

The Krein signature, Krein eigenvalues, and the Krein Oscillation Theorem

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Abstract. In this paper the problem of locating eigenvalues of negative Krein signature is considered for operators of the form $\mathcal{J}\mathcal{L}$, where \mathcal{J} is skew-symmetric with bounded inverse and \mathcal{L} is self-adjoint. A finite-dimensional matrix, hereafter referred to as the Krein matrix, associated with the eigenvalue problem $\mathcal{J}\mathcal{L}u = \lambda u$ is constructed with the property that if the Krein matrix has a nontrivial kernel for some z_0 , then $\pm\sqrt{-z_0} \in \sigma(\mathcal{J}\mathcal{L})$. The eigenvalues of the Krein matrix, i.e., the Krein eigenvalues, are real meromorphic functions of the spectral parameter, and have the property that their derivative at a zero is directly related to the Krein signature of the eigenvalue. The Krein Oscillation Theorem relates the number of zeros of a Krein eigenvalue to the number of eigenvalues with negative Krein signature. Because the construction of the Krein matrix is functional analytic in nature, it can be used for problems posed in more than one space dimension. This feature is illustrated in an example for which the spectral stability of the dipole solution to the Gross-Pitaevski equation is considered.

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1. INTRODUCTION

When attempting to understand the dynamics near the critical points of an energy surface for a Hamiltonian system, one must study eigenvalue problems of the form

$$\mathcal{J}\mathcal{L}u = \lambda u, \quad (1.1)$$

where \mathcal{J} is a skew-symmetric operator with bounded inverse and \mathcal{L} is a self-adjoint operator (see [9] for the case where $\ker(\mathcal{J})$ is nontrivial). The operator \mathcal{L} is the linearization of the energy surface about a critical point, and $\sigma(\mathcal{L})$ corresponds to the local curvature of the surface. If $\sigma(\mathcal{L}) \subset \mathbb{R}^+$, then under appropriate assumptions the critical point is a minimizer of the energy surface, and is hence stable. If $\sigma(\mathcal{L}) \not\subset \mathbb{R}^+$, then the critical point is not a minimizer. It may be a constrained minimizer, or it may instead be energetically unstable. In the latter case one wishes to better understand the manner in which being energetically unstable influences $\sigma(\mathcal{J}\mathcal{L})$.

If $\text{Im}(\mathcal{J}\mathcal{L}) = 0$, then it was seen in [21] that (1.1) is equivalent to the canonical case

$$\mathcal{J} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} \mathcal{L}_+ & 0 \\ 0 & \mathcal{L}_- \end{pmatrix}, \quad (1.2)$$

upon setting $\mathcal{L}_+ = \mathcal{L}$ and $\mathcal{L}_- = -\mathcal{J}\mathcal{L}\mathcal{J}$. If $\text{Im}(\mathcal{J}\mathcal{L}) \neq 0$, then one must be careful when comparing the two problems; in particular, the reduction of the four-fold eigenvalue symmetry for (1.1) when $\text{Im}(\mathcal{J}\mathcal{L}) = 0$ to a two-fold symmetry when $\text{Im}(\mathcal{J}\mathcal{L}) \neq 0$ causes some difficulties. It will henceforth be assumed that (1.1) is in the canonical form of (1.2). Furthermore, it will be assumed that:

Assumption 1.1. The operators \mathcal{J} and \mathcal{L} in (1.2) satisfy a variant of [13, Assumption 2.1(b)-(d)], i.e.,

- (a) \mathcal{L}_\pm are self-adjoint with compact resolvent
- (b) $\sigma(\mathcal{L}_\pm) \cap \mathbb{R}^-$ is a finite set
- (c) there is a self-adjoint operator \mathcal{L}_0 with compact resolvent such that:

- (1) $\mathcal{L}_\pm = \mathcal{L}_0 + \mathcal{A}_\pm$, where \mathcal{A}_\pm are \mathcal{L}_0 -compact and satisfy

$$\|\mathcal{A}_\pm u\| \leq a\|u\| + b\|\mathcal{L}_0\|^r u\|$$

for some positive constants a, b , and $r \in [0, 1)$

- (2) the increasing sequence of nonzero eigenvalues ω_j of \mathcal{L}_0 satisfy for some $s \geq 1$,

$$\sum_{j=1}^{\infty} |\omega_j|^{-s} < \infty$$

- (3) there exists a subsequence of eigenvalues $\{\omega_{n_k}\}_{k \in \mathbb{N}}$ and positive constants $c > 0$ and $r' > r$ such that

$$\omega_{n_k+1} - \omega_{n_k} \geq c\omega_{n_k+1}^{r'}.$$

- (d) regarding $\text{gker}(\mathcal{J}\mathcal{L})$ one has:

- (1) $\ker(\mathcal{L}_+) \perp \ker(\mathcal{L}_-)$
- (2) $\dim[\text{gker}(\mathcal{J}\mathcal{L})] = 2 \dim[\ker(\mathcal{L})]$ (see [20, Lemma 3.1] for a simple condition which ensures this dimension count).

Remark 1.2. The assumption of compactness is not necessary, e.g., see [10, 11, 20, 27] and the references therein, but it simplifies the analysis by removing the possibility of having eigenvalues embedded in the continuous spectrum. The compactness of the operators was recently exploited in [13].

Assumption 1.1, unlike [13, Assumption 2.1], does not assume that \mathcal{L} is invertible. However, the invertibility assumption can be recovered in the following manner. Upon using **Assumption 1.1(d:1)** let $\Pi : H \mapsto [\ker(\mathcal{L}_+) \oplus \ker(\mathcal{L}_-)]^\perp$ be the orthogonal projection, where H is the appropriate Hilbert space for the problem. It was seen in [20, Section 3] that solving (1.1) for nonzero eigenvalues is equivalent to solving the system

$$-\Pi\mathcal{L}_+\Pi u = \lambda v, \quad \Pi\mathcal{L}_-\Pi v = \lambda u. \quad (1.3)$$

The operators $\Pi\mathcal{L}_\pm\Pi$ are self-adjoint and nonsingular under **Assumption 1.1(d:2)**.

Set

$$\mathcal{R} := \Pi\mathcal{L}_+\Pi, \quad \mathcal{S}^{-1} := \Pi\mathcal{L}_-\Pi, \quad z := -\lambda^2 \quad (-\pi/2 < \arg \lambda \leq \pi/2), \quad (1.4)$$

and note that the derivation leading to (1.5) yields that \mathcal{R} and \mathcal{S}^{-1} are nonsingular self-adjoint operators with compact resolvent on a Hilbert space H endowed with inner-product $\langle \cdot, \cdot \rangle$. (1.3) is equivalent to the eigenvalue problem

$$(\mathcal{R} - z\mathcal{S})u = 0 \quad (1.5)$$

in the following sense (the mapping $z = -\lambda^2$ is illustrated in Figure 1). The nonzero spectrum associated with (1.2) has the four-fold symmetry $\{\pm\lambda, \pm\bar{\lambda}\}$. Eigenvalues with positive real part and nonzero imaginary part are mapped in a one-to-one fashion to eigenvalues with nonzero imaginary part, and instead of the four-fold symmetry there is now only the reflection symmetry $\{z, \bar{z}\}$. Positive real eigenvalues are mapped to negative real eigenvalues. Consequently, all unstable eigenvalues for (1.3) are captured by considering (1.5) with

- $z \in \mathbb{R}^-$ if and only if $\lambda \in \mathbb{R}^+$
- $\text{Im } z \neq 0$ if and only if $\text{Im } \lambda \neq 0$ and $\text{Re } \lambda > 0$.

With respect to the purely imaginary eigenvalues, one has that $\lambda \in i\mathbb{R}^+$ is mapped to \mathbb{R}^+ ; however, $i\mathbb{R}^-$ is not included in the domain of the mapping, and hence these eigenvalues are not captured. Since purely imaginary eigenvalues come in the pair $\pm\lambda$, one has that if there is an eigenvalue $z \in \mathbb{R}^+$ for (1.5), then for (1.4) there will be the eigenvalue pair $\pm\lambda \in i\mathbb{R}$.

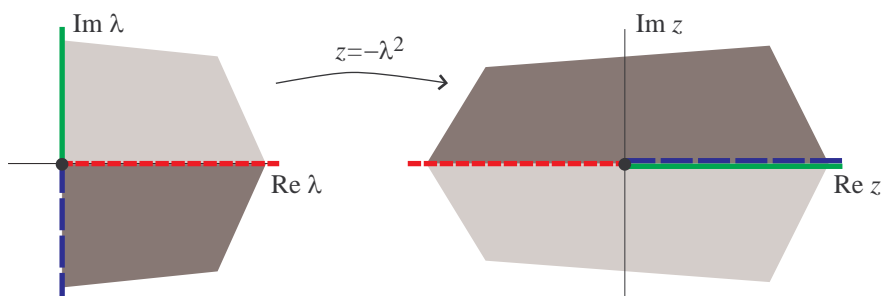


Figure 1: (color online) The left panel is the spectral set $-\pi/2 < \arg(\lambda) \leq \pi/2$ for (1.3), and the right panel is the spectral set $0 < \arg(z) \leq 2\pi$ for (1.5). These two sets are related by the mapping $z = -\lambda^2$.

There has been a great deal of recent work on (1.3). Because of the equivalence between (1.3) and (1.5), the results will be stated in the context of (1.5). Most recently it was shown that under Assumption 1.1,

$$k_r + 2k_c + 2k_i^- = n(\mathcal{R}) + n(\mathcal{S}) \quad (1.6)$$

[13, Theorem 2.25]. Here $n(\mathcal{A})$ is the number of strictly negative eigenvalues (counted with algebraic multiplicities) of the self-adjoint operator \mathcal{A} , k_r refers to the number of negative real eigenvalues, k_c is the number of eigenvalues with $\text{Im } z < 0$, and k_i^- is the number of positive real eigenvalues with negative Krein signature.

The Krein signature of an eigenvalue $z \in \mathbb{R}^+$ with eigenspace E_z says something important about the nature of the flow in the direction of E_z along the energy surface. If the signature is positive, then the energy surface is positive definite in this direction. On the other hand, if the signature is negative, then the surface is negative definite in the direction of some nontrivial subspace of E_z ; however, the vector field associated with the linearized flow (and perhaps the nonlinear flow, see [9, Section 4]) is orthogonal to those energetically unstable directions. The signature is computed via

$$k_i^-(z) = n(\mathcal{S}|_{E_z}), \quad k_i^- = \sum_{z \in \sigma(\mathcal{S}^{-1}\mathcal{R}) \cap \mathbb{R}^+} k_i^-(z);$$

in particular, if $z \in \mathbb{R}^+$ is geometrically and algebraically simple with eigenfunction u , then

$$k_i^-(z) = \begin{cases} 0, & \langle u, \mathcal{S}u \rangle > 0 \\ 1, & \langle u, \mathcal{S}u \rangle < 0. \end{cases} \quad (1.7)$$

If $\langle u, \mathcal{S}u \rangle = 0$ then one knows that the eigenspace has a Jordan block [10, Theorem 2.3].

