The Evans Function: A Primer

Todd Kapitula

Department of Mathematics and Statistics
University of New Mexico
www.math.unm.edu/∼kapitula

Workshop: Stability and Instability of Nonlinear Waves (2006)

Introductory material can be found in

- *Eigenvalues and resonances using the Evans function*, co-authored with B. Sandstede, Discrete and Continuous Dynamical Systems **10**(4):857-869 (2004)
- *Stability analysis of pulses via the Evans function: dissipative systems*, Lecture Notes in Physics **661**:407-427 (2005)

Introductory material can be found in

- *Eigenvalues and resonances using the Evans function*, co-authored with B. Sandstede, Discrete and Continuous Dynamical Systems **10**(4):857-869 (2004)
- *Stability analysis of pulses via the Evans function: dissipative systems*, Lecture Notes in Physics **661**:407-427 (2005)

Some people who have used the Evans function in their work:

- C.K.R.T. Jones, R. Gardner, J. Alexander (analysis vs. topology)
- B. Sandstede, A. Scheel (various interesting extensions)
- A. Doelman, T. Kaper (NLEP)
- K. Zumbrun, P. Howard (conservation laws)
- R. Pego, M. Weinstein (dispersive systems)
- my apologies to those not listed

Introductory material can be found in

- *Eigenvalues and resonances using the Evans function*, co-authored with B. Sandstede, Discrete and Continuous Dynamical Systems **10**(4):857-869 (2004)
- *Stability analysis of pulses via the Evans function: dissipative systems*, Lecture Notes in Physics **661**:407-427 (2005)

Some people who have used the Evans function in their work:

- C.K.R.T. Jones, R. Gardner, J. Alexander (analysis vs. topology)
- B. Sandstede, A. Scheel (various interesting extensions)
- A. Doelman, T. Kaper (NLEP)
- K. Zumbrun, P. Howard (conservation laws)
- R. Pego, M. Weinstein (dispersive systems)
- my apologies to those not listed

Introductory material can be found in

- *Eigenvalues and resonances using the Evans function*, co-authored with B. Sandstede, Discrete and Continuous Dynamical Systems **10**(4):857-869 (2004)
- *Stability analysis of pulses via the Evans function: dissipative systems*, Lecture Notes in Physics **661**:407-427 (2005)

Some people who have used the Evans function in their work:

- C.K.R.T. Jones, R. Gardner, J. Alexander (analysis vs. topology)
- B. Sandstede, A. Scheel (various interesting extensions)
- A. Doelman, T. Kaper (NLEP)
- K. Zumbrun, P. Howard (conservation laws)
- R. Pego, M. Weinstein (dispersive systems)
- my apologies to those not listed

Introductory material can be found in

- *Eigenvalues and resonances using the Evans function*, co-authored with B. Sandstede, Discrete and Continuous Dynamical Systems **10**(4):857-869 (2004)
- *Stability analysis of pulses via the Evans function: dissipative systems*, Lecture Notes in Physics **661**:407-427 (2005)

Some people who have used the Evans function in their work:

- C.K.R.T. Jones, R. Gardner, J. Alexander (analysis vs. topology)
- B. Sandstede, A. Scheel (various interesting extensions)
- A. Doelman, T. Kaper (NLEP)
- K. Zumbrun, P. Howard (conservation laws)
- R. Pego, M. Weinstein (dispersive systems)
- my apologies to those not listed

Introductory material can be found in

- *Eigenvalues and resonances using the Evans function*, co-authored with B. Sandstede, Discrete and Continuous Dynamical Systems **10**(4):857-869 (2004)
- *Stability analysis of pulses via the Evans function: dissipative systems*, Lecture Notes in Physics **661**:407-427 (2005)

Some people who have used the Evans function in their work:

- C.K.R.T. Jones, R. Gardner, J. Alexander (analysis vs. topology)
- B. Sandstede, A. Scheel (various interesting extensions)
- A. Doelman, T. Kaper (NLEP)
- K. Zumbrun, P. Howard (conservation laws)
- R. Pego, M. Weinstein (dispersive systems)
- my apologies to those not listed

Introductory material can be found in

- *Eigenvalues and resonances using the Evans function*, co-authored with B. Sandstede, Discrete and Continuous Dynamical Systems **10**(4):857-869 (2004)
- *Stability analysis of pulses via the Evans function: dissipative systems*, Lecture Notes in Physics **661**:407-427 (2005)

Some people who have used the Evans function in their work:

- C.K.R.T. Jones, R. Gardner, J. Alexander (analysis vs. topology)
- B. Sandstede, A. Scheel (various interesting extensions)
- A. Doelman, T. Kaper (NLEP)
- K. Zumbrun, P. Howard (conservation laws)
- R. Pego, M. Weinstein (dispersive systems)
- my apologies to those not listed

Outline

1. **Construction of the Evans function**
   - Simple example: the linear Schrödinger equation
   - General construction
2. **The orientation index**
   - Simple example: direct calculation
   - Simple example: connection to phase space geometry
3. **Edge bifurcations**
   - Simple example: explicit computation
   - Simple example: perturbation calculation at the branch point
   - Extension: nonlinear Schrödinger equation
4. **Singular perturbations**

Outline

1. **Construction of the Evans function**
   - Simple example: the linear Schrödinger equation
   - General construction
2. **The orientation index**
   - Simple example: direct calculation
   - Simple example: connection to phase space geometry
3. **Edge bifurcations**
   - Simple example: explicit computation
   - Simple example: perturbation calculation at the branch point
   - Extension: nonlinear Schrödinger equation
4. **Singular perturbations**

In order to fully understand the local dynamics associated with a travelling wave $U(x-ct)$, one first must determine the spectrum, $\sigma(\mathcal{L})$, of the linearized operator

$$\mathcal{L} := D\frac{\mathrm{d}^2}{\mathrm{d}y^2} + c\frac{\mathrm{d}}{\mathrm{d}y} + F'(U), \quad y := x - ct.$$

For the sake of clarity it will be assumed that:

$$U(y) \to U_\infty, \quad y \to \pm\infty.$$

However, the subsequent theory is also applicable to the cases:

- $|U(y) - U_\pm| \to 0$ as $y \to \pm\infty$
- $|U(y) - U_{\mathrm{per}}(y)| \to 0$ as $y \to \pm\infty$.

