x, y) = \alpha \sqrt{\mu_1} h_n(\sqrt{\mu_1} y) g_{21}(x) + O(\mu_1).$$

Remark 3.5. In [12, 26], as well as some of the references therein, the solution structure for the ODE given in (3.8) was studied without the higher-order terms. In these works the physically relevant restriction that $d_{21} = d_{12}$ was assumed.

3.2.2. Multipulse transverse profiles

In this section we construct multipulse solutions for the system (3.5). We use the approach developed in [9] for a very similar situation. Most parts of the proofs are identical, and we therefore only indicate the main steps and point out the differences. We make the following assumption on the coefficients d_{ij} of the reduced system.

Hypothesis 3.6. The coefficients d_{ij} of the reduced system (3.5) have the following properties:

$$\begin{aligned}\alpha^2 &:= \frac{d_{22} - d_{12}}{d_{11} - d_{21}} > 0 \text{ and } \alpha \neq 1; \\ \beta^2 &:= \frac{d_{11}d_{22} - d_{21}d_{12}}{d_{11} - d_{21}} > 0; \\ \lambda &:= \frac{2(3d_{11}d_{22} - 2d_{11}d_{12} - 2d_{22}d_{21} + d_{21}d_{12})}{d_{11}d_{22} - d_{21}d_{12}} \notin \mathbb{N}.\end{aligned}$$

The multipulse homoclinic solutions are constructed for parameters $\mu_1 \sim \mu_2 > 0$. We therefore set $\mu_1 = \mu > 0$, $\mu_2 = \mu(1 + \delta) > 0$, and as in the previous section we introduce the scaled variables

$$z = \sqrt{\mu}y, \quad a_1(y) = \sqrt{\mu}U_1(z), \quad a_2(y) = \sqrt{\mu}U_2(z), \quad a_3(y) = \sqrt{\mu^3}U_3(z), \quad a_4(y) = \sqrt{\mu^3}U_4(z).$$

We then obtain the system

$$\begin{aligned}U_{1z} &= U_3 \\ U_{2z} &= U_4 \\ U_{3z} &= U_1 - (d_{11}U_1^2 + d_{12}U_2^2)U_1 + O(\mu) \\ U_{4z} &= (1 + \delta)U_2 - (d_{21}U_1^2 + d_{22}U_2^2)U_2 + O(\mu).\end{aligned}\tag{3.12}$$

This system is of the same form as the system (59)–(62) considered in [9]. The major difference between the two systems concerns the number of discrete symmetries. While the system in [9] is reversible and possesses one reflection symmetry, our system is reversible and possesses two distinct reflection symmetries. As a consequence, we shall find here a larger family of multipulse solutions.

As in the case of the single equation, the reflection in y of (3.1) implies that the vector field in (3.12) anti-commutes with the reversor

$$\mathcal{R} = \text{diag}(1, 1, -1, -1),$$

and as a consequence of the phase invariance it commutes with the reflectors

$$\mathcal{S}_1 = \text{diag}(-1, 1, -1, 1), \quad \mathcal{S}_2 = \text{diag}(1, -1, 1, -1).$$

In addition to these discrete symmetries, a crucial role in this approach to multipulse solutions is played by the Hamiltonian structure of the reduced system. It is not difficult to check that the full dynamical system (3.2) possesses a Hamiltonian structure. As it is well-known, the Hamiltonian structure is preserved by the center manifold reduction [23], so that the reduced system (3.12) is hamiltonian. For our purposes, it is enough to compute the quadratic part of its Hamiltonian given by

$$H_2^\delta(U_3, U_4; U_1, U_2) = \frac{1}{2}U_3^2 + \frac{1}{2}U_4^2 - \frac{1}{2}U_1^2 - \frac{1}{2}(1 + \delta)U_2^2.\tag{3.13}$$