From (1.6) one necessarily has that $k_r \geq 1$ if $n(\mathcal{R}) + n(\mathcal{S})$ is odd (see [12] for a similar result for (1.1)). Thus, one has a simple instability criterion, which for (1.5) can be refined in the following manner. For a self-adjoint operator \mathcal{H} define the negative cone $C(\mathcal{H})$ by

$$C(\mathcal{H}) := \{u : \langle u, \mathcal{H}u \rangle < 0\} \cup \{0\},$$

and let $\dim[C(\mathcal{H})]$ denote the dimension of the maximal subspace in $C(\mathcal{H})$. First suppose that each negative real-valued eigenvalue is algebraically simple. By [13, Corollary 2.24] one knows that (1.6) can be refined as

$$\begin{aligned} k_r &= |n(\mathcal{R}) - n(\mathcal{S})| + 2(\min\{n(\mathcal{R}), n(\mathcal{S})\} - \dim[C(\mathcal{R}) \cap C(\mathcal{S})]) \\ k_c + k_i^- &= \dim[C(\mathcal{R}) \cap C(\mathcal{S})]. \end{aligned} \quad (1.8)$$

If the assumption that negative real-valued eigenvalues are algebraically simple is removed, then the equalities are removed in favor of inequalities:

$$\begin{aligned} k_r &\geq |n(\mathcal{R}) - n(\mathcal{S})| + 2(\min\{n(\mathcal{R}), n(\mathcal{S})\} - \dim[C(\mathcal{R}) \cap C(\mathcal{S})]) \\ k_c + k_i^- &\leq \dim[C(\mathcal{R}) \cap C(\mathcal{S})]. \end{aligned} \quad (1.9)$$

A close examination of (1.8) yields some important insights concerning the location of the spectra for (1.5). First, if $n(\mathcal{S}) = 0$, then $C(\mathcal{S}) = \{0\}$, so that $k_r = n(\mathcal{R})$ and $k_i^- = k_c = 0$. Consequently, it will henceforth be assumed in this paper that $n(\mathcal{S}) \geq 1$. Second, the first line in either (1.8) or (1.9) yields the lower bound

$$k_r \geq |n(\mathcal{R}) - n(\mathcal{S})| \quad (1.10)$$

(for the earliest proofs see [11, 14]). Finally, the inequality in (1.10) is closed upon considering the cone intersection $C(\mathcal{R}) \cap C(\mathcal{S})$. If $C(\mathcal{R}) \cap C(\mathcal{S}) = \{0\}$, then one has that $k_r = n(\mathcal{R}) + n(\mathcal{S})$, which by (1.6) implies that all of the unstable spectra is purely real. As the dimension of this cone intersection increases, the number of negative real eigenvalues decreases in favor of either (a) eigenvalues with nonzero imaginary part, or (b) positive real eigenvalues.

As a consequence of (1.8) one has a precise count of k_r . It would be useful if one could also precisely count k_c , for it would then be the case that one would have an exact count of the total number of spectrally unstable eigenvalues. Alternatively, one could determine k_i^- , and then use (1.6) to determine k_c . As it is seen in the second line of (1.8), the two quantities k_i^- and k_c are closely related by the cone intersection; furthermore, one has the upper bound

$$k_c + k_i^- \leq \min\{n(\mathcal{R}), n(\mathcal{S})\} \quad (1.11)$$

(this upper bound is also valid for (1.9)). It is, and should be, difficult to further refine the second line of (1.8). It is well-known that if $1 \leq k_i^-(z) \leq \dim(E_z) - 1$, i.e., there is a collision of a positive real eigenvalues of opposite sign, then there is generically a Hamiltonian-Hopf bifurcation. Under this bifurcation a small change of the vector field decreases the count k_i^- while increasing the count k_c . The dimension of the cone intersection is unchanged under small perturbations, so that the total count $k_c + k_i^-$ remains unchanged.

In this paper a certain matrix, called the *Krein matrix*, will be constructed, and it will have the property that its eigenvalues, called the *Krein eigenvalues*, can be used to locate all of those eigenvalues making up the subcount $k_c + k_i^-$. These eigenvalues are realized as zeros of the determinant of the Krein matrix, which in turn implies that they are realized as zeros of one or more of the Krein eigenvalues. For positive real zeros which do not coincide with removable singularities of the Krein matrix the Krein signature of the eigenvalue will be directly related to the sign of the derivative of a Krein eigenvalue. A careful study of the graphical asymptotics of the Krein eigenvalues leads to the *Krein Oscillation Theorem*, which relates the number of zeros of the Krein eigenvalue in a given interval for which it is real analytic to the number of negative eigenvalues, or to the number of positive eigenvalues with negative signature.

The Evans function is a holomorphic function of the eigenvalue parameter, the zeros of the Evans function are located precisely at the eigenvalues, and the order of the zero is the algebraic multiplicity of the eigenvalue (see [3, 15, 18, 29] and the references therein). There is one way in which the Krein matrix behaves like the Evans function: the zeros of its determinant will correspond to eigenvalues. However, it will differ from the Evans function in many important respects. On the other hand, for the Krein matrix and Krein eigenvalues one has:

- (a) the Krein matrix and Krein eigenvalues are meromorphic
- (b) the determinant of the Krein matrix does not necessarily vanish for all real eigenvalues
- (c) the order of the zero of the Krein eigenvalue will match the algebraic multiplicity of the eigenvalue only for those eigenvalues not located at removable singularities.

A final important point of distinction is that the construction of the Krein matrix is functional analytic in nature, and hence it can be formally constructed for problems posed in many spatial dimensions. Alternatively, the Evans function is constructed via ODE techniques, and is therefore applicable to only those problems which can be somehow posed on a subset of the line.

Property (a) of the Krein eigenvalues is a necessary condition if one wishes to relate the derivative of the Krein eigenvalue to the signature of the eigenvalue. Regarding (b), it appears at first glance that this property would seem to preclude the usefulness of the Krein matrix and the Krein

eigenvalues. However, the matrix will be constructed in such a manner that its determinant will be zero for all of those eigenvalues for which the associated eigenspace has a nontrivial intersection with $C(\mathcal{S})$. As was seen in [13], this then means it will necessarily be zero whenever the eigenvalue has nonzero imaginary part, or is positive and real with negative signature. Hence, the determinant of the Krein matrix will detect all of those eigenvalues making up the subcount $k_c + k_1^-$. The only eigenvalues that will potentially be missed are either negative real eigenvalues, or the positive real ones with positive Krein signature. It will be seen that those eigenvalues which are not realized as a zero of a Krein eigenvalue will correspond to a removable singularity of the Krein matrix.

The paper will be organized as follows. In Section 2 the Krein matrix will be constructed, and the properties leading to the Krein Oscillation Theorem will be derived. In Section 3 the theory will be used to study the spectral problem associated with waves of the Gross-Pitaevski equation in both one and two space dimensions.

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2. THE KREIN MATRIX AND KREIN EIGENVALUES

The construction of the Krein matrix follows the ideas presented in [10, 27]. In contrast to [10, 27] there will be the fundamental difference that \mathcal{R} and \mathcal{S}^{-1} are both assumed to have compact resolvent, and hence there will be no issue herein with continuous spectrum. In [27] it was also assumed that $n(\mathcal{S}^{-1}) = 1$, which was relevant to the problem solved therein. Those properties of the Krein matrix and the Krein eigenvalues which follow directly from either of [10, 27] will be appropriately cited.

2.1. Construction

It will be assumed for all self-adjoint operators that the eigenfunctions form an orthonormal basis. For the self-adjoint operator \mathcal{S} with eigenfunction basis $\{\phi_j\}$, let the associated eigenvalues be denoted by λ_j . Since \mathcal{S} is nonsingular and compact with $1 \leq n(\mathcal{S}) < \infty$, one has that $\sigma(\mathcal{S})$ can be ordered as $\lambda_1 \leq \lambda_2 \leq \dots \lambda_n < 0$, and $\lambda_j > 0$ for $j \geq n + 1$. Set

$$N(\mathcal{S}) := \text{span}\{\phi_1, \dots, \phi_n\},$$

and let $P : H \mapsto N(\mathcal{S})^\perp$ and $Q = \mathbb{1} - P$ be orthogonal projections, i.e.,

$$Pu = u - \sum_{j=1}^n \langle u, \phi_j \rangle \phi_j, \quad Qu = \sum_{j=1}^n \langle u, \phi_j \rangle \phi_j. \quad (2.1)$$

Define the operators

$$\mathcal{R}_2 := PRP, \quad \mathcal{S}_2 := PSP, \quad (2.2)$$

and note that (a) \mathcal{R}_2 and \mathcal{S}_2^{-1} have compact resolvent, and (b) $n(\mathcal{S}_2) = 0$ with $n(\mathcal{R}_2) \leq n(\mathcal{R})$. Upon setting $p := Pu$ and writing $u = p + \sum \alpha_j \phi_j$, by applying the projection P to (1.5) one gets the system

$$(\mathcal{R}_2 - z\mathcal{S}_2)p + \sum_{j=1}^n \alpha_j PR\phi_j = 0. \quad (2.3)$$

Applying the projection Q , taking the inner product of the resulting equation with ϕ_ℓ for $\ell = 1, \dots, n$, and using the fact that P, Q are commuting self-adjoint operators yields the system of equations

$$\langle \mathcal{R}p, \phi_\ell \rangle + \sum_{j=0}^n \alpha_j \langle \mathcal{R}\phi_j, \phi_\ell \rangle = \alpha_\ell z \lambda_\ell. \quad (2.4)$$

The system of (2.3) and (2.4) is the one to be studied.

In order to close (2.4), one must solve for p in (2.3) and substitute that expression into (2.4). Since \mathcal{S}_2 is compact, self-adjoint, and positive definite, the expression $\mathcal{S}_2 = \mathcal{S}_2^{1/2} \mathcal{S}_2^{1/2}$ is well-defined. Write

$$\mathcal{R}_2 - z\mathcal{S}_2 = \mathcal{S}_2^{1/2}(\tilde{\mathcal{R}} - z\mathbb{1})\mathcal{S}_2^{1/2}, \quad \tilde{\mathcal{R}} := \mathcal{S}_2^{-1/2}\mathcal{R}_2\mathcal{S}_2^{-1/2} : N(\mathcal{S})^\perp \mapsto N(\mathcal{S})^\perp, \quad (2.5)$$

and note that $\tilde{\mathcal{R}}$ is self-adjoint with compact resolvent and satisfies $n(\mathcal{R}_2) = n(\tilde{\mathcal{R}})$. Upon substituting (2.5) into (2.3) one sees that

$$p = \mathcal{S}_2^{-1/2}k - \sum_{j=1}^n \alpha_j \mathcal{S}_2^{-1/2}(\tilde{\mathcal{R}} - z\mathbb{1})^{-1} \mathcal{S}_2^{-1/2} P \mathcal{R} \phi_j, \quad (2.6)$$

where

$$k \in \ker(\tilde{\mathcal{R}} - z\mathbb{1}), \quad \mathcal{S}_2^{-1/2} P \mathcal{R} \phi_j \in [\ker(\tilde{\mathcal{R}} - z\mathbb{1})]^\perp.$$

Since $\tilde{\mathcal{R}}$ is self-adjoint with a compact resolvent, the real-valued eigenvalues, say μ_j , can be ordered as a monotone increasing sequence with $\mu_j \rightarrow +\infty$ as $j \rightarrow \infty$. Clearly (2.6) makes sense for all $z \notin \sigma(\tilde{\mathcal{R}})$, and care must be taken only when $z = \mu_j$ for some j .