One has that

$$\sigma(\mathcal{L}) = \sigma_{\rm p}(\mathcal{L}) \cup \sigma_{\rm e}(\mathcal{L}),$$

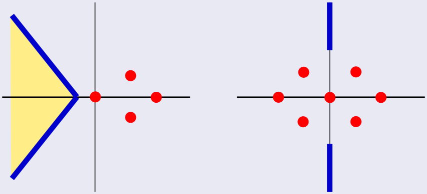
where

$$\sigma_{\rm p}(\mathcal{L}) = \{\lambda \in \sigma(\mathcal{L}) \,:\, \lambda \text{ is isolated with finite multiplicity}\}$$
$$\sigma_{\rm e}(\mathcal{L}) = \sigma(\mathcal{L}) \backslash \sigma_{\rm p}(\mathcal{L}).$$

The boundary(ies) of $\sigma_{\rm e}(\mathcal{L})$ satisfy

$$\det[(\lambda - {\rm i}ck)\mathbb{1} + k^2 D - F'(U_\infty)] = 0, \quad k \in \mathbb{R}.$$

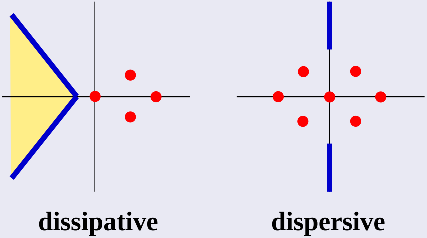
Regarding $\sigma(\mathcal{L})$, there are two possible cases of interest today:



dissipative **dispersive**

As a consequence of underlying symmetries (e.g., spatial translation), one has that $\lambda = 0 \in \sigma_{\mathrm{p}}(\mathcal{L})$. If the system is Hamiltonian, then $m_{\mathrm{a}}(0) = 2m_{\mathrm{g}}(0)$.

Regarding $\sigma(\mathcal{L})$, there are two possible cases of interest today:



As a consequence of underlying symmetries (e.g., spatial translation), one has that $\lambda = 0 \in \sigma_{\mathrm{p}}(\mathcal{L})$. If the system is Hamiltonian, then $m_{\mathrm{a}}(0) = 2m_{\mathrm{g}}(0)$.

Question: What is $\sigma_{\mathrm{p}}(\mathcal{L})\backslash\{0\}$?

Goal

Construct an analytic function, $E(\lambda)$ (the Evans function), whose set of zeros coincides with $\sigma_{\mathrm{p}}(\mathcal{L})$. Furthermore, the multiplicity of the eigenvalue must equal the order of the zero.

Why do this?

Goal

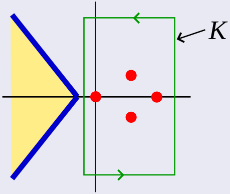
Construct an analytic function, $E(\lambda)$ (the Evans function), whose set of zeros coincides with $\sigma_{\mathrm{p}}(\mathcal{L})$. Furthermore, the multiplicity of the eigenvalue must equal the order of the zero.

Why do this?

Assuming no arbitrarily large eigenvalues,

$$W[E(K)] := \frac{1}{2\pi \mathrm{i}} \oint_K \frac{E'(\lambda)}{E(\lambda)} \, \mathrm{d}\lambda$$

gives a total count of the number of eigenvalues.

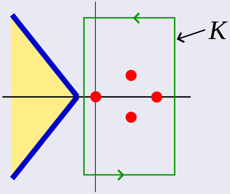


Goal

Construct an analytic function, $E(\lambda)$ (the Evans function), whose set of zeros coincides with $\sigma_{\mathrm{p}}(\mathcal{L})$. Furthermore, the multiplicity of the eigenvalue must equal the order of the zero.

Why do this?

Possible simplification of numerical computations, i.e., perhaps it is easier to compute $W[E(K)]$ than it is to directly locate the eigenvalues.

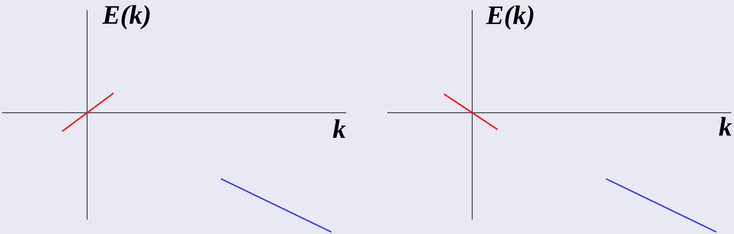


Goal

Construct an analytic function, $E(\lambda)$ (the Evans function), whose set of zeros coincides with $\sigma_{\mathrm{p}}(\mathcal{L})$. Furthermore, the multiplicity of the eigenvalue must equal the order of the zero.

Why do this?

Instability results via the orientation index (parity index).

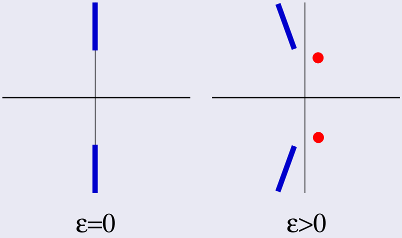


Goal

Construct an analytic function, $E(\lambda)$ (the Evans function), whose set of zeros coincides with $\sigma_{\mathrm{p}}(\mathcal{L})$. Furthermore, the multiplicity of the eigenvalue must equal the order of the zero.