The first step in the construction of multipulse solutions consists in constructing unipulse transverse homoclinic solutions to (3.12) for small μ and δ . At $\mu = \delta = 0$, the system has the invariant subspace

$$(U_1, U_3) = \alpha(U_2, U_4),$$

in which we find the homoclinic solution

$$(U_2(z), U_4(z)) = \left(\frac{\sqrt{2}}{\beta} \text{sech}(z), -\frac{\sqrt{2}}{\beta} \tanh(z) \text{sech}(z) \right).\tag{3.14}$$

Here $\alpha > 0$ and $\beta > 0$ are the constants given in Hypothesis 3.6. Following the construction of transverse homoclinic solutions in [9, Proposition 3.6], we obtain the following result.

Proposition 3.7. *Assume that the coefficients d_{ij} in the reduced system satisfy Hypothesis 3.6. Then there exists a branch of transverse homoclinic solutions $u_*^{\mu, \delta}$ to (3.12) for (μ, δ) in a neighborhood of the origin in \mathbb{R}^2 which contains the homoclinic solution (3.14) at $\mu = \delta = 0$. In addition, there exist three further branches of homoclinic solutions obtained by applying the reflectors \mathcal{S}_1 , \mathcal{S}_2 , and $\mathcal{S}_1 \circ \mathcal{S}_2$ to the above branch.*

Proof: Following the proof of [9, Proposition 3.6], we linearize (3.12) at the explicit homoclinic (3.14) and obtain the equations

$$\begin{pmatrix} U_{1zz} - U_1 \\ U_{2zz} - U_2 \end{pmatrix} = -\frac{2}{\beta^2} \operatorname{sech}(z) \begin{pmatrix} 3\alpha^2 d_{11} + d_{12} & 2\alpha d_{12} \\ 2\alpha d_{21} & 3d_{22} + \alpha^2 d_{21} \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}.$$

As in [9, Proposition 3.6], the matrix in the right hand side of the above formula has real eigenvalues

$$\lambda_1 = 3\beta^2, \quad \lambda_2 = \frac{\lambda\lambda_1}{6},$$

in which λ is given in Hypothesis 3.6. The remainder of the proof is identical to the proof of [9, Proposition 3.6], and we therefore omit it. \square

Remark 3.8. Due to the additional reflection symmetry of our system, we have here four branches of homoclinic orbits, whereas the system in [9] possesses only two branches of homoclinic orbits. As a consequence, we shall find a larger class of N -pulse homoclinic solutions (2^{N+1} such solutions instead of 2).

The construction of multipulse homoclinic solutions relies upon the following general lemma:

Lemma 3.9. [9, Lemma 4.2] *Consider a Hamiltonian system*

$$\frac{du}{dz} = f^\delta(u), \quad (3.15)$$

with Hamiltonian function $H^\delta(u)$, where u and δ lie in a neighborhood of the origin in respectively \mathbb{R}^4 and \mathbb{R} . Suppose that the system (3.15) has four real, nonzero eigenvalues $\pm\lambda_1^\delta, \pm\lambda_2^\delta$. Further, assume that (3.15) possesses transverse homoclinic orbits $u_1^\delta, \dots, u_\ell^\delta$, and consider a cycle of (not necessarily distinct) transverse homoclinic orbits $u_1^\delta, \dots, u_N^\delta$. Then for sufficiently small values of δ and any sequence $\mathbf{L} = \{L_j\}_{j=0}^N$ of positive numbers with $L := \min\{L_j, j = 1, \dots, N-1\}$ sufficiently large and $L_0 = L_N = \infty$, there exists a unique set $\{U_j^\pm\}_{j=0}^N$ of functions with the following properties:

- (a) U_j^\pm are solutions to (3.15) on respectively $[0, L_j]$ and $[-L_{j-1}, 0]$ which lie close to u_j^δ ;
- (b) $U_j^+(L_j) = U_{j+1}^-(-L_j)$;
- (c) $U_j^-(0) - U_j^+(0) = \xi_j(\mathbf{L}, \delta)\Psi_j^\delta(0)$, in which $\Psi_j^\delta(z) = \nabla H^\delta(u_j^\delta(z))$, and the jump sizes $\xi_j(\mathbf{L}, \delta)$ are given by

$$\xi_j(\mathbf{L}, \delta) = \langle \Psi_j^\delta(-L_{j-1}), u_{j-1}^\delta(L_{j-1}) \rangle - \langle \Psi_j^\delta(L_j), u_{j+1}^\delta(-L_j) \rangle + \mathcal{R}_j(\mathbf{L}, \delta).$$

The remainder term $\mathcal{R}_j(\mathbf{L}, \delta)$ is a continuously differentiable function of \mathbf{L} which depends smoothly upon δ and satisfies $\mathcal{R}_j(\mathbf{L}, \delta), \partial_1 \mathcal{R}_j(\mathbf{L}, \delta) = O(e^{-3\lambda_s L})$, as $L \rightarrow \infty$, where $\lambda_s = \min\{|\lambda_1^\delta|, |\lambda_2^\delta|\}$.

The multipulse homoclinic orbits are now obtained by concatenating the solutions U_j^\pm . For this one solves the bifurcation equations

$$\xi_j(\mathbf{L}, \delta) = 0, \quad j = 1, \dots, N.$$

It is shown in [9, Section 4.1] that these equations are equivalent to

$$\eta_j(\mathbf{L}, \delta) = \langle \nabla^2 H^\delta(0) \mathcal{R} u_j^\delta(L_j), u_{j+1}^\delta(L_j) \rangle + \widehat{\mathcal{R}}_j(\mathbf{L}, \delta) = 0, \quad j = 1, \dots, N-1,$$

in which the remainder term $\widehat{\mathcal{R}}_j(\mathbf{L}, \delta)$ satisfies the same asymptotic estimate as $\mathcal{R}_j(\mathbf{L}, \delta)$. We now proceed as in [9, Section 4.2], and start by expanding these equations. In the following we fix μ , sufficiently small, and use δ as bifurcation parameter.

The primary homoclinic orbits $u_1^\delta, \dots, u_N^\delta$ in Lemma 3.9 are chosen among the four homoclinic solutions $\{u_*^{\mu,\delta}, \mathcal{S}_1 u_*^{\mu,\delta}, \mathcal{S}_2 u_*^{\mu,\delta}, \mathcal{S}_1 \circ \mathcal{S}_2 u_*^{\mu,\delta}\}$ found in Proposition 3.7. Consequently, we have

$$\eta_j(\mathbf{L}, \delta) = \langle \nabla^2 H^\delta(0) \mathcal{R} T_j u_*^{\mu,\delta}(L_j), T_{j+1} u_*^{\mu,\delta}(L_j) \rangle + \widehat{\mathcal{R}}_j(\mathbf{L}, \delta) = 0, \quad j = 1, \dots, N-1,$$

in which $T_j \in \{\text{id}, \mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_1 \circ \mathcal{S}_2\}$. From the formula (3.13) we find

$$\nabla^2 H^\delta(0) = \text{diag}(-1, -(1+\delta), 1, 1).$$

Next, using [9, Lemma 4.1], a straightforward calculation gives

$$u_*^{\mu,\delta}(z) = b^{\mu,\delta} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} e^{-z} + d^{\mu,\delta} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -(1+\delta)^{1/2} \end{pmatrix} e^{-(1+\delta)^{1/2}z} + \mathcal{O}(e^{-2\lambda_s z}),$$

as $z \rightarrow \infty$, where $b^{\mu,\delta}$ and $d^{\mu,\delta}$ are smooth functions of μ and δ which satisfy

$$b^{0,0} = 2 \left(\frac{2(d_{22} - d_{12})}{d_{11}d_{22} - d_{21}d_{12}} \right)^{1/2}, \quad d^{0,0} = 2 \left(\frac{2(d_{11} - d_{21})}{d_{11}d_{22} - d_{21}d_{12}} \right)^{1/2}.$$