Substitution of (2.6) into (2.4) yields the algebraic system for $\ell = 1, \dots, n$,

$$\langle \mathcal{R} \mathcal{S}_2^{-1/2} k, \phi_\ell \rangle + \sum_{j=1}^n \alpha_j \langle \mathcal{R} \phi_j, \phi_\ell \rangle - \sum_{j=1}^n \alpha_j \langle \mathcal{S}_2^{-1/2}(\tilde{\mathcal{R}} - z\mathbb{1})^{-1} \mathcal{S}_2^{-1/2} P \mathcal{R} \phi_j, \mathcal{R} \phi_\ell \rangle = \alpha_\ell z \lambda_\ell. \quad (2.7)$$

If $z \neq \mu_j$ for all j , then $k = 0$ and the first term on the left-hand side in (2.7) disappears. If $z = \mu_j$, and if $\mathcal{S}_2^{-1/2} P \mathcal{R} \phi_\ell \in [\ker(\tilde{\mathcal{R}} - \mu_j \mathbb{1})]^\perp$, then since $\mathcal{S}_2^{-1/2}$ and \mathcal{R} are self-adjoint one has

$$\langle \mathcal{R} \mathcal{S}_2^{-1/2} k, \phi_\ell \rangle = \langle \mathcal{S}_2^{-1/2} k, \mathcal{R} \phi_\ell \rangle = \langle \mathcal{S}_2^{-1/2} k, P \mathcal{R} \phi_\ell \rangle = \langle k, \mathcal{S}_2^{-1/2} P \mathcal{R} \phi_\ell \rangle = 0. \quad (2.8)$$

The second equality follows from the fact that $\mathcal{S}_2 : N(\mathcal{S})^\perp \mapsto N(\mathcal{S})^\perp$. If $\mathcal{S}_2^{-1/2} P \mathcal{R} \phi_\ell \notin [\ker(\tilde{\mathcal{R}} - \mu_j \mathbb{1})]^\perp$, then (2.3) cannot be solved, and hence it makes no sense to consider (2.7). In conclusion, (2.7) can be reduced to

$$\sum_{j=1}^n \alpha_j \langle \mathcal{R} \phi_j, \phi_\ell \rangle - \sum_{j=1}^n \alpha_j \langle \mathcal{S}_2^{-1/2}(\tilde{\mathcal{R}} - z\mathbb{1})^{-1} \mathcal{S}_2^{-1/2} P \mathcal{R} \phi_j, \mathcal{R} \phi_\ell \rangle = \alpha_\ell z \lambda_\ell, \quad z \notin \sigma(\tilde{\mathcal{R}}). \quad (2.9)$$

As a consequence of (2.5) one has

$$\langle \mathcal{S}_2^{-1/2}(\tilde{\mathcal{R}} - z\mathbb{1})^{-1} \mathcal{S}_2^{-1/2} P \mathcal{R} \phi_j, \mathcal{R} \phi_\ell \rangle = \langle (\mathcal{S}_2^{-1/2}(\tilde{\mathcal{R}} - z\mathbb{1})^{-1} \mathcal{S}_2^{-1/2} P \mathcal{R} \phi_j, P \mathcal{R} \phi_\ell) \rangle.$$

If one defines the matrices $\mathbf{R}, \mathbf{D}, \mathbf{C}(z)$ by

$$\mathbf{R}_{j\ell} = \langle \phi_j, \mathcal{R} \phi_\ell \rangle, \quad \mathbf{D} = \text{diag}(\lambda_1, \dots, \lambda_n), \quad \mathbf{C}(z)_{j\ell} = \langle (\mathcal{S}_2^{-1/2}(\tilde{\mathcal{R}} - z\mathbb{1})^{-1} \mathcal{S}_2^{-1/2} P \mathcal{R} \phi_j, P \mathcal{R} \phi_\ell) \rangle, \quad (2.10)$$

then upon setting $\mathbf{a} := (\alpha_1, \dots, \alpha_n)^\top$ (2.9) is equivalent to

$$\mathbf{K}(z)\mathbf{a} = \mathbf{0}, \quad \mathbf{K}(z) := \mathbf{R} - z\mathbf{D} - \mathbf{C}(z). \quad (2.11)$$

In (2.11) note that $\mathbf{K}(z)$ is symmetric, i.e. $\mathbf{K}(z)^\top = \mathbf{K}(z)$ for all $z \in \mathbb{C}$. In particular, $\mathbf{K}(z)$ is symmetric for $z \in \mathbb{R}$.

Definition 2.1. For (1.5) the Krein matrix $\mathbf{K}(z)$ is given in (2.11).

In order to better understand the implications associated with (2.11), it is first necessary to rewrite $\mathbf{C}(z)$. For $\mu_j \in \sigma(\tilde{\mathcal{R}})$ let v_j represent the corresponding normalized eigenfunction, so that $\{v_j\}$ is an orthonormal basis. Upon using the fact that $\mathcal{S}_2^{-1/2}$ is self-adjoint, a standard expansion yields that the entries of $\mathbf{C}(z)$ can be rewritten as

$$\mathbf{C}(z)_{j\ell} = \sum_{i=1}^{\infty} \frac{\langle \mathcal{S}_2^{-1/2} P\mathcal{R}\phi_j, v_i \rangle \overline{\langle \mathcal{S}_2^{-1/2} P\mathcal{R}\phi_\ell, v_i \rangle}}{\mu_i - z}. \quad (2.12)$$

Upon setting

$$\mathbf{c}^i := (c_1^i, c_2^i, \dots, c_n^i)^\top, \quad c_j^i := \langle \mathcal{S}_2^{-1/2} P\mathcal{R}\phi_j, v_i \rangle, \quad (2.13)$$

one has the compact form

$$\mathbf{C}(z) = \sum_{i=1}^{\infty} \frac{\mathbf{c}^i (\mathbf{c}^i)^\mathbf{H}}{\mu_i - z}, \quad (2.14)$$

where $\mathbf{v}^\mathbf{H}$ refers to the Hermitian of the vector \mathbf{v} .

Consider solutions to (2.11) near a simple pole μ_i . Suppose that $\mu_i = \mu_{i+1} = \dots = \mu_{i+\ell}$ for some finite $\ell \in \mathbb{N}_0$. Upon examining (2.14) one sees that a Laurent expansion of $\mathbf{C}(z)$ in a neighborhood of $z = \mu_i$ is given by

$$\mathbf{C}(z) = \frac{\sum_{j=0}^{\ell} \mathbf{c}^{i+j} (\mathbf{c}^{i+j})^\mathbf{H}}{\mu_i - z} + \mathbf{C}^{(0)} + \dots$$

Now, (2.3) is solvable for $\mathbf{a} \neq \mathbf{0}$ at $z = \mu_i$ if and only if $\mathbf{c}^{i+j} = \mathbf{0}$ for all $0 \leq j \leq \ell$; in other words, (2.3) is solvable at $z = \mu_i$ if and only if $z = \mu_i$ is a removable singularity for $\mathbf{K}(z)$. If the singularity is removable, then (2.3) has a nontrivial solution with $\mathbf{a} = \mathbf{0}$; furthermore, (2.4) is automatically satisfied with $\mathbf{a} = \mathbf{0}$ as a consequence of (2.6) and (2.8). Since $\mathbf{a} = \mathbf{0}$ one sees that $\langle u, \mathcal{S}u \rangle = \langle u, \mathcal{S}_2 u \rangle > 0$; hence, if $z \in \mathbb{R}^+$ then by (1.7) the eigenvalue has positive signature. In order to see if there are solutions with $\mathbf{a} \neq \mathbf{0}$, one must consider the matrix

$$\mathbf{K}(\mu_i) := \lim_{z \rightarrow \mu_i} \mathbf{K}(z) = \mathbf{R} - \mu_i \mathbf{D} - \mathbf{C}^{(0)}.$$

If $\det \mathbf{K}(\mu_i) \neq 0$, then one has (2.11) is solved only with $\mathbf{a} = \mathbf{0}$, so that all solutions lie in $\text{span}\{\mathcal{S}_2^{-1/2} k : k \in \ker(\tilde{\mathcal{R}} - \mu_i \mathbb{1})\}$. If $\det \mathbf{K}(\mu_i) = 0$, then in addition to the $(\ell + 1)$ -dimensional solution set found with $\mathbf{a} = \mathbf{0}$ there will be a set of solutions with $\mathbf{a} \neq \mathbf{0}$. If in (2.11) $\mathbf{a} = \mathbf{0}$, then upon examining (2.3) this implies that for a nontrivial solution one needs $z \in \sigma(\tilde{\mathcal{R}})$. This scenario has been covered above. Thus, one can conclude that for solutions with $z \notin \sigma(\tilde{\mathcal{R}})$ it is necessarily true that $\mathbf{a} \neq \mathbf{0}$. In conclusion, if an eigenvalue exists for which (2.11) is solved with $\mathbf{a} = \mathbf{0}$, then (a) it is a real eigenvalue, (b) coincides with an eigenvalue of $\tilde{\mathcal{R}}$ and is a removable singularity for $\mathbf{K}(z)$, and (c) if positive has positive Krein signature.

For nontrivial \mathbf{a} (2.11) has a solution if and only if $\det \mathbf{K}(z) = 0$. From the above discussion one has that eigenvalues with nonzero imaginary part, and positive real eigenvalues with negative signature, will be realized as zeros of $\det \mathbf{K}(z)$. The following lemma has now been proved:

Lemma 2.2. All eigenvalues of (1.5) are realized as either (a) solutions of $\det \mathbf{K}(z) = 0$, or (b) removable singularities of $\mathbf{K}(z)$. An eigenvalue $z \in \mathbb{C}$ of (1.5) which satisfies either $\text{Im } z \neq 0$, or $z \in \mathbb{R}^+$ with negative Krein sign, will be realized as a solution of $\det \mathbf{K}(z) = 0$.

Remark 2.3. It follows immediately from the construction of $\mathbf{K}(z)$ that if $\det \mathbf{K}(z) = 0$ with $z \notin \sigma(\tilde{\mathcal{R}})$, then the geometric multiplicity of the eigenvalue is bounded above by $n(\mathcal{S})$.

Remark 2.4. If $\sigma(\tilde{\mathcal{R}}) \subset \mathbb{R}^+$, then it is clear that solving $\det \mathbf{K}(z) = 0$ will capture all of the eigenvalues associated with the count of (1.6). This assumption will be true for the applications considered in Section 2.5.

Remark 2.5. In a manner similar to that presented above one can construct a Krein matrix for the eigenvalue problem

$$(\mathcal{S} - \xi \mathcal{R})u = 0, \quad \xi = \frac{1}{z}. \quad (2.15)$$

Instead of projecting off $N(\mathcal{S})$, one instead projects off $N(\mathcal{R})$; otherwise, the construction is exactly the same. For $\xi = 1/\mu_j < 0$ with $\mathbf{a} = \mathbf{0}$ in (2.11) one will have

$$\text{sign}(\langle u, \mathcal{R}u \rangle) = -\text{sign}(\langle u, \mathcal{S}u \rangle) < 0;$$

hence, under the formulation of (2.15) the negative real eigenvalue will be captured as a zero of the determinant of the Krein matrix. Consequently, if one wishes to definitively capture all of the negative real eigenvalues via the determinant of a Krein matrix, one must simultaneously construct a Krein matrix for (1.5) and another one for (2.15) and consider the zero set associated with the determinant of each.

Remark 2.6. When considering the spectral stability of localized waves, it was shown in [31] that by constructing a matrix $\mathbf{E}(\lambda) \in \mathbb{C}^{N \times N}$ using the solutions of an N -dimensional linear ODE which have certain spatial asymptotic properties, then $\det \mathbf{E}(\lambda) = 0$ if and only if λ was an eigenvalue. It was shown in [7] that $\det \mathbf{E}(\lambda)$ is equivalent to the Evans function [3]. When considering the spectral stability of periodic waves with large period for systems which possessed m symmetries, it was shown in [30] that the critical eigenvalues were realized as the zeros of the determinant of a particular matrix $\mathbf{E}(\lambda, \gamma) \in \mathbb{C}^{m \times m}$, where $\lambda \in \mathbb{C}$ is the spectral parameter and $\gamma \in \mathbb{R}$ is the Floquet multiplier.

Solving $\det \mathbf{K}(z) = 0$ is equivalent to solving (for a particular j , and not necessarily all j)

$$r_j(z) = 0, \quad j = 1, \dots, n,$$

where $r_j(z)$ is an eigenvalue of $\mathbf{K}(z)$ and will henceforth be referred to as a Krein eigenvalue. Regarding the Krein eigenvalues, the following can be deduced from [25] upon using the fact that $\mathbf{K}(z)$ is meromorphic and symmetric for $z \in \mathbb{R}$:

Proposition 2.7. Suppose that $z \notin \sigma(\tilde{\mathcal{R}})$. Then:

- (a) the Krein eigenvalues $r_j(z)$ are real analytic, as are the associated eigenvectors $\mathbf{v}_j(z)$
- (b) for $\text{Im } z \neq 0$ the Krein eigenvalues are analytic as long as they are distinct; furthermore, any branch points correspond to algebraic singularities.