Why do this?

Detection of Hopf bifurcations for perturbations of dispersive systems (edge bifurcations).



Goal

Construct an analytic function, $E(\lambda)$ (the Evans function), whose set of zeros coincides with $\sigma_{\mathrm{p}}(\mathcal{L})$. Furthermore, the multiplicity of the eigenvalue must equal the order of the zero.

Why do this?

Reduction of difficult eigenvalue problems into a product of simpler ones. This is often possible if the solution has pieces on different time scales. For an example to be considered later, consider the Fitzhugh-Nagumo equations

$$\begin{aligned} u_t &= u_{yy} + cu_y + f(u) - w \\ w_t &= cw_y + \epsilon(u - \gamma w) \end{aligned} \qquad (y := x - ct),$$

where $0 < \epsilon \ll 1$ and $c = \mathcal{O}(1)$.

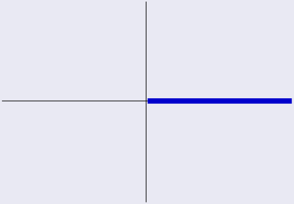
Simple example: the linear Schrödinger equation

Consider the linear Schrödinger operator

$$\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + U(x), \quad |U(x)| \leq C\mathrm{e}^{-2\rho_0|x|}.$$

Here one has that

$$\sigma_{\mathrm{e}}(\mathcal{L}) = \{\lambda \in \mathbb{C} : \lambda = k^2, \ k \in \mathbb{R}\}.$$



For the eigenvalue problem $\mathcal{L}v = \lambda v$, an eigenfunction $v_\lambda(x)$ corresponding to the eigenvalue λ will satisfy $|v_\lambda(x)| \leq C\mathrm{e}^{-\kappa(\lambda)|x|}$. Thus, we wish to solve the boundary value problem

$$\mathcal{L}v = \lambda v, \qquad \lim_{x\to\pm\infty} v(x) = 0.$$

For the eigenvalue problem $\mathcal{L}v = \lambda v$, an eigenfunction $v_\lambda(x)$ corresponding to the eigenvalue λ will satisfy $|v_\lambda(x)| \leq C\mathrm{e}^{-\kappa(\lambda)|x|}$. Thus, we wish to solve the boundary value problem

$$\mathcal{L}v = \lambda v, \qquad \lim_{x\to\pm\infty} v(x) = 0.$$

Setting $\boldsymbol{W} := (v, v')^{\mathrm{T}}$ recasts the eigenvalue problem as

$$\boldsymbol{W}' = \boldsymbol{A}(x,\lambda)\boldsymbol{W}, \quad \boldsymbol{A}(x,\lambda) := \underbrace{\begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix}}_{\boldsymbol{A}_\infty(\lambda)} + \underbrace{\begin{pmatrix} 0 & 0 \\ U(x) & 0 \end{pmatrix}}_{\boldsymbol{R}(x)}.$$

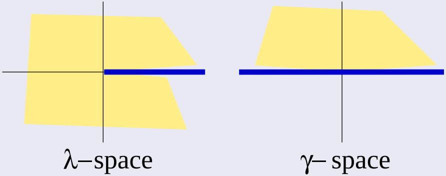
Note that $|\boldsymbol{R}(x)| \to 0$ as $x \to \pm\infty$. Further note that the desired solutions satisfy $|\boldsymbol{W}(x,\lambda)| \to 0$ as $|x| \to \infty$.

Since $|\boldsymbol{R}(x)| \to 0$ as $x \to \pm\infty$, it is useful to first consider

$$\boldsymbol{W}' = \boldsymbol{A}_\infty(\lambda)\boldsymbol{W}.$$

Set

$$\gamma^2 := \lambda; \quad 0 \le \arg(\lambda) < 2\pi.$$

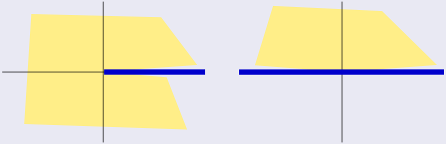


Since $|\boldsymbol{R}(x)| \to 0$ as $x \to \pm\infty$, it is useful to first consider

$$\boldsymbol{W}' = \boldsymbol{A}_\infty(\lambda)\boldsymbol{W}.$$

Set

$$\gamma^2 := \lambda; \quad 0 \le \arg(\lambda) < 2\pi.$$



One has the linearly independent solutions of

$$\boldsymbol{W}_\pm^\infty(x,\gamma) := \mathrm{e}^{\mp\mathrm{i}\gamma x}\begin{pmatrix} 1 \\ \mp\mathrm{i}\gamma \end{pmatrix}, \quad |\boldsymbol{W}_\pm^\infty(x,\gamma)| \to 0 \text{ as } x \to \pm\infty.$$

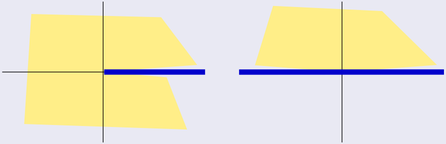
For fixed x that these solutions are analytic in γ.