We can now compute the $\eta_j(\mathbf{L}, \delta)$ and find that

$$\eta_j(\mathbf{L}, \delta) = 2\varepsilon_j (b^{\mu,\delta})^2 e^{-2L_j} + 2\kappa_j (1+\delta) (d^{\mu,\delta})^2 e^{-2(1+\delta)^{1/2}L_j} + \mathcal{O}(e^{-3\lambda_s L_j}),$$

in which

$$\varepsilon_j = \begin{cases} 1, & \text{if } T_j T_{j+1} \in \{\mathcal{S}_1, \mathcal{S}_1 \circ \mathcal{S}_2\} \\ -1, & \text{if } T_j T_{j+1} \in \{\text{id}, \mathcal{S}_2\} \end{cases}, \quad \kappa_j = \begin{cases} 1, & \text{if } T_j T_{j+1} \in \{\mathcal{S}_2, \mathcal{S}_1 \circ \mathcal{S}_2\} \\ -1, & \text{if } T_j T_{j+1} \in \{\text{id}, \mathcal{S}_1\} \end{cases}.$$

The bifurcation equations $\eta_j(\mathbf{L}, \delta) = 0$ are exactly of the same form as the ones in [9, Section 4.2], and they have a local branch of solutions $L_j = L_j(\delta)$, with $L_j(0) = \infty$, for $\delta < 0$ if $\alpha < 1$, and for $\delta > 0$ if $\alpha > 1$, provided $\varepsilon_j \kappa_j < 0$. This yields the following existence result for multipulse homoclinic solutions.

Theorem 3.10. *Assume that Hypothesis 3.6 holds and that $\mu > 0$ is sufficiently small. Then the system (3.12) possesses 2^{N+1} homoclinic solutions with N excursions away from the origin, for small $\delta < 0$ if $\alpha < 1$ (resp. $\delta > 0$ if $\alpha > 1$). Their N excursions away from the origin lie close to those of either $u_*^{\mu,\delta}$ or $\mathcal{S}_1 \circ \mathcal{S}_2 u_*^{\mu,\delta}$, and either $\mathcal{S}_1 u_*^{\mu,\delta}$ or $\mathcal{S}_2 u_*^{\mu,\delta}$, in a strictly alternating sequence.*

4. EXAMPLES

In this section we consider some physically relevant examples for which the preceding theory holds.

4.1. Bose-Einstein condensates: one component

As a first example, consider the governing equations for a one-component Bose-Einstein condensate, i.e., the equations (1.4) with $N = 1$. Clearly, steady solutions satisfy a system of the form (2.1). Assuming that the potential V is one-dimensional and satisfies Hypothesis 2.2, the result in Theorem 2.5 holds. In particular, this holds true for potentials of the form

$$V(x) = -V_0 x^2 + V_1 \cos 2x,$$

where the term $V_0 x^2$ corresponds to the application of a magnetic trap, and the term $V_1 \cos 2x$ models the creation of an optical lattice via counter-propagating laser beams (e.g., see [13] and the references therein).

4.2. Bose-Einstein condensates: two components

Next, consider the governing equations for a two-component Bose-Einstein condensate, i.e., the equations (1.4) with $N = 2$. Clearly, steady solutions satisfy a system of the form (3.1). Assuming that the potential V is one-dimensional and satisfies Hypothesis 3.2, the result in Theorem 3.4 holds under the assumption that the intra-species and inter-species interactions are attractive, i.e., that $a_{jk} > 0$. If $V_1(x) = V_2(x)$, then the existence criteria for the coefficients can be simplified (see [14] for a discussion on physically appropriate parameter values for the matrix \mathbf{A} in (1.4); see also Section 5.3 for the case of three spatial dimensions and some more details).