2.2. Order of zeros and the algebraic multiplicity

The immediate goal is to relate the order of a zero of a Krein eigenvalue with the size of the associated Jordan chain. Since the Krein eigenvalues and associated eigenvectors are real analytic

by [Proposition 2.7](#), for a given $z_0 \notin \sigma(\tilde{\mathcal{R}})$ there is an appropriate Taylor series. The coefficients can be found in the following manner. Suppose that $r_j(z_0) = 0$, and assume that $|\mathbf{v}_j(z_0)| = 1$. Upon differentiating

$$\mathbf{K}(z)\mathbf{v}_j(z) = r_j(z)\mathbf{v}_j(z)$$

with respect to z and evaluating at z_0 one sees that

$$\mathbf{K}(z_0)\mathbf{v}'_j(z_0) = r'_j(z_0)\mathbf{v}_j(z_0) - \mathbf{K}'(z_0)\mathbf{v}_j(z_0). \quad (2.16)$$

(2.16) has the solvability condition

$$r'_j(z_0) = \mathbf{v}_j(z_0)^{\mathbf{H}}\mathbf{K}'(z_0)\mathbf{v}_j(z_0), \quad (2.17)$$

which fixes $r'_j(z_0)$, and $\mathbf{v}'_j(z_0)$ is then uniquely determined by solving (2.16) under the requirement that $\mathbf{v}'_j(z_0) \in [\ker(\mathbf{K}(z_0) - r_j(z_0)\mathbb{1})]^\perp$. If one now assumes that $r'_j(z_0) = 0$, then upon differentiating a second time and evaluating at z_0 one sees that

$$\mathbf{K}(z_0)\mathbf{v}''_j(z_0) = r''_j(z_0)\mathbf{v}_j(z_0) - \mathbf{K}''(z_0)\mathbf{v}_j(z_0) - 2\mathbf{K}'(z_0)\mathbf{v}'_j(z_0). \quad (2.18)$$

The solvability condition for (2.18) is

$$r''_j(z_0) = \mathbf{v}_j(z_0)^{\mathbf{H}}\mathbf{K}''(z_0)\mathbf{v}_j(z_0) + 2\mathbf{v}_j(z_0)^{\mathbf{H}}\mathbf{K}'(z_0)\mathbf{v}'_j(z_0). \quad (2.19)$$

Of course, one can repeat this process in order to determine the coefficients at any order.

Suppose that for (1.5) there is the finite Jordan chain

$$(\mathcal{R} - z_0\mathcal{S})\psi_j = \mathcal{S}\psi_{j-1} \quad (\psi_0 = 0), \quad j = 1, \dots, k. \quad (2.20)$$

Solvability and the finiteness of the chain implies that

$$\langle \psi_k, \mathcal{S}\psi_1 \rangle \neq 0, \quad \langle \psi_j, \mathcal{S}\psi_1 \rangle = 0, \quad j = 0, \dots, k-1. \quad (2.21)$$

In order to relate (2.21) to the derivatives of the Krein eigenvalues, use the decomposition $\psi_1 = \sum \alpha_j \phi_j + p$ and the expression for p given in (2.6) to write

$$\psi_1 = \sum_{j=1}^n \alpha_j^1 \phi_j - \sum_{j=1}^n \alpha_j^1 \mathcal{S}_2^{-1/2} (\tilde{\mathcal{R}} - z_0\mathbb{1})^{-1} \mathcal{S}_2^{-1/2} P\mathcal{R}\phi_j, \quad (2.22)$$

where $\mathbf{a}_1 = (\alpha_1^1, \dots, \alpha_n^1)^{\mathbf{T}}$ satisfies $|\mathbf{a}_1| = 1$. Using (2.22) and simplifying yields

$$\langle \psi_1, \mathcal{S}\psi_1 \rangle = \mathbf{a}_1^{\mathbf{H}}(\mathbf{D} + \mathbf{C}'(z_0))\mathbf{a}_1 = -\mathbf{a}_1^{\mathbf{H}}\mathbf{K}'(z_0)\mathbf{a}_1. \quad (2.23)$$

Since $\mathbf{v}_j(z_0) = \mathbf{a}_1$, one sees from (2.17) that

$$r'_j(z_0) = -\langle \psi_1, \mathcal{S}\psi_1 \rangle; \quad (2.24)$$

in other words, if $z_0 \in \mathbb{R}^+$ the sign of the derivative of the Krein eigenvalue at a zero is the negative of the Krein signature.

Now suppose that $k \geq 2$ in (2.20), which by (2.21) and (2.24) implies that $r'_j(z_0) = 0$. Writing $\psi_2 = \sum \alpha_j^2 \phi_j + p_2$ in (2.20) and projecting with P yields

$$(\mathcal{R}_2 - z_0\mathcal{S}_2)p_2 + \sum_{j=1}^n \alpha_j^2 P\mathcal{R}\phi_j = P\mathcal{S}\psi_1.$$

Substituting the expression for ψ_1 given in (2.22) into the above, solving for p_2 using $(\mathcal{R}_2 - z_0 \mathcal{S}_2)^{-1} = \mathcal{S}_2^{-1/2}(\tilde{\mathcal{R}} - z_0 \mathbb{1})^{-1} \mathcal{S}_2^{-1/2}$, and simplifying yields

$$\psi_2 = \sum_{j=1}^n \alpha_j^2 \phi_j - \sum_{j=1}^n \alpha_j^2 \mathcal{S}_2^{-1/2} (\tilde{\mathcal{R}} - z_0 \mathbb{1})^{-1} \mathcal{S}_2^{-1/2} P \mathcal{R} \phi_j - \sum_{j=1}^n \alpha_j^1 \mathcal{S}_2^{-1/2} (\tilde{\mathcal{R}} - z_0 \mathbb{1})^{-2} \mathcal{S}_2^{-1/2} P \mathcal{R} \phi_j.$$

In order to solve for the α_j^2 one uses the projection Q in (2.20) (see (2.4)) and eventually finds that these coefficients satisfy

$$\mathbf{K}(z_0) \mathbf{a}_2 = (\mathbf{C}'(z_0) + \mathbf{D}) \mathbf{a}_1 = -\mathbf{K}'(z_0) \mathbf{a}_1, \quad \mathbf{a}_2 = (\alpha_1^2, \dots, \alpha_n^2)^\top. \quad (2.25)$$

(2.25) can be solved if and only if $\mathbf{a}_1^\top \mathbf{K}'(z_0) \mathbf{a}_1 = 0$, which, by (2.17) with $\mathbf{v}(z_0) = \mathbf{a}_1$ yields $r_j'(z_0) = 0$; furthermore, the solution is $\mathbf{v}_j'(z_0) = \mathbf{a}_2$. Continuing in this fashion one finds that for $\psi_3 = \sum \alpha_j^3 \phi_j + p$ the linear system to be solved is

$$\mathbf{K}(z_0) \mathbf{a}_3 = -\frac{1}{2} (\mathbf{K}''(z_0) \mathbf{a}_1 + 2\mathbf{K}'(z_0) \mathbf{a}_2), \quad \mathbf{a}_3 = (\alpha_1^3, \dots, \alpha_n^3)^\top. \quad (2.26)$$

An examination of (2.19) reveals that (2.26) can be solved if and only if $r_j''(z_0) = 0$; furthermore, $\mathbf{v}_j''(z_0) = 2\mathbf{a}_3$. Thus, $k \geq 3$ if and only if $r_j(z_0) = r_j'(z_0) = r_j''(z_0) = 0$. An induction argument eventually leads to the conclusion that

$$r_j(z_0) = r_j'(z_0) = \dots = r_j^{(k-1)}(z_0) = 0, \quad r_j^{(k)}(z_0) \neq 0. \quad (2.27)$$

The next goal is to determine an upper bound on the algebraic multiplicity of the zero. Suppose that (2.20) is satisfied for $z_0 \in \mathbb{R} \setminus \sigma(\mathcal{R})$. A small perturbation of the form $\mathcal{S} \mapsto \mathcal{S} + \epsilon \mathcal{S}_\epsilon$ will generically break this eigenvalue of algebraic multiplicity k into k simple eigenvalues (e.g., see [32, Section 3] for a concrete example). Upon using (2.27) and the fact that the coefficients of the Taylor expansion for $r_j(z)$ for real z will vary smoothly as a function of ϵ one has that to leading order the equation to solve to locate these eigenvalues is

$$\frac{r_j^{(k)}(z_0)}{k!} (z - z_0)^k + \partial_\epsilon r_j(z_0) \epsilon = 0. \quad (2.28)$$

Solving (2.28) yields

$$(z - z_0)^k = \zeta \epsilon, \quad \zeta := -\frac{k! \partial_\epsilon r_j(z_0)}{r_j^{(k)}(z_0)} \in \mathbb{R} \setminus \{0\}, \quad (2.29)$$

so that the k eigenvalues satisfy $z = z_0 + \mathcal{O}(\epsilon^{1/k})$.

Without loss of generality suppose that $\zeta \in \mathbb{R}^+$. If $z_0 \in \mathbb{R}^-$, then one sees from (2.30) that

$$z = z_j = z_0 + \zeta^{1/k} e^{i2\pi j/k} \epsilon^{1/k}, \quad j = 0, \dots, k-1,$$

from which it can be concluded that $k_r + 2k_c = k$. Upon using (1.6) one gets the upper bound for the algebraic multiplicity is $k \leq n(\mathcal{R}) + n(\mathcal{S})$. Suppose that $z_0 \in \mathbb{R}^+$. If $k = 2\ell$ is even, then upon using (2.24) one has for (2.28) that

$$k_c = \begin{cases} \ell, & k_1^- = 0 \\ \ell - 1, & k_1^- = 1; \end{cases}$$

in other words, $k_c + k_i^- = k$. If $k = 2\ell + 1$ is odd, then $k_c = \ell$, and

$$k_i^- = \begin{cases} 0, & r_j^{(k)}(z_0) > 0 \\ 1, & r_j^{(k)}(z_0) < 0, \end{cases}$$

which implies that

$$k_c + k_i^- = \begin{cases} \ell, & r_j^{(k)}(z_0) > 0 \\ \ell + 1, & r_j^{(k)}(z_0) < 0. \end{cases}$$

Upon using (1.11) one gets the upper bound on the algebraic multiplicity for $\ell \in \mathbb{N}$:

$$\begin{aligned} k = 2\ell &\implies \ell \leq \min\{n(\mathcal{R}), n(\mathcal{S})\} \\ k = 2\ell + 1 &\implies \ell \leq \begin{cases} \min\{n(\mathcal{R}), n(\mathcal{S})\}, & r_j^{(k)}(z_0) > 0 \\ \min\{n(\mathcal{R}), n(\mathcal{S})\} - 1, & r_j^{(k)}(z_0) < 0. \end{cases} \end{aligned} \quad (2.30)$$

Finally, it is of interest to determine $k_i^-(z_0)$ for $k \geq 2$. If $z_0 \in \mathbb{R}^+$ the following can be shown using the argument leading to [13, Corollary 2.26]. Let $E_{z_0}^\epsilon$ correspond to the subspace formed by projecting onto the eigenspaces associated with the k eigenvalues satisfying $|z - z_0| = \mathcal{O}(|\epsilon|^{1/k})$. One has for $|\epsilon| > 0$ that

$$n(\mathcal{S}|_{E_{z_0}^\epsilon}) = k_c + k_i^-.$$

Since the negative index is robust, it is then necessarily true that when $\epsilon = 0$ with $k \in \{2\ell, 2\ell + 1\}$,

$$k_i^-(z_0) = n(\mathcal{S}|_{E_{z_0}^0}) = \begin{cases} \ell, & k = 2\ell \\ \ell, & k = 2\ell + 1 \text{ with } r_j^{(k)}(z_0) > 0 \\ \ell + 1, & k = 2\ell + 1 \text{ with } r_j^{(k)}(z_0) < 0. \end{cases} \quad (2.31)$$

The interested reader should note that [32] contains the first statement of (2.31); however, [32] does not relate $k_i^-(z_0)$ to the sign of $r_j^{(k)}(z_0)$ when k is odd.