Since $|\boldsymbol{R}(x)| \to 0$ as $x \to \pm\infty$, it is useful to first consider

$$\boldsymbol{W}' = \boldsymbol{A}_\infty(\lambda)\boldsymbol{W}.$$

Set

$$\gamma^2 := \lambda; \quad 0 \le \arg(\lambda) < 2\pi.$$



It can be shown that for $\boldsymbol{W}' = [\boldsymbol{A}_\infty(\lambda) + \boldsymbol{R}(x)]\boldsymbol{W}$ there exist solutions $\boldsymbol{W}_\pm(x,\gamma)$, analytic in γ for each fixed x, such that

$$|\boldsymbol{W}_\pm(x,\gamma) - \boldsymbol{W}_\pm^\infty(x,\gamma)| \to 0, \quad x \to \pm\infty.$$

One has the asymptotics for $x \gg 1$,

$$\boldsymbol{W}_{-}(x,\gamma) \sim a(\gamma)\mathrm{e}^{-\mathrm{i}\gamma x}\begin{pmatrix} 1 \\ -\mathrm{i}\gamma \end{pmatrix} + b(\gamma)\mathrm{e}^{+\mathrm{i}\gamma x}\begin{pmatrix} 1 \\ +\mathrm{i}\gamma \end{pmatrix},$$

where $a(\gamma)$ is the transmission coefficient and $b(\gamma)$ is the reflection coefficient. One will have an eigenfunction if and only if $a(\gamma) = 0$.

Observations

One has the asymptotics for $x \gg 1$,

$$\boldsymbol{W}_-(x,\gamma) \sim a(\gamma)\mathrm{e}^{-\mathrm{i}\gamma x}\begin{pmatrix} 1 \\ -\mathrm{i}\gamma \end{pmatrix} + b(\gamma)\mathrm{e}^{+\mathrm{i}\gamma x}\begin{pmatrix} 1 \\ +\mathrm{i}\gamma \end{pmatrix},$$

where $a(\gamma)$ is the transmission coefficient and $b(\gamma)$ is the reflection coefficient. One will have an eigenfunction if and only if $a(\gamma) = 0$.

Observations

Note that

$$\lim_{x\to+\infty} \det(\boldsymbol{W}_-, \boldsymbol{W}_+)(x,\gamma) = \mathrm{i}2\gamma a(\gamma).$$

One has the asymptotics for $x \gg 1$,

$$\boldsymbol{W}_-(x,\gamma) \sim a(\gamma)\mathrm{e}^{-\mathrm{i}\gamma x}\begin{pmatrix} 1 \\ -\mathrm{i}\gamma \end{pmatrix} + b(\gamma)\mathrm{e}^{+\mathrm{i}\gamma x}\begin{pmatrix} 1 \\ +\mathrm{i}\gamma \end{pmatrix},$$

where $a(\gamma)$ is the transmission coefficient and $b(\gamma)$ is the reflection coefficient. One will have an eigenfunction if and only if $a(\gamma) = 0$.

Observations

Consider the adjoint problem, $\boldsymbol{W}' = -\boldsymbol{A}(x,\gamma)^{\mathrm{H}}\boldsymbol{W}$, and consider the adjoint solution $\boldsymbol{W}_+^{\mathrm{A}}(x,\gamma)$ which satisfies

$$\lim_{x\to+\infty}\left|\boldsymbol{W}_+^{\mathrm{A}}(x,\gamma) - \mathrm{e}^{-\mathrm{i}\gamma^* x}\begin{pmatrix} \mathrm{i}\gamma^* \\ 1 \end{pmatrix}\right| = 0.$$

Note that

$$\lim_{x\to+\infty}\langle \boldsymbol{W}_-, \boldsymbol{W}_+^{\mathrm{A}}\rangle(x,\gamma) = -\mathrm{i}2\gamma a(\gamma).$$

The Evans function

The Evans function for the linear Schrödinger operator can be given by either

$$E(\gamma) := \det(\boldsymbol{W}_-, \boldsymbol{W}_+)(x, \gamma), \quad \operatorname{Im}\gamma > 0,$$

or

$$E(\gamma) := \langle \boldsymbol{W}_-, \boldsymbol{W}_+^{\mathrm{A}} \rangle(x, \gamma), \quad \operatorname{Im}\gamma > 0.$$

The Evans function

The Evans function for the linear Schrödinger operator can be given by either

$$E(\gamma) := \det(\boldsymbol{W}_-, \boldsymbol{W}_+)(x,\gamma), \quad \mathrm{Im}\,\gamma > 0,$$

or

$$E(\gamma) := \langle \boldsymbol{W}_-, \boldsymbol{W}_+^{\mathrm{A}} \rangle(x,\gamma), \quad \mathrm{Im}\,\gamma > 0.$$

Observations

In either definition the Evans function has the properties that

- it is analytic for $\mathrm{Im}\,\gamma > 0$
- $E(\gamma') = 0$ if and only if $\gamma' \in \sigma_{\mathrm{p}}(\mathcal{L})$
- the order of the zero is the multiplicity of the eigenvalue.

The Evans function

The Evans function for the linear Schrödinger operator can be given by either

$$E(\gamma) := \det(\boldsymbol{W}_-, \boldsymbol{W}_+)(x,\gamma), \quad \operatorname{Im}\gamma > 0,$$

or

$$E(\gamma) := \langle \boldsymbol{W}_-, \boldsymbol{W}_+^{\mathrm{A}} \rangle(x,\gamma), \quad \operatorname{Im}\gamma > 0.$$

Observations

In either definition the Evans function has the properties that

- it is analytic for $\operatorname{Im}\gamma > 0$
- $E(\gamma') = 0$ if and only if $\gamma' \in \sigma_{\mathrm{p}}(\mathcal{L})$
- the order of the zero is the multiplicity of the eigenvalue.