4.3. Photorefractive media

Consider the two-dimensional steady-state problem

$$\Delta Q - \omega Q - \frac{E_0}{(1 + I_0(x))(1 + |Q|^2)} Q = 0, \quad (4.1)$$

in which I_0 is a one-dimensional potential depending upon x . The equation (4.1) arises as the steady-state problem in the study of the propagation of light in a photorefractive crystal (see [15] and the references therein). Though not exactly of the form (2.1), the approach in Section 2 can be easily extended to this equation, provided I_0 is such that the operator

$$\mathcal{L} = \partial_{xx} - \frac{E_0}{1 + I_0(x)},$$

satisfies Hypothesis 2.2. The resulting reduced equation is of the form (2.11) with

$$c_1 = E_0 \int_{\mathbb{R}} \frac{g_1^4(x)}{1 + I_0(x)} dx > 0,$$

showing, in particular, that the result in Theorem 2.5 holds in this case.

5. DISCUSSION

We conclude with a brief discussion of some possible extensions of this approach.

5.1. Periodic potentials

The present approach can be adapted to the case of periodic potentials. The major difference consists then in the choice of the function space in the spatial dynamics formulation, the spaces of functions defined on the whole real line should be replaced by spaces of periodic functions. The period of these functions can be any integer multiple of the period of the potential. As a consequence, the two-dimensional waves constructed in this way are periodic in x . Their profile in y is determined again by the nature of the bounded orbits of the reduced system (localized for homoclinic orbits, periodic for periodic orbits, etc.).

5.2. Boundary conditions

The focus of this paper was on spots and stripes, but the present approach allows to construct other types of waves, as for instance the waves which correspond to quasi-periodic orbits or to homoclinic orbits to periodic orbits in the reduced system. In addition to such waves, one can also construct solutions which in the direction y satisfy certain boundary conditions such as Dirichlet or Neumann boundary conditions. Consider, for example, the simplest situation of the first bifurcation in Section 2.2 and restrict to real-valued

solutions. In this case the reduced system is the second order ODE (2.13) for which the phase portrait is entirely known. In particular, for suitable μ and c_1 , this equation has even orbits with the property that $a_1(-y_*) = a_1(y_*) = 0$, for some y_* . Upon using the discrete symmetries of the equation, going backwards the steps in the reduction procedure one can show that these orbits correspond to solutions to (2.1) which satisfy the Dirichlet boundary conditions $Q(-y_*) = Q(y_*) = 0$, on the interval $[-y_*, y_*]$. Similarly, one can construct solutions satisfying Neumann boundary conditions.

5.3. Higher dimensions

The equations considered in this paper are all planar, i.e., the spatial domain is \mathbb{R}^2 . It is not difficult to see that the same results hold in higher space dimensions \mathbb{R}^{d+1} , $d > 1$, when $x \in \mathbb{R}$ is replaced by $\mathbf{x} \in \mathbb{R}^d$, provided the potential V is nearly d -dimensional and satisfies the spectral Hypothesis 2.2 or Hypothesis 3.2. In particular, consider the results of Section 3.2 in the case of $d = 2$ with $V_1(x, z) = V_2(x, z) = x^2 + z^2$. In this case $\gamma_1 = -2$ with associated eigenfunction of

$$g_{11} = \sqrt{\frac{1}{\pi}} e^{-r^2/2}, \quad r^2 = x^2 + z^2.$$

The coefficients d_{ij} after (3.5) satisfy

$$d_{ij} = a_{ij} \iint_{\mathbb{R}^2} g_{11}^4 \, dA = 2\pi a_{ij} \int_0^\infty r g_{11}^4(r) \, dr.$$