Since

$$\det \mathbf{K}(z) = \prod_{i=1}^n r_i(z),$$

the following result has now been proven.

Lemma 2.8. *For $z \in \mathbb{R} \setminus \sigma(\tilde{\mathcal{R}})$ a Krein eigenvalue is zero only if z is an eigenvalue. The order of the zero is equal to the size of the Jordan block. The algebraic multiplicity of an eigenvalue is the order of the zero of the determinant of the Krein matrix. If $z \in \mathbb{R}^+$ and the zero of the Krein eigenvalue is simple, then the sign of the eigenvalue is the opposite the sign of the derivative of the Krein eigenvalue evaluated at the zero.*

Remark 2.9. It is clear that the proof, and consequently the results, of Lemma 2.8 relating the order of the zero of the Krein eigenvalue to the size of the Jordan block, and the order of the zero of the determinant of the Krein matrix to the algebraic multiplicity of the eigenvalue can be extended to $\text{Im } z \neq 0$ as long as the Krein eigenvalues remain distinct.

Remark 2.10. If $z \in \sigma(\tilde{\mathcal{R}})$ is an eigenvalue such that $\det \mathbf{K}(z) = 0$, i.e., it corresponds to a removable singularity of the Krein matrix, then the analysis relating the order of the zero of the Krein eigenvalue to the size of the Jordan chain is still valid, as is (2.31). Since there will be eigenvalues which will not be captured by the Krein matrix, however, the relationship between the order of the zero of $\det \mathbf{K}(z)$ and the algebraic multiplicity of the eigenvalue is no longer valid.

2.3. The Krein Oscillation Theorem

A significant consequence of [Lemma 2.8](#) is the Krein Oscillation Theorem ([Theorem 2.12](#)). Before it can be stated, however, some graphical properties of the Krein eigenvalues must be derived. It is helpful to first consider the case of $n = 1$, which in particular implies that $k_1^- \leq 1$ (see [\[27\]](#) for this case when the compactness assumption is removed). Upon using the formulation of [\(2.12\)](#) one has that

$$r_1(z) = \langle \phi_1, \mathcal{R}\phi_1 \rangle - \lambda_1 z - \sum_{i=1}^{\infty} \frac{|\langle \mathcal{S}_2^{-1/2} P\mathcal{R}\phi_1, v_i \rangle|^2}{\mu_i - z}. \quad (2.32)$$

Differentiation yields

$$r_1'(z) = -\lambda_1 - \sum_{i=1}^{\infty} \frac{|\langle \mathcal{S}_2^{-1/2} P\mathcal{R}\phi_1, v_i \rangle|^2}{(\mu_i - z)^2}, \quad r_1''(z) = -2 \sum_{i=1}^{\infty} \frac{|\langle \mathcal{S}_2^{-1/2} P\mathcal{R}\phi_1, v_i \rangle|^2}{(\mu_i - z)^3} \quad (2.33)$$

For $z < \mu_1$ one has that

$$r_1''(z) < 0, \quad \lim_{z \rightarrow -\infty} \frac{r_1(z)}{z} = -\lambda_1 > 0, \quad \lim_{z \rightarrow \mu_1^-} r_1(z) = -\infty. \quad (2.34)$$

One can then conclude that $r_1(z) = 0$ has at most two real-valued solutions in $(-\infty, \mu_1)$. If there is only one solution, say $z_1 \in (-\infty, \mu_1)$, then as a consequence of [\(2.34\)](#) it must be true that $r_1'(z_1) = 0$; furthermore, from [\(2.30\)](#) and [\(2.31\)](#) one can conclude that the eigenvalue has a Jordan block of size two with $k_1^-(z_0) = 1$. For $i \in \mathbb{N}$ one has that

$$\lim_{z \rightarrow \mu_i^\pm} r_1(z) = \pm\infty; \quad (2.35)$$

hence, in each interval (μ_i, μ_{i+1}) there is an odd number (counting multiplicity) of real-valued zeros of $r_1(z)$.

Remark 2.11. The structural form of $r_1(z)$ for $z \in (-\infty, \mu_1)$ is exactly that detailed in [\[27\]](#). The difference is that here μ_1 corresponds to an eigenvalue of $\tilde{\mathcal{R}}$, whereas in [\[27\]](#) μ_1 is the edge of the continuous spectrum. The presence of the continuous spectrum precluded any additional analysis in [\[27\]](#) of $r_1(z)$ for real-valued $z > \mu_1$.

A cartoon of the situation is depicted in the left panel of [Figure 2](#). Note by [Lemma 2.8](#) that the middle zero of $r_1(z)$ in the interval (μ_1, μ_2) corresponds to the eigenvalue with negative Krein signature, and that the other two zeros correspond to eigenvalues with positive signature. From [\(1.11\)](#) one has that $k_1^- \leq 1$; hence, all other zeros of $r_1(z)$ will be simple with negative derivative.

Now suppose that $n \geq 2$. One has

$$\lim_{z \rightarrow -\infty} \frac{\mathbf{K}(z)}{z} = -\mathbf{D};$$

consequently,

$$\lim_{z \rightarrow -\infty} \frac{r_j(z)}{z} = -\lambda_j > 0, \quad j = 1, \dots, n. \quad (2.36)$$

In considering the behavior near the simple poles $\mu_j \in \sigma(\tilde{\mathcal{R}})$ one has that

$$\lim_{z \rightarrow \mu_j} (z - \mu_j) \mathbf{K}(z) = \lim_{z \rightarrow \mu_j} (\mu_j - z) \mathbf{C}(z) = -\operatorname{Res}_{z=\mu_j} \mathbf{C}(z).$$

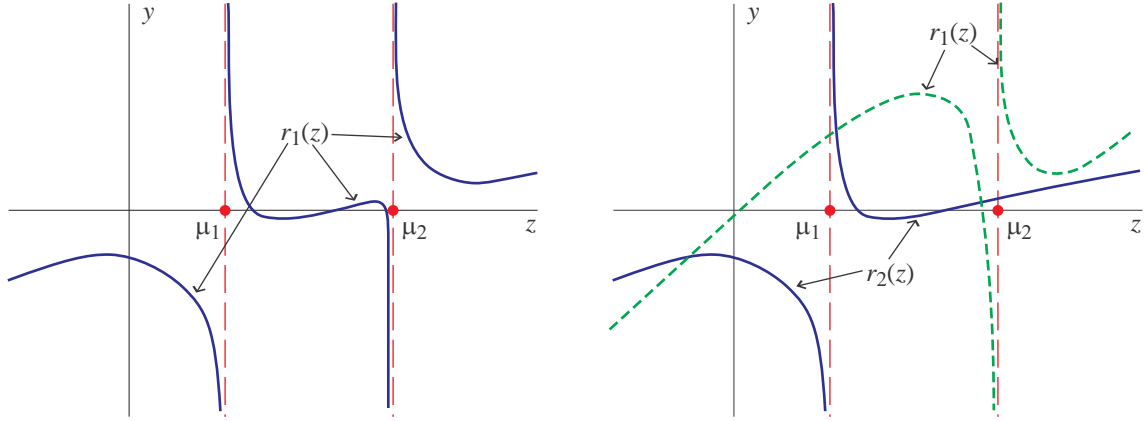


Figure 2: (color online) The Krein eigenvalue $r_1(z)$ for $n = 1$ (left panel), and the Krein eigenvalues $r_1(z)$ and $r_2(z)$ for $n = 2$ (right panel). Note that in the left panel $k_1^- = 1$, whereas in the right panel $k_1^- = 2$.

If μ_j is algebraically simple, then by using (2.14) one sees that

$$\operatorname{Res}_{z=\mu_j} \mathbf{C}(z) = -\mathbf{c}^j (\mathbf{c}^j)^H.$$

Since $\mathbf{c}^j (\mathbf{c}^j)^H$ is a rank-one matrix with $\ker(\mathbf{c}^j (\mathbf{c}^j)^H) = \operatorname{span}\{\mathbf{c}^j\}^\perp$, there will be a $1 \leq \ell_j \leq n$ for which one will see the asymptotic behavior

$$|r_{\ell_j}(z)| = \mathcal{O}(|\mu_j - z|^{-1}); \quad |r_j(z)| = \mathcal{O}(1), \quad j \neq \ell_j.$$

In other words, even though $\mathbf{K}(z)$ has a simple pole, for all of the Krein eigenvalues but one the pole is a removable singularity. Since

$$\operatorname{trace} \mathbf{c}^j (\mathbf{c}^j)^H = \sum_i |c_i^j|^2 > 0,$$

one has

$$\operatorname{Res}_{z=\mu_j} r_{\ell_j}(z) = \sum_i |c_i^j|^2 > 0,$$

so that

$$\lim_{z \rightarrow \mu_j^\pm} r_{\ell_j}(z) = \pm\infty \tag{2.37}$$

(compare to (2.35)).

Now suppose that $\mu_{j+1} = \mu_{j+2} = \dots = \mu_{j+\ell}$ for some $\ell \in \mathbb{N}_0$. One then has that

$$\operatorname{Res}_{z=\mu_j} \mathbf{C}(z) = -\left(\mathbf{c}^j (\mathbf{c}^j)^H + \dots + \mathbf{c}^{j+\ell} (\mathbf{c}^{j+\ell})^H \right).$$

Since each $\mathbf{c}^{j+i} (\mathbf{c}^{j+i})^H$ is a rank-one matrix, for each $1 \leq i \leq \ell$, $\ker(\mathbf{c}^{j+i} (\mathbf{c}^{j+i})^H) = \operatorname{span}\{\mathbf{c}^{j+i}\}^\perp$. Thus,

$$\ker(\operatorname{Res}_{z=\mu_j} \mathbf{C}(z)) \subset \operatorname{span}\{\mathbf{c}^j, \dots, \mathbf{c}^{j+\ell}\}^\perp \implies \dim[\ker(\operatorname{Res}_{z=\mu_j} \mathbf{C}(z))] \geq n - \ell.$$

It will then be the case that at least $n - \ell$ of the eigenvalues of $\mathbf{K}(z)$ will have a removable singularity at the pole. This case of multiple eigenvalues is degenerate and can be removed via the application

of a finite rank perturbation to $\widetilde{\mathcal{R}}$; hence, it will be assumed that all of the eigenvalues are simple. A cartoon of the situation is depicted in the right panel of [Figure 2](#) when $n = 2$.

The Krein Oscillation Theorem requires the following assumptions and definitions. Assume that $\sigma(\widetilde{\mathcal{R}}) = \{\mu_i\}_{i=1}^{\infty}$ is such that all of the eigenvalues are algebraically simple. Let the sequence $\{\mu_i^j\}_{i=0}^{\infty} \subset \sigma(\mathcal{R})$, $\mu_0^j = -\infty$, denote the simple poles of $r_j(z)$. For $i = 0, \dots, \infty$ and $j = 1, \dots, n$ define the open intervals $I_i^j = (\mu_i^j, \mu_{i+1}^j)$. Set

- $k_r(I_i^j)$ to be the number of negative real eigenvalues detected by $r_j(z)$ in I_i^j
- $k_i^-(I_i^j)$ to be the number of positive real eigenvalues with negative Krein signature detected by $r_j(z)$ in I_i^j
- $\#r_j(I_i^j)$ to be the number of zeros (counting multiplicity) of $r_j(z)$ in I_i^j .