The Evans function

The Evans function for the linear Schrödinger operator can be given by either

$$E(\gamma) := \det(\boldsymbol{W}_-, \boldsymbol{W}_+)(x,\gamma), \quad \mathrm{Im}\,\gamma > 0,$$

or

$$E(\gamma) := \langle \boldsymbol{W}_-, \boldsymbol{W}_+^{\mathrm{A}}\rangle(x,\gamma), \quad \mathrm{Im}\,\gamma > 0.$$

Observations

In either definition the Evans function has the properties that

- it is analytic for $\mathrm{Im}\,\gamma > 0$
- $E(\gamma') = 0$ if and only if $\gamma' \in \sigma_{\mathrm{p}}(\mathcal{L})$
- the order of the zero is the multiplicity of the eigenvalue.

The Evans function

The Evans function for the linear Schrödinger operator can be given by either

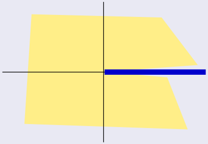
$$E(\gamma) := \det(\boldsymbol{W}_-, \boldsymbol{W}_+)(x,\gamma), \quad \operatorname{Im}\gamma > 0,$$

or

$$E(\gamma) := \langle \boldsymbol{W}_-, \boldsymbol{W}_+^{\mathrm{A}}\rangle(x,\gamma), \quad \operatorname{Im}\gamma > 0.$$

Observations

The set $\sigma_{\mathrm{e}}(\mathcal{L})$ is the absolute spectrum for $\mathcal{L}$; in fact, it is a branch cut for $E(\lambda)$. This observation holds true in general.



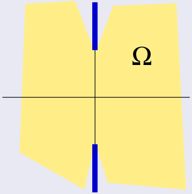
λ–space

Rewrite the general eigenvalue problem $\mathcal{L}v = \lambda v$ as

$$\boldsymbol{W}' = [\boldsymbol{A}_\infty(\lambda) + \boldsymbol{R}(x)]\boldsymbol{W}, \quad |\boldsymbol{R}(x)| \le Ce^{-2\rho_0|x|}.$$

Let $\Omega \subset \mathbb{C}$ be open such that for $\lambda \in \Omega$, $\sigma(\boldsymbol{A}_\infty(\lambda)) \cap \mathrm{i}\mathbb{R} = \emptyset$. Order $\nu_j(\lambda) \in \sigma(\boldsymbol{A}_\infty(\lambda))$, $j = 1, \ldots, n$, as

$$\begin{aligned} \mathrm{Re}\,\nu_j &> 0, \quad j = 1, \ldots, m \\ \mathrm{Re}\,\nu_j &< 0, \quad j = m+1, \ldots, n. \end{aligned}$$

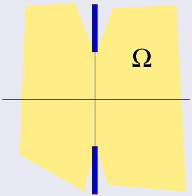


Rewrite the general eigenvalue problem $\mathcal{L}v = \lambda v$ as

$$\boldsymbol{W}' = [\boldsymbol{A}_\infty(\lambda) + \boldsymbol{R}(x)]\boldsymbol{W}, \quad |\boldsymbol{R}(x)| \le C\mathrm{e}^{-2\rho_0|x|}.$$

Let $\Omega \subset \mathbb{C}$ be open such that for $\lambda \in \Omega$, $\sigma(\boldsymbol{A}_\infty(\lambda)) \cap \mathrm{i}\mathbb{R} = \emptyset$. Order $\nu_j(\lambda) \in \sigma(\boldsymbol{A}_\infty(\lambda))$, $j = 1, \dots, n$, as

$$\begin{aligned} \operatorname{Re}\nu_j &> 0, \quad j = 1, \dots, m \\ \operatorname{Re}\nu_j &< 0, \quad j = m+1, \dots, n. \end{aligned}$$



Assumption

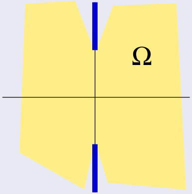
The eigenvalues are simple for $\lambda \in \Omega$.

Rewrite the general eigenvalue problem $\mathcal{L}v = \lambda v$ as

$$\boldsymbol{W}' = [\boldsymbol{A}_\infty(\lambda) + \boldsymbol{R}(x)]\boldsymbol{W}, \quad |\boldsymbol{R}(x)| \le C\mathrm{e}^{-2\rho_0|x|}.$$

Let $\Omega \subset \mathbb{C}$ be open such that for $\lambda \in \Omega$, $\sigma(\boldsymbol{A}_\infty(\lambda)) \cap \mathrm{i}\mathbb{R} = \emptyset$. Order $\nu_j(\lambda) \in \sigma(\boldsymbol{A}_\infty(\lambda))$, $j = 1, \dots, n$, as

$$\begin{aligned} \mathrm{Re}\,\nu_j &> 0, \quad j = 1, \dots, m \\ \mathrm{Re}\,\nu_j &< 0, \quad j = m+1, \dots, n. \end{aligned}$$



Consequence

The eigenvalues are analytic. Furthermore, the associated eigenvectors $\boldsymbol{w}_j(\lambda)$ can be chosen to be analytic.

Construct solutions $\boldsymbol{W}_j(x,\lambda)$, analytic in λ for fixed x, which satisfy

$$\lim_{x\to-\infty} |\boldsymbol{W}_j(x,\lambda)| = 0 \quad j = 1,\dots,m$$
$$\lim_{x\to+\infty} |\boldsymbol{W}_j(x,\lambda)| = 0 \quad j = m+1,\dots,n.$$

Construct solutions $\boldsymbol{W}_j(x,\lambda)$, analytic in λ for fixed x, which satisfy

$$\lim_{x\to-\infty} |\boldsymbol{W}_j(x,\lambda)| = 0 \quad j = 1,\dots,m$$
$$\lim_{x\to+\infty} |\boldsymbol{W}_j(x,\lambda)| = 0 \quad j = m+1,\dots,n.$$

The Evans function

For $\lambda \in \Omega$ the Evans function is given by

$$E(\lambda) := \mathrm{e}^{-\int^x \mathrm{tr}(\boldsymbol{A}(s,\lambda))\,\mathrm{d}s} \det(\underbrace{\boldsymbol{W}_1,\dots,\boldsymbol{W}_m}, \underbrace{\boldsymbol{W}_{m+1},\dots,\boldsymbol{W}_n})(x,\lambda).$$

It has the properties that

- it is analytic for $\lambda \in \Omega$
- $E(\gamma') = 0$ if and only if $\gamma' \in \sigma_{\mathrm{p}}(\mathcal{L})$
- the order of the zero is the multiplicity of the eigenvalue.