The physically relevant assumption is $a_{12} = a_{21}$; hence, for $j = 1, 2$ set

$$a_j := \frac{a_{jj}}{a_{12}}.$$

Since $a_{12} > 0$ the assumptions under Hypothesis 3.6 eventually become $a_1 \neq a_2$ with

$$(a_1 - 1)(a_2 - 1) > 0; \quad a_2 \neq \frac{a_1 - (2 - c_k)}{c_k a_1 - 1}, \quad c_k := \frac{6 - k}{4} \text{ for } k \in \{1, 3, 4, 5\}.$$

As an application of Theorem 3.10 one then has the existence of N -spots of the form

$$U_j(r, z) \propto \sqrt{\mu} g_{11}(r) \phi_{j,N}(y),$$

where $\phi_{j,N}$ is one of the N -pulse solutions described in Theorem 3.10.

5.4. Multiple eigenvalues

The main hypothesis on the potentials considered here is on the existence of a finite number of isolated eigenvalues for the operator \mathcal{L} that we assumed to be simple. However, the latter assumption can be dropped and the same approach can be used in the case of eigenvalues with finite multiplicities. In such a situation one then expects higher dimensional center manifolds; for example, for the first bifurcation the dimension would be twice the multiplicity of the eigenvalue for the real-valued solutions, and four times the multiplicity of the eigenvalue for the complex-valued solutions. However, on such manifolds the question of persistence of bounded orbits may require a very different analysis.

As an illustration, using the notations from Section 2 assume that γ_2 is algebraically and geometrically double, and that γ_1 is simple. We look at the second bifurcation at $\omega = \gamma_2$, so that we set $\omega = \gamma_2 + \mu$. We then have the system in (2.16), and in (2.17) the eigenvalue zero is now algebraically quadruple and geometrically double:

$$\ker(\mathcal{A}_2) = \text{span}(\mathbf{e}_1, \mathbf{f}_1), \quad \text{gker}(\mathcal{A}_2) = \text{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{f}_1, \mathbf{f}_2),$$

where

$$\mathbf{e}_1 = \begin{pmatrix} g_2 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ g_2 \end{pmatrix}; \quad \mathbf{f}_1 = \begin{pmatrix} g_3 \\ 0 \end{pmatrix}, \quad \mathbf{f}_2 = \begin{pmatrix} 0 \\ g_3 \end{pmatrix},$$

and g_2, g_3 form an orthonormal basis for the eigenspace associated with the double eigenvalue γ_2 .

Upon continuing to follow the argument leading to (2.18) one defines the spectral projection as

$$P\mathbf{Q} = \langle \mathbf{Q}, \mathbf{e}_1 \rangle \mathbf{e}_1 + \langle \mathbf{Q}, \mathbf{e}_2 \rangle \mathbf{e}_2 + \langle \mathbf{Q}, \mathbf{f}_1 \rangle \mathbf{f}_1 + \langle \mathbf{Q}, \mathbf{f}_2 \rangle \mathbf{f}_2 + \langle \mathbf{Q}, \mathbf{f}_+^{\text{ad}} \rangle \mathbf{f}_+ + \langle \mathbf{Q}, \mathbf{f}_-^{\text{ad}} \rangle \mathbf{f}_-,$$

where $\mathbf{f}_\pm, \mathbf{f}_\pm^{\text{ad}}$ are as in Section 2.3. The Ansatz for the reduction now becomes

$$\mathbf{Q}(y) = a_1(y)\mathbf{e}_1 + a_2(y)\mathbf{e}_2 + b_1(y)\mathbf{f}_1 + b_2(y)\mathbf{f}_2 + B(y)\mathbf{f}_+ + \overline{B}(y)\mathbf{f}_- + \mathbf{R}(y), \quad (5.1)$$

where $P\mathbf{R}(y) = 0$ for all $y \in \mathbb{R}$ and

- a_1, a_2, b_1, b_2 are real-valued
- B is complex-valued.