Theorem 2.12 (Krein Oscillation Theorem). *Suppose that not all of the singularities of the Krein matrix are removable. For each $j = 1, \dots, n$, $\#r_j(I_0^j)$ is even, and if $n = 1$, then $\#r_1(I_0^1) \in \{0, 2\}$. If $\mu_{i+1}^j < +\infty$ for $i \in \mathbb{N}$, then $\#r_j(I_i^j)$ is odd; furthermore,*

$$\begin{aligned} I_i^j \subset \mathbb{R}^- &\implies k_r(I_i^j) = \#r_j(I_i^j) \\ I_i^j \subset \mathbb{R}^+ &\implies k_i^-(I_i^j) = \frac{\#r_j(I_i^j) - 1}{2}. \end{aligned}$$

Finally, if $n = 1$ and $\{0\} \subset I_0^1$ with $\#r_1(I_0^1) = 2$, then either $k_r(I_0^1) = 2$ or $k_r(I_0^1) + k_i^-(I_0^1) = 1$.

Proof: The statement regarding the number of zeros in a given interval is a consequence of the asymptotics given in [\(2.34\)](#) and [\(2.35\)](#) ($n = 1$), and [\(2.36\)](#) and [\(2.37\)](#) $n \geq 2$. The statement that each real negative zero corresponds to an eigenvalue follows immediately from [Lemma 2.8](#). The statement regarding the number of eigenvalues of negative Krein sign follows from both the fact that $r'_j(z_0) > 0$ if and only if $z_0 \in \mathbb{R}^+$ is a simple eigenvalue with negative signature, as well as [\(2.31\)](#) in the event of a nontrivial Jordan block. The final statement follows from the fact that if $r_1(z_j) = 0$ for $j = 1, 2$ with $z_1 < 0 < z_2$, then from [\(2.33\)](#) it is necessarily true that $r'_1(z_2) < 0$, whereas if $0 < z_1 < z_2$, then $r'_1(z_1) > 0$ and $r'_1(z_2) < 0$. \square

Remark 2.13. If $I_i^j \subset \mathbb{R}^+$ and all of the zeros of $r_j(z)$ are simple on this interval, then for the eigenvalues detected by $r_j(z)$ one has the ordering

$$\mu_i^j < p < n < p < \dots < n < p < \mu_{i+1}^j.$$

Here p represents an eigenvalue with positive Krein signature, and n is an eigenvalue with negative Krein signature.

Remark 2.14. Suppose that all of the singularities of the Krein matrix are removable. This implies that $\mathbf{C}(z) \equiv 0$, which in turn implies that $\mathbf{K}(z) = \mathbf{R} - z\mathbf{D}$. Since \mathbf{D} is negative definite, $\det \mathbf{K}(z) = 0$ will have $n(\mathbf{R})$ positive real solutions and $n(\mathcal{S}) - n(\mathbf{R})$ negative real solutions. Letting $z_0 \in \mathbb{R}^+$ be a solution, since \mathbf{D} is negative definite one has that

$$-\mathbf{a}^H \mathbf{K}'(z_0) \mathbf{a} = -\mathbf{a}^H \mathbf{D} \mathbf{a} > 0, \quad \mathbf{a} \in \mathbb{C}^n.$$

Upon using [\(2.17\)](#) and [\(2.24\)](#) this then implies z_0 is an eigenvalue with negative Krein signature. Upon using [Lemma 2.8](#) and the count of [\(1.6\)](#) one can conclude that

$$k_i^- = n(\mathbf{R}), \quad k_r = |n(\mathcal{R}) - n(\mathcal{S})| + 2[n(\mathcal{R}) - n(\mathbf{R})].$$

Note that the Krein matrix detects $n(\mathcal{S}) - n(\mathbf{R})$ of the negative real-valued eigenvalues; hence, $n(\mathcal{R}) - n(\mathbf{R})$ of the negative real-valued eigenvalues correspond to removable singularities. It is also interesting to note that

$$n(\mathbf{R}) = n(\mathcal{R}|_{N(\mathcal{S})}),$$

which implies in particular that if $n(\mathcal{R}) = n(\mathcal{S})$, then

$$k_i^- = n(\mathcal{R}|_{N(\mathcal{S})}), \quad k_r = 2[n(\mathcal{R}) - n(\mathcal{R}|_{N(\mathcal{S})})].$$

Remark 2.15. Suppose that all but a finite number of the singularities of the Krein matrix are removable, which implies that each Krein eigenvalue is a rational function. Then $r_j(z) \sim -\lambda_j z > 0$ as $z \rightarrow +\infty$, which implies that there is an $R_0 \in \mathbb{R}^+$ such that for $z > R_0$ all eigenvalues correspond to removable singularities of the Krein matrix.

2.4. Asymptotic results

It is possible to make a general statement regarding the location of the eigenvalues in the limit $z \rightarrow +\infty$. Since $n(\mathcal{R}) + n(\mathcal{S})$ is finite, there is an $R_0 \in \mathbb{R}^+$ such that the only eigenvalues for $|z| > R_0$ are positive and real with positive Krein signature. As a consequence of the Krein Oscillation Theorem this then implies that for each j there is an i_j^+ such that $\#r_j(I_i^j) = 1$ for $i \geq i_j^+$. For each j ,

$$\mathcal{S}_2^{-1/2} P\mathcal{R}\phi_j = \sum_{i=1}^{\infty} \langle \mathcal{S}_2^{-1/2} P\mathcal{R}\phi_j, v_i \rangle v_i, \quad \langle \mathcal{S}_2^{-1/2} P\mathcal{R}\phi_j, \mathcal{S}_2^{-1/2} P\mathcal{R}\phi_j \rangle = \sum_{i=1}^{\infty} |\langle \mathcal{S}_2^{-1/2} P\mathcal{R}\phi_j, v_i \rangle|^2.$$

The finiteness of the sums imply that

$$\lim_{i \rightarrow \infty} |\langle \mathcal{S}_2^{-1/2} P\mathcal{R}\phi_j, v_i \rangle| = 0;$$

hence, an examination of (2.13) reveals that

$$\lim_{i \rightarrow \infty} \mathbf{c}^i (\mathbf{c}^i)^H = \mathbf{0}.$$

Since the residues of $\mathbf{K}(z)$ can be made arbitrarily small, as z increases the effect of the pole will be seen only in smaller and smaller neighborhoods of $\sigma(\tilde{\mathcal{R}})$. Consequently, the zero of $r_j(z)$ which is effected by the pole at $z = \mu_\ell^j$ will be in a small neighborhood of μ_ℓ^j . In conclusion, one then has that for large z the spectrum of (1.5) is well-approximated by $\sigma(\tilde{\mathcal{R}})$, and that these large eigenvalues will all have positive Krein signature.

2.5. Perturbative results

In examples for which the nonlinearity is weak one has the situation that

$$\mathcal{R} = \mathcal{A}_0 + \epsilon \mathcal{R}_1, \quad \mathcal{S}^{-1} = \mathcal{A}_0 + \epsilon \mathcal{S}_1, \quad 0 < \epsilon \ll 1 \tag{2.38}$$

where \mathcal{A}_0 is a self-adjoint operator with a compact resolvent and $n(\mathcal{A}_0) = n$, and $\mathcal{R}_1, \mathcal{S}_1$ are \mathcal{A}_0 -compact. When $\epsilon = 0$ the spectral problem is well-understood. Letting λ_j , $j = 1, 2, \dots$, represent the eigenvalues of \mathcal{A}_0^{-1} , one has that when $\epsilon = 0$,

$$N(\mathcal{R}) = N(\mathcal{S}), \quad \sigma(\mathcal{S}^{-1}\mathcal{R}) = \{\lambda_j^{-2}\}_{j=1}^{\infty}, \quad \sigma(\tilde{\mathcal{R}}) = \{\lambda_j^{-2}\}_{j=n+1}^{\infty}.$$

There are eigenvalues of negative sign at $\lambda_1^{-2} \leq \dots \leq \lambda_n^{-2}$, and all of the others have positive sign. Since $\mathbf{c}^i(\mathbf{c}^i)^H = \mathbf{0}$ for all $i \geq n+1$, $\mathbf{K}(z)$ has removable singularities for all $z \in \sigma(\tilde{\mathcal{R}})$; consequently,

$$\mathbf{K}(z) = \text{diag}(1/\lambda_1 - \lambda_1 z, \dots, 1/\lambda_n - \lambda_n z). \quad (2.39)$$

Remark 2.16. It can certainly be the case that for $\epsilon > 0$ one has either or both of $n(\mathcal{R}), n(\mathcal{S}) > n$ (e.g., see [16, 19]). The increase in the number of negative eigenvalues is related to nonzero eigenvalues of $\mathcal{O}(\epsilon^{1/2})$ being created by a symmetry-breaking bifurcation. In this case the analysis that follows will still be valid if one first projects onto the subspace associated with the $\mathcal{O}(1)$ eigenvalues of \mathcal{S} . In order to simplify the subsequent analysis, it will henceforth be assumed that $n(\mathcal{R}) = n(\mathcal{S}) = n$ for all $\epsilon > 0$ sufficiently small.

(2.39) implies that for $j = 1, \dots, n$, $r_j(z) = 0$ if and only if $z = z_j := \lambda_j^{-2}$. Furthermore, $r'_j(z_j) > 0$, so by Lemma 2.8 all of these eigenvalues have negative Krein sign. All of the eigenvalues associated with (1.5) which have positive sign are not captured by $\det \mathbf{K}(z) = 0$ because each of these is associated with a removable singularity. Since $\sigma(\tilde{\mathcal{R}}) \subset \mathbb{R}^+$ when $\epsilon = 0$, the same will be true for small ϵ ; hence, following Remark 2.4 one knows that for $\epsilon > 0$ the zeros of the determinant of the Krein matrix will capture all of the eigenvalues associated with the count of (1.6).

The goal is to understand what happens for $\epsilon > 0$. Since $C(\mathcal{R}) = C(\mathcal{S})$ when $\epsilon = 0$, by the robustness of the cone intersection one has by (1.8) that $k_c + k_1^- = n$ for $\epsilon > 0$ sufficiently small. Thus, the question is the manner in which the zeros of the Krein eigenvalues capture this result. If $z_i \notin \sigma(\tilde{\mathcal{R}})$ when $\epsilon = 0$, so that the zero is not located on a removable singularity, then by the Implicit Function Theorem one can conclude that the zero persists smoothly with $r'_i(z_i) > 0$. Now suppose that $z_i = \mu \in \sigma(\tilde{\mathcal{R}})$ for some $1 \leq i \leq n$ when $\epsilon = 0$. First suppose that $\mu \in \sigma(\tilde{\mathcal{R}})$ is geometrically and algebraically simple. For $\epsilon > 0$ one will generically have $|\mathbf{c}^i(\mathbf{c}^i)^H| = \mathcal{O}(\epsilon)$; in particular, this implies that the effect of the pole on the graph of $r_i(z)$ will be felt only within an $\mathcal{O}(\epsilon)$ neighborhood of z_i . Outside of a small neighborhood of the pole regular perturbation theory yields

$$r_i(z) \sim -(\lambda_i + \mathcal{O}(\epsilon))z + (1/\lambda_i + \mathcal{O}(\epsilon)),$$

whereas in the neighborhood of λ_i one will have the expansion

$$r_i(z) \sim -(\lambda_i + \mathcal{O}(\epsilon))z + (1/\lambda_i + \mathcal{O}(\epsilon)) + \frac{\mathcal{O}(\epsilon)}{1/\lambda_i^2 + \mathcal{O}(\epsilon) - z}. \quad (2.40)$$

Consequently, $r_i(z)$ will be nonzero outside a small neighborhood of μ , whereas in a neighborhood of μ , $r_i(z) = 0$ will have precisely two solutions (counting multiplicity). If the zeros are simple, then they will be either be real-valued and have derivatives of opposite sign, or they will be complex-conjugates with nonzero imaginary parts.