Construct solutions $\boldsymbol{W}_j^{\mathrm{A}}(x,\lambda)$ to the adjoint system, analytic in λ for fixed x, which satisfy

$$\lim_{x\to+\infty} |\boldsymbol{W}_j^{\mathrm{A}}(x,\lambda)| = 0, \ \ (j=1,\dots,m).$$

The Evans matrix is given by

$$\mathbb{D}(\lambda) := \begin{pmatrix} \langle \boldsymbol{W}_1, \boldsymbol{W}_1^{\mathrm{A}}\rangle & \cdots & \langle \boldsymbol{W}_1, \boldsymbol{W}_m^{\mathrm{A}}\rangle \\ \vdots & \ddots & \vdots \\ \langle \boldsymbol{W}_m, \boldsymbol{W}_1^{\mathrm{A}}\rangle & \cdots & \langle \boldsymbol{W}_m, \boldsymbol{W}_m^{\mathrm{A}}\rangle \end{pmatrix}(x,\lambda).$$

Construct solutions $\mathbf{W}_j^{\mathrm{A}}(x,\lambda)$ to the adjoint system, analytic in λ for fixed x, which satisfy

$$\lim_{x\to+\infty} |\mathbf{W}_j^{\mathrm{A}}(x,\lambda)| = 0, \;\; (j = 1,\ldots,m).$$

The Evans matrix is given by

$$\mathbb{D}(\lambda) := \begin{pmatrix} \langle \mathbf{W}_1, \mathbf{W}_1^{\mathrm{A}} \rangle & \cdots & \langle \mathbf{W}_1, \mathbf{W}_m^{\mathrm{A}} \rangle \\ \vdots & \ddots & \vdots \\ \langle \mathbf{W}_m, \mathbf{W}_1^{\mathrm{A}} \rangle & \cdots & \langle \mathbf{W}_m, \mathbf{W}_m^{\mathrm{A}} \rangle \end{pmatrix} (x,\lambda).$$

Equivalence

Set $D(\lambda) := \det(\mathbb{D}(\lambda))$. There exists a nonzero analytic function $C(\lambda)$ such that $D(\lambda) = C(\lambda)E(\lambda)$.

The assumption that the eigenvalues of $\boldsymbol{A}_\infty(\lambda)$ are simple is done for convenience only. If the assumption is violated, then one can still construct an Evans function by using

- exterior products
- exponential dichotomies and analytic projections.

The assumption that the eigenvalues of $\boldsymbol{A}_\infty(\lambda)$ are simple is done for convenience only. If the assumption is violated, then one can still construct an Evans function by using

- exterior products
- exponential dichotomies and analytic projections.

Equivalence

One also has the topological interpretation that

$$\frac{1}{2\pi\mathrm{i}}\oint_K \frac{E'(\lambda)}{E(\lambda)}\,\mathrm{d}\lambda = c_1(\mathcal{E}(K)),$$

where $c_1(\mathcal{E}(K))$ is the first Chern number of the unstable bundle $\mathcal{E}(K)$. This formulation is extremely useful in singular perturbation problems.

Outline

1. Construction of the Evans function
   - Simple example: the linear Schrödinger equation
   - General construction
2. The orientation index
   - Simple example: direct calculation
   - Simple example: connection to phase space geometry
3. Edge bifurcations
   - Simple example: explicit computation
   - Simple example: perturbation calculation at the branch point
   - Extension: nonlinear Schrödinger equation
4. Singular perturbations

Assume that

- $\lambda = 0$ is a simple eigenvalue (spatial translation)
- there is an $R > 0$ such that there is no eigenvalue for $|\lambda| > R$.

Assume that

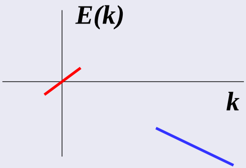
- $\lambda = 0$ is a simple eigenvalue (spatial translation)
- there is an $R > 0$ such that there is no eigenvalue for $|\lambda| > R$.

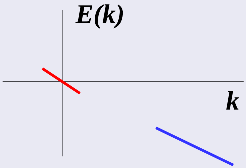
Consequently, one has

$$E(0) = 0, \ \ E'(0) \neq 0$$
$$E(\lambda) \neq 0, \quad |\lambda| > R.$$

The Evans function can be constructed to be real-valued for $\lambda \in \mathbb{R}$. Upon doing so one has that for real eigenvalues,

$$\underbrace{E'(0)E(+\infty)}_{\text{orientation index}} \begin{cases} > 0, & ? \\ < 0, & \text{unstable}. \end{cases}$$





Consider

$$u_t = u_{xx} - u + 2u^3.$$

The equation has a pulse solution which is given by $U(x) := \operatorname{sech}(x)$. Linearizing yields the linear eigenvalue problem

$$\mathcal{L}v = \lambda v, \quad \mathcal{L} := \frac{\mathrm{d}^2}{\mathrm{d}x^2} - (1 - 6U^2(x)).$$

Spatial translation invariance implies that $\lambda = 0$ is an eigenvalue with associated eigenfunction $U'(x)$.