After using the center-manifold reduction one finds a six-dimensional reduced system which is reversible, i.e., the vector field anti-commutes with the symmetry

$$\mathcal{R}(a_1, a_2, b_1, b_2, B) = (a_1, -a_2, b_1, -b_2, \overline{B}),$$

and which commutes with the symmetry

$$\mathcal{S}(a_1, a_2, b_1, b_2, B) = (-a_1, -a_2, -b_1, -b_2, -B).$$

Since $\langle g_j, g_k \rangle = \delta_{jk}$, the truncated normal form for the equations on the center manifold is eventually seen to be of the form

$$\begin{aligned} a_{1y} &= a_2 \\ a_{2y} &= \mu a_1 - c_0 (\langle g_2^3, g_2 \rangle a_1^3 + 3\langle g_2^2 g_3, g_2 \rangle a_1^2 b_1 + 3\langle g_2 g_3^2, g_2 \rangle a_1 b_1^2 + \langle g_3^3, g_2 \rangle b_1^3) \\ b_{1y} &= b_2 \\ b_{2y} &= \mu b_1 - c_0 (\langle g_2^3, g_3 \rangle a_1^3 + 3\langle g_2^2 g_3, g_3 \rangle a_1^2 b_1 + 3\langle g_2 g_3^2, g_3 \rangle a_1 b_1^2 + \langle g_3^3, g_3 \rangle b_1^3) \\ B_y &= i\sqrt{\gamma_2 - \gamma_1} B. \end{aligned} \quad (5.2)$$

(compare to (2.19)). As a consequence of the symmetries \mathcal{R} and \mathcal{S} all other terms in the expansion are necessarily of higher order.

Now let us consider a concrete example for which the above scenario holds. Consider (1.4) when $N = 1$ and $d = 2$ with $V(x, z) = x^2 + z^2$. The spectra and eigenfunctions for $-\Delta + V$ is well-understood; in particular, the first two eigenvalues, as well as the associated eigenfunctions, are given by in polar coordinates by

$$\begin{aligned} \gamma_1 &= 2, \quad g_1 = \sqrt{\frac{1}{\pi}} e^{-r^2/2} \\ \gamma_2 &= 4, \quad g_2 = \sqrt{\frac{2}{\pi}} r e^{-r^2/2} \cos \theta, \quad g_3 = \sqrt{\frac{2}{\pi}} r e^{-r^2/2} \sin \theta. \end{aligned}$$

The inner products in the coefficients of the system (5.2) can then be explicitly computed as

$$\langle g_2^3 g_3 \rangle = \langle g_2 g_3^3 \rangle = 0, \quad \langle g_2^3, g_2 \rangle = \langle g_3^3, g_3 \rangle = 3\langle g_2^2, g_3^2 \rangle = c_1; \quad c_1 = \frac{3}{128\pi} c_0.$$

The system (5.2) then collapses to

$$\begin{aligned} a_{1y} &= a_2 \\ a_{2y} &= \mu a_1 - c_1 a_1 (a_1^2 + b_1^2) \\ b_{1y} &= b_2 \\ b_{2y} &= \mu b_1 - c_1 b_1 (a_1^2 + b_1^2) \\ B_y &= i\sqrt{2} B, \end{aligned} \quad (5.3)$$

which upon setting $A_j = a_j + ib_j$, $j = 1, 2$, becomes

$$\begin{aligned} A_{1y} &= A_2 \\ A_{2y} &= \mu A_1 - c_1 A_1 |A_1|^2 \\ B_y &= i\sqrt{2} B. \end{aligned}$$

Clearly, the first two equations give the NLS equation (2.11), so that for $B = 0$ one finds the solutions in the list following (2.11). For $A_1 = 0$, one finds again the periodic solutions $B(y) = e^{i\sqrt{2}y} B(0)$. It is expected that the periodic solutions persist for the full normal form, whereas the homoclinic orbit to zero only persists as a homoclinic orbit to periodic orbits, just as in the case of the second bifurcation in Section 2.3.

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