Now suppose that μ is no longer geometrically simple when $\epsilon = 0$. For $\epsilon > 0$ one can generically expect that the eigenvalues of $\tilde{\mathcal{R}}$ can be ordered as $\mu_0(\epsilon) < \dots < \mu_\ell(\epsilon)$, and that each eigenvalue is geometrically and algebraically simple. The analogous expansion to (2.40) will read

$$r_i(z) \sim -(\lambda_i + \mathcal{O}(\epsilon))z + (1/\lambda_i + \mathcal{O}(\epsilon)) + \sum_{j=0}^{\ell} \frac{\mathcal{O}_j(\epsilon)}{\mu_j(\epsilon) - z}. \quad (2.41)$$

If all of the residues are nonzero, then there will be $\ell + 2$ solutions to $r_i(z) = 0$ within an $\mathcal{O}(\epsilon)$ neighborhood of z_i . If $1 \leq \ell' \leq \ell + 1$ of the residues are zero, then the number of solutions reduces to $(\ell - \ell') + 2$; furthermore, since the total number of eigenvalues within the $\mathcal{O}(\epsilon)$ must be invariant under small perturbation, there will be ℓ' eigenvalues which are contained in $\sigma(\tilde{\mathcal{R}})$.

3. APPLICATION: GROSS-PITAEVSKI EQUATION

The previous results and perspective of [Section 2.5](#) will be applied to solutions of a system which is currently undergoing a great deal of mathematical and experimental study. The governing equations for an N -component Bose-Einstein condensate are given by

$$i\partial_t U_j + \Delta U_j + \omega_j U_j + \sum_{k=1}^N a_{jk} |U_k|^2 U_j = V(\mathbf{x}) U_j, \quad j = 1, \dots, N, \quad (3.1)$$

where $U_j \in \mathbb{C}$ is the mean-field wave-function of species j , $\mathbf{A} = (a_{jk}) \in \mathbb{R}^{N \times N}$ is symmetric, Δ represents the Laplacian, $\omega_j \in \mathbb{R}$ are free parameters which represent the chemical potential, and $V : \mathbb{R}^n \mapsto \mathbb{R}$ represents the trapping potential (see [[2](#), [4-6](#), [8](#), [17](#), [22](#), [24](#), [26](#), [28](#)] and the references therein for further details). If $a_{jk} \in \mathbb{R}^+$ for all j, k , then the intra-species and inter-species interactions are attractive, whereas if $a_{jk} \in \mathbb{R}^-$ for all j, k the intra-species and inter-species interactions are repulsive.

Suppose that $N = 1$, and for $0 < \delta \ll 1$ consider a potential of the form

$$V(\mathbf{x}) = |\mathbf{y}|^2 + \frac{1}{\delta^4} |\mathbf{z}|^2, \quad (\mathbf{y}, \mathbf{z}) \in \mathbb{R}^{d_y \times d_z}.$$

It was shown in [[1](#)] that for repulsive interactions, i.e., $a_{11} < 0$, then if $n = d_y + d_z < 4$ solutions to (3.1) for $\mathbf{x} \in \mathbb{R}^n$ are well-approximated over long time intervals by solutions of the form

$$U(t, \mathbf{x}) \sim \pi^{-d_z/4} e^{-id_z t/\delta^2} u(t, \mathbf{y}) e^{-|\mathbf{z}|^2/2},$$

where $u(t, \mathbf{y})$ satisfies (3.1) in the \mathbf{y} -variables only, i.e.,

$$iu_t + \Delta u + \omega u + \tilde{a}_{11} |u|^2 u = |\mathbf{y}|^2 u, \quad \tilde{a}_{11} := \frac{a_{11}}{\pi^{d_z}} \int_{\mathbb{R}^{d_z}} e^{-2|\mathbf{z}|^2} d\mathbf{z}. \quad (3.2)$$

Thus, the new governing equation is simply the old one with the number of spatial variables being reduced. The case of $d_y = 1$ is referred to as the cigar trap, and the case $d_y = 2$ is call the pancake trap. In the case of attractive interactions the reduction is rigorous only if $d_y = 1$; however, in practice it is still assumed that the $d_y = 2$ reduction is valid. In all of the examples that follow it will be assumed that $N = 1$ with $a_{11} < 0$.

3.1. Cigar trap

In the case of $d_y = 1$ (3.2) can be rewritten after some rescalings as

$$iu_t + u_{xx} + \omega u - \epsilon |u|^2 u = x^2 u, \quad 0 < \epsilon. \quad (3.3)$$

For $0 < \epsilon \ll 1$ weakly nonlinear solutions steady-state solutions can be constructed via an elementary Lyapunov-Schmidt reduction procedure. The spectrum and eigenfunctions for the operator

$$\mathcal{H} = -\frac{d^2}{dx^2} + x^2$$

are well-known; in particular, the eigenvalues are given by $\mu_\ell = 1 + 2\ell$ for $\ell \in \mathbb{N}_0$, and the associated eigenfunctions are given by $\phi_\ell(x) = H_\ell(x) e^{-x^2/2}$, where $H_\ell(x)$ is the Gauss-Hermite polynomial of order ℓ is scaled so that

$$\int_{\mathbb{R}} \phi_\ell^2(x) dx = 1, \quad j \in \mathbb{N}_0.$$

For $0 < \epsilon \ll 1$ it is known that for each $\ell \in \mathbb{N}_0$ there are weakly nonlinear solutions of the form

$$U_\ell(x) \propto \phi_\ell(x) + \mathcal{O}(\epsilon), \quad 0 < \omega - \mu_\ell = \mathcal{O}(\epsilon).$$

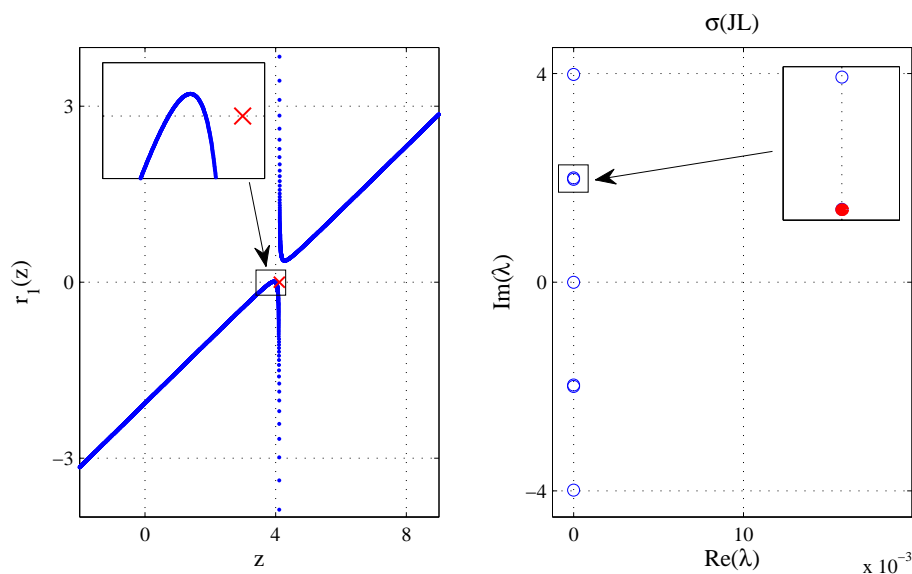


Figure 3: (color online) The eigenvalue $r_1(z)$ for $n = 1$ (left panel), and $\sigma(\mathcal{JL})$ (right panel). In the left panel the (red) cross indicates the location of a pole of the Krein matrix. In the right panel the eigenvalue with negative signature is that associated with the filled (red) circle in the insert.

The linear operators associated with the spectral stability problem in the framework of (1.2) are given by

$$\mathcal{L}_+ = \mathcal{H} - \omega + 3\epsilon U_\ell^2, \quad \mathcal{L}_- = \mathcal{H} - \omega + \epsilon U_\ell^2. \quad (3.4)$$

One has that $\ker(\mathcal{L}_-) = \text{span}\{U_\ell\}$, whereas \mathcal{L}_+ is nonsingular for $\epsilon > 0$. When $\epsilon = 0$ one has $\mathcal{L}_+ = \mathcal{L}_-$ with $\ker(\mathcal{L}_-) = \text{span}\{\phi_\ell\}$ and $n(\mathcal{L}_-) = \ell$; hence, upon projecting off the kernel via the projection operator P_ℓ one has in the formulation of (2.38) that $\mathcal{A}_0 = P_\ell \mathcal{L}_- P_\ell$ with $n(\mathcal{A}_0) = \ell$. Because the interactions are repulsive, for $0 < \epsilon \ll 1$ one continues to have that $n(\mathcal{L}_\pm) = \ell$. Upon projecting off $\ker(\mathcal{L}_-)$ for $\epsilon > 0$ one sees that $n(\mathcal{R}) = n(\mathcal{S}) = \ell$.

The system is of the form of (2.38). Since the bifurcation arises from $\omega = \mu_\ell$, when $\epsilon = 0$ one has for $i \neq \ell$, $\lambda_i = (\mu_i - \mu_\ell)^{-1} = [2(i - \ell)]^{-1}$, and each eigenvalue is geometrically and algebraically simple. As a consequence one has that when $\epsilon = 0$, $z_i \in \sigma(\tilde{\mathcal{R}})$; furthermore, z_i is geometrically and algebraically simple. Now

$$r_i(z) = 2(i - \ell) - \frac{1}{2(i - \ell)}z, \quad i = 0, 1, \dots, \ell - 1.$$

The simple zeros occur at $z_i = 4(i - \ell)^2$, which for (1.2) correspond to the purely imaginary eigenvalues $\pm i2(i - \ell)$. For $\epsilon > 0$ one will generically have near these simple zeros of $r_i(z)$ a pair of purely real zeros which correspond to eigenvalues of opposite sign, or a pair of eigenvalues with nonzero imaginary parts.

Figure 3 gives the spectral output in the case $\ell = 1$, and Figure 4 gives the spectral output in the case $\ell = 2$. In both cases $\epsilon = 0.6$. The left panels show a plot of the eigenvalue(s) for

$\mathbf{K}(z)$, while the right panels show the spectrum for (1.2). The purely imaginary eigenvalue in the upper half of the complex plane with negative signature is associated with the filled (red) circle, and the (red) crosses in the left panel denote the location of the poles of $\mathbf{K}(z)$. In both figures the Hamiltonian-Hopf bifurcation does not occur near $z = 4$, whereas in the second figure that bifurcation does occur near $z = 16$. As a consequence of (1.11) one knows that $k_c + k_1^- \leq \ell$; hence, it is known that the figures capture the location of all of the eigenvalues associated with the count of (1.6).