Upon setting $\boldsymbol{W} = (v, v')^{\mathrm{T}}$, the eigenvalue problem becomes

$$\boldsymbol{W}' = [\boldsymbol{A}_\infty(\lambda) + \boldsymbol{R}(x)]\boldsymbol{W},$$

where

$$\boldsymbol{A}_\infty(\lambda) = \begin{pmatrix} 0 & 1 \\ 1+\lambda & 0 \end{pmatrix} \quad \boldsymbol{R}(x) = \begin{pmatrix} 0 & 0 \\ -6U^2(x) & 0 \end{pmatrix}.$$

Simple example: direct calculation

Set

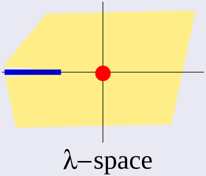
$\gamma^2 := 1 + \lambda, \quad -\pi < \arg(1+\lambda) < \pi.$

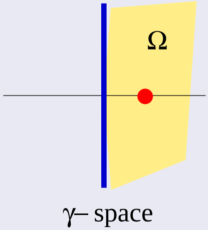
The relevant solutions $\boldsymbol{W}_{\pm}(x,\gamma)$ satisfy

$$\lim_{x\to\mp\infty} |\boldsymbol{W}_{\mp}(x,\gamma) - e^{\pm\gamma x}\begin{pmatrix} 1 \\ \pm\gamma \end{pmatrix}| = 0,$$

and for $\gamma \in \Omega$ the Evans function is given by

$$E(\gamma) = \det(\boldsymbol{W}_-, \boldsymbol{W}_+)(x,\gamma).$$





Calculation of $E(+\infty)$

Rescale via

$$s := |\gamma|x, \quad \boldsymbol{Y} := \mathrm{diag}(1, |\gamma|^{-1})\boldsymbol{W}.$$

In the limit of $|\gamma| \to +\infty$ one gets the system

$$\frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{Y} = \begin{pmatrix} 0 & 1 \\ \mathrm{e}^{\mathrm{i}2\arg(\gamma)} & 0 \end{pmatrix}\boldsymbol{Y}.$$

For $\arg(\gamma) = 0$ the Evans function then satisfies

$$E(+\infty) = \det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} < 0.$$

By continuity this sign condition holds for $\gamma \in \mathbb{R}$ with $\gamma \gg 1$.

Calculation of $E'(1)$

One has that $E(1) = 0$; hence, the relevant solutions can be chosen so that

$$\boldsymbol{W}_-(x, 1) = \boldsymbol{W}_+(x, 1) = \begin{pmatrix} U'(x) \\ U''(x) \end{pmatrix}.$$

Now,

$$E'(1) = \det(\partial_\gamma[\boldsymbol{W}_- - \boldsymbol{W}_+], \boldsymbol{W}_+)(x, 1).$$

Thus, it is crucial that one determines $\partial_\gamma(\boldsymbol{W}_- - \boldsymbol{W}_+)(x, 1)$.

Calculation of $\partial_\gamma(\boldsymbol{W}_- - \boldsymbol{W}_+)(x,1)$

At $\gamma = 1$ on has that

$$\partial_\gamma \boldsymbol{W}_\pm' = [\boldsymbol{A}_\infty(1) + \boldsymbol{R}(x)]\partial_\gamma \boldsymbol{W}_\pm + \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} U'(x) \\ U''(x) \end{pmatrix}.$$

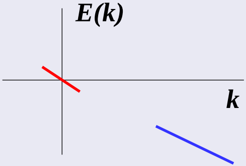
Regarding solutions to the homogeneous equation, let $\boldsymbol{U}_1(x) := \boldsymbol{W}_-(x,1)$, and let $\boldsymbol{U}_2(x)$ be the solution at $\gamma = 1$ such that $\det(\boldsymbol{U}_1, \boldsymbol{U}_2)(x) = 1$. Upon using the variation of parameters one then sees that

$$\partial_\gamma(\boldsymbol{W}_- - \boldsymbol{W}_+)(x,1) = 2\int_{-\infty}^{+\infty} (U'(x))^2 \,\mathrm{d}x\, \boldsymbol{U}_2(x) + C_1 \boldsymbol{U}_1(x)$$

for some $C_1 \in \mathbb{R}$.

Substitution yields that

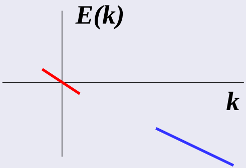
$$E'(1) = \det(\partial_\gamma[\boldsymbol{W}_- - \boldsymbol{W}_+], \boldsymbol{W}_+)(x,1)$$
$$= \left[2\int_{-\infty}^{+\infty} (U'(x))^2 \, \mathrm{d}x\right] \times$$
$$\det(\boldsymbol{U}_2, \boldsymbol{U}_1)(x) < 0.$$



In conclusion, one has that $E'(1)E(+\infty) > 0$.

Substitution yields that

$$E'(1) = \det(\partial_\gamma[\boldsymbol{W}_- - \boldsymbol{W}_+], \boldsymbol{W}_+)(x,1)$$
$$= \left[2\int_{-\infty}^{+\infty} (U'(x))^2 \, \mathrm{d}x\right] \times$$
$$\det(\boldsymbol{U}_2, \boldsymbol{U}_1)(x) < 0.$$



In conclusion, one has that $E'(1)E(+\infty) > 0$.

What went wrong?

By Sturm-Liouville theory one knows that there is **one** real positive eigenvalue. Why does the above calculation suggest that there is an even number?