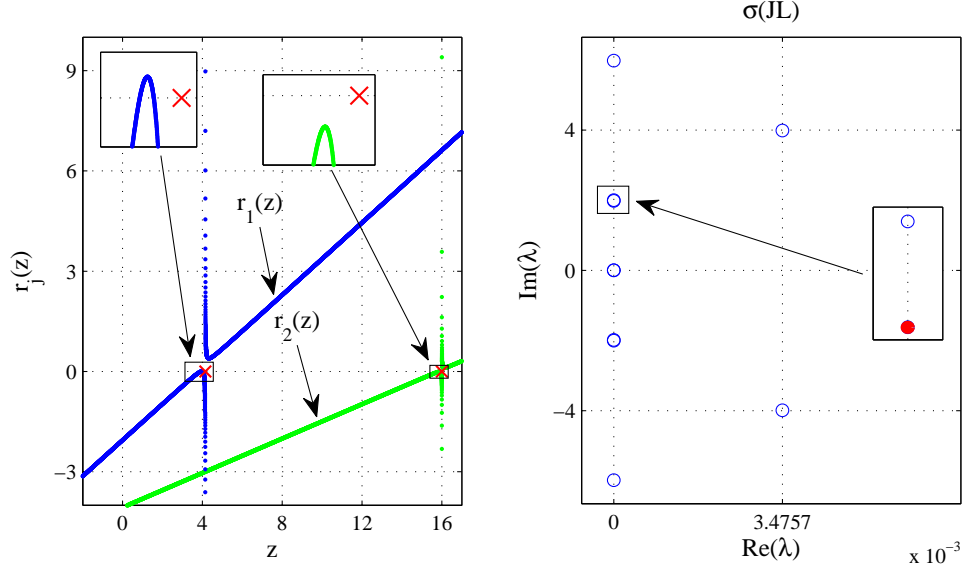


Figure 4: (color online) The Krein eigenvalues $r_j(z)$ for $j = 1, 2$ (left panel), and $\sigma(\mathcal{JL})$ (right panel). In the left panel the (red) cross indicates the location of a pole of the Krein matrix. In the right panel the eigenvalue with negative signature is that associated with the filled (red) circle in the insert.

3.2. Pancake trap

In the case of $d_y = 2$ (3.1) can be rewritten after some rescalings as

$$iu_t + \Delta u + \omega u - \epsilon|u|^2 u = (x^2 + y^2)u, \quad 0 < \epsilon. \quad (3.5)$$

For $0 < \epsilon \ll 1$ weakly nonlinear solutions steady-state solutions can be constructed via a Lyapunov-Schmidt reduction procedure. The spectrum and eigenfunctions for the operator

$$\mathcal{H} = -\Delta + x^2 + y^2$$

are well-known; in particular, the eigenvalues are given by $\mu_{\ell,m} = 2 + 2(\ell + m)$ for $\ell, m \in \mathbb{N}_0$, and the associated eigenfunctions are given by $\phi_{\ell,m}(x, y) = H_\ell(x)H_m(y)e^{-(x^2+y^2)/2}$, where $H_i(\cdot)$ is the Gauss-Hermite polynomial as in the previous subsection. The situation for $0 < \epsilon \ll 1$ is more complicated than the $d_y = 1$ case, for the eigenvalue $\mu_{\ell,m}$ has multiplicity $\ell + m + 1$; hence, the resulting bifurcation equations are more complicated to solve (see [23] for the case $\ell + m = 2$).

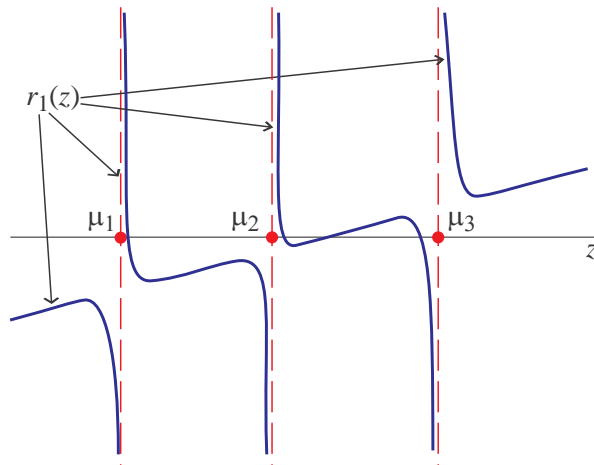


Figure 5: (color online) A cartoon of $r_1(z)$ near $z = 4$ for the case $0 < \mu_3 - \mu_2, \mu_2 - \mu_1 = \mathcal{O}(\epsilon)$ when $0 < \epsilon \ll 1$ under the assumption that the residue for each pole is nonzero. From this graph there will be one real positive eigenvalue with negative sign, and three positive real eigenvalues with positive sign.

Instead of giving a general discussion, the focus instead will be on the easiest case: $\ell + m = 1$. The real-valued solution is the dipole, and is given in polar coordinates by

$$U(x, y) \propto \sqrt{\frac{2}{\pi}} r e^{-r^2/2} \cos \theta + \mathcal{O}(\epsilon), \quad 0 < \omega - 4 = \mathcal{O}(\epsilon).$$

(e.g., see [19]).

The linear operators associated with the spectral stability problem in the framework of (1.2) are given by

$$\mathcal{L}_+ = \mathcal{H} - \omega + 3\epsilon U_\ell^2, \quad \mathcal{L}_- = \mathcal{H} - \omega + \epsilon U_\ell^2. \quad (3.6)$$

One has that $\ker(\mathcal{L}_-) = \text{span}\{U\}$, and $\ker(\mathcal{L}_+) = \text{span}\{U_\theta\}$. When $\epsilon = 0$ one has $\mathcal{L}_+ = \mathcal{L}_-$ with $\ker(\mathcal{L}_-) = \text{span}\{\phi_{0,1}, \phi_{1,0}\}$ and $\mathfrak{n}(\mathcal{L}_-) = 1$. For $0 < \epsilon \ll 1$ one will have that for \mathcal{L}_\pm there will be an eigenvalue which is $\mathcal{O}(\epsilon)$. However, these small eigenvalues do not figure in the spectral calculation because $\dim[\text{gker}(\mathcal{J}\mathcal{L})] = 4$ for all $\epsilon > 0$, so that no small eigenvalues are created for $\sigma(\mathcal{J}\mathcal{L})$ when $\epsilon > 0$. After projecting off of the kernels one will eventually see that $\mathfrak{n}(\mathcal{R}) = \mathfrak{n}(\mathcal{S}) = 1$ for all $\epsilon > 0$ sufficiently small.

The system is of the form of (2.38). Since the bifurcation arises from $\omega = 4$, when $\epsilon = 0$ one has for $i + j \neq 1$, $\lambda_{i,j} = [2(i + j - 1)]^{-1}$. As a consequence one has that when $\epsilon = 0$, $4 = z_1 \in \sigma(\tilde{\mathcal{R}})$; furthermore, z_1 is algebraically simple with geometric multiplicity three. One has

$$r_1(z) = 2 - \frac{1}{2}z,$$

and there are removable singularities at $z = 4$. For $\epsilon > 0$ it is generically expected that (a) the poles will become simple, and (b) the residue associated with each pole will be nonzero. Following the argument associated with (2.41) with $\ell = 2$ one can conclude that there will be at most four zeros of $r_1(z)$ which are within $\mathcal{O}(\epsilon)$ of $z = 4$ (see the cartoon in Figure 5 for a possible scenario).

Figure 6 gives the numerically generated spectrum when $\epsilon = 0.6$. The left panel shows a plot of the eigenvalue $r_1(z)$, while the right panel shows the spectrum for (1.2). The purely imaginary eigenvalue in the upper half of the complex plane with negative signature is associated with the filled (red) circle, and the (red) crosses in the left panel denote the location of the poles of $r_1(z)$. When one compares Figure 6 to the cartoon of Figure 5 it is clear that there is a significant difference: in the cartoon the effect of all three poles is clearly present, whereas in the numerical output the effect of the second pole at μ_2 is missing.

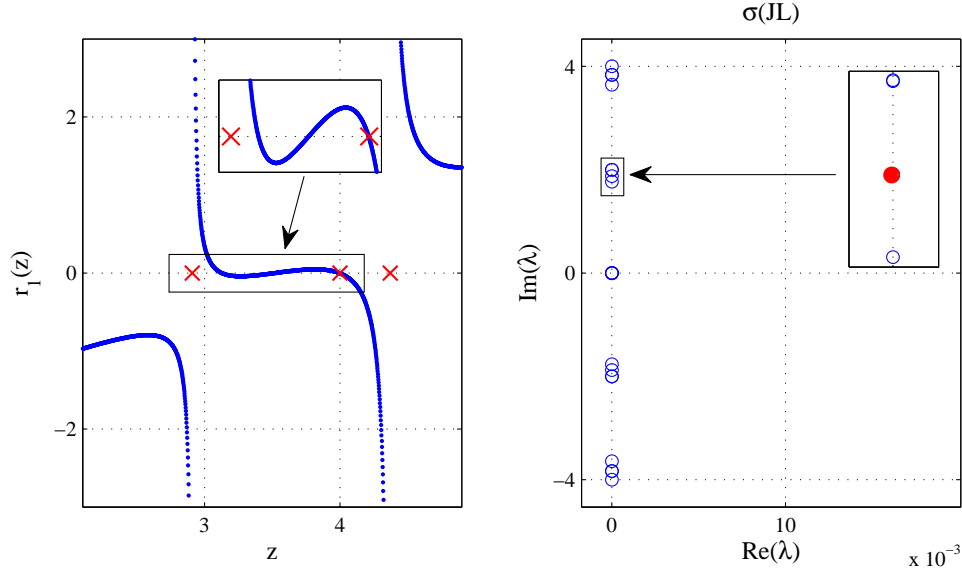


Figure 6: (color online) The Krein eigenvalue $r_1(z)$ (left panel), and $\sigma(\mathcal{JL})$ (right panel). In the left panel the (red) cross indicates the location of a pole of the Krein matrix. In the right panel the eigenvalue with negative signature is that associated with the filled (red) circle in the insert.

The missing zero can be explained with the help of Figure 7. Therein one sees $\sigma(\mathcal{S}^{-1}\mathcal{R})$ (blue diamond) overlaid with the singularities of $\mathbf{K}(z)$ (red cross). Using the ordering $z_j < z_{j+1}$ for $z_j \in \sigma(\mathcal{S}^{-1}\mathcal{R})$, note that z_4 , the only one of the four eigenvalues which is not captured as a zero of $r_1(z)$, practically coincides with μ_2 ; in fact, one has $|z_4 - \mu_2| = \mathcal{O}(10^{-12})$. Recall that for positive real eigenvalues there are two possibilities: (a) the eigenvalue will coincide with a zero of $r_1(z)$, or (b) the eigenvalue will coincide with a removable singularity and be of positive sign. Numerically one sees that the residue of $\mathbf{K}(z)$ at μ_2 is $\mathcal{O}(10^{-29})$. Thus, one can conclude that the location of eigenvalue z_4 falls under scenario (b). In conclusion, the residues associated with μ_1 and μ_3 are nonzero and that associated with μ_2 is zero; hence, by the Krein Oscillation Theorem one can expect that $r_1(z)$ will have an odd number of zeros in the interval (μ_1, μ_3) . In fact, there are three simple zeros, and by Remark 2.13 the middle zero corresponds to an eigenvalue of negative signature. The fourth eigenvalue is of positive signature and coincides with the pole μ_2 .

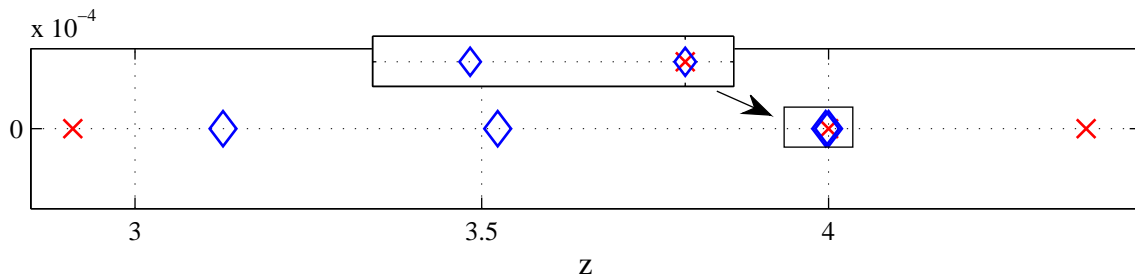


Figure 7: (color online) The eigenvalues of $\mathcal{S}^{-1}\mathcal{R}$ (blue diamond) and the poles of the Krein matrix (red cross). Note that z_4 and μ_2 coincide.

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