Orient the bases

The calculation of $E'(1)$ is **local**, and does not take into account the manner in which $E(+\infty)$ was calculated. Recall that

$$\lim_{x\to\mp\infty}\left|\mathbf{W}_{\mp}(x,\gamma)e^{\mp\gamma x}-\begin{pmatrix}1\\ \pm\gamma\end{pmatrix}\right|=0.$$

In particular, for $\gamma\in\mathbb{R}$ the asymptotic basis vectors have the orientation

$$\mathcal{O}(\gamma):=\det\begin{pmatrix}1 & 1\\ \gamma & -\gamma\end{pmatrix}<0.$$

In calculating $E(+\infty)$, care was taken to preserve the orientation of the asymptotic basis vectors, i.e., $\mathcal{O}(+\infty) < 0$. In the calculation of $E'(1)$ the solutions $\boldsymbol{W}_{\pm}(x,1)$ were chosen so that

$$\lim_{x\to-\infty}\left|\boldsymbol{W}_{-}(x,1)\mathrm{e}^{-x}-\begin{pmatrix}2\\2\end{pmatrix}\right|=0$$
$$\lim_{x\to+\infty}\left|\boldsymbol{W}_{+}(x,1)\mathrm{e}^{+x}-\begin{pmatrix}-2\\2\end{pmatrix}\right|=0;$$

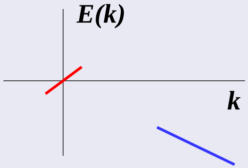
in other words,

$$\mathcal{O}(1):=\det\begin{pmatrix}2&-2\\2&2\end{pmatrix}>0.$$

The calculation must take into account the manner in which the basis vectors are chosen. Consequently, the correct calculation of the orientation index is

$$\underbrace{E'(1)}_{<0}\,\underbrace{\mathcal{O}(1)}_{>0}\,\underbrace{E(+\infty)}_{<0}\,\underbrace{\mathcal{O}(+\infty)}_{<0} < 0,$$

from which one deduces the correct result of an odd number of positive real eigenvalues.



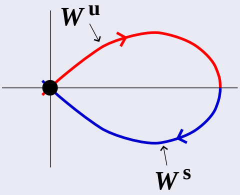
We now show that $E'(1)$ is intimately connected to the manner in which the wave is constructed.

Setting $y = x - ct$ yields

$$u_t = u_{yy} + cu_y - u + 2u^3.$$

Looking for steady-state solutions gives the system

$$u' = v, \quad v' = -cv + u - 2u^3 \qquad \left(' := \frac{\mathrm{d}}{\mathrm{d}y}\right)$$



A pulse solution corresponds to a solution which is homoclinic to $(u, v) = (0, 0)$, and is realized as the intersection of the stable ($W^{\mathrm{s}}(c)$) and unstable ($W^{\mathrm{u}}(c)$) manifolds.

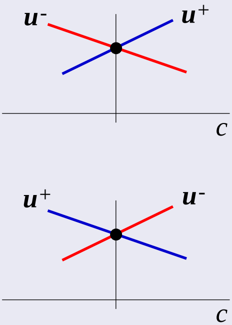
Parameterize the one-dimensional manifolds as

$$W^{\mathrm{u}}(c) = \begin{pmatrix} u^-(y,c) \\ v^-(y,c) \end{pmatrix}$$

$$W^{\mathrm{s}}(c) = \begin{pmatrix} u^+(y,c) \\ v^+(y,c) \end{pmatrix}$$

with $v^\pm(0,c) = 0$. Set

$$d(c) := u^-(0,c) - u^+(0,c).$$



Note that a pulse will exist if an only if $d(c) = 0$. One clearly has that $d(0) = 0$. If $d'(0) \neq 0$, i.e., if the intersection is transverse, then the manifolds will continue to intersect for small perturbations of the vector field.

Upon setting

$$\boldsymbol{Y}_\pm := \partial_c \begin{pmatrix} u^\pm(y,0) \\ v^\pm(y,0) \end{pmatrix},$$

one sees from the equations of variation that

$$\frac{\mathrm{d}}{\mathrm{d}y}\boldsymbol{Y}_\pm = [\boldsymbol{A}_\infty(1) + \boldsymbol{R}(x)]\boldsymbol{Y}_\pm - \begin{pmatrix} 0 \\ U'(x) \end{pmatrix}.$$

Using the same notation as in the calculation of $\partial_\gamma[\boldsymbol{W}_- - \boldsymbol{W}_+](x,1)$ one finds that

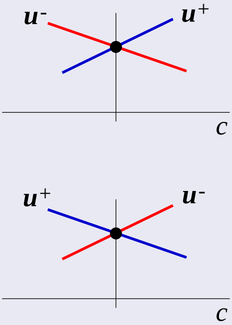
$$(\boldsymbol{Y}_- - \boldsymbol{Y}_+)(x) = -\int_{-\infty}^{+\infty} (U'(x))^2\,\mathrm{d}x\,\boldsymbol{U}_2(x) + C_1\,\boldsymbol{U}_1(x).$$

Simple example: connection to phase space geometry

Consequently, one has that

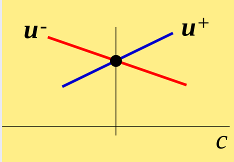
$$\begin{aligned} d'(0) &= \frac{1}{U''(0)} \det(\boldsymbol{Y}_- - \boldsymbol{Y}_+, \boldsymbol{U}_1)(0) \\ &= \frac{1}{U''(0)} \int_{-\infty}^{+\infty} (U'(x))^2 \, \mathrm{d}x. \\ &= -\frac{1}{2U''(0)} E'(1). \end{aligned}$$

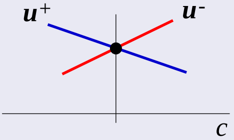
- The manner in which the manifolds intersect encodes information related to the stability of the wave.



Consequently, one has that

$$\begin{aligned} d'(0) &= \frac{1}{U''(0)} \det(\boldsymbol{Y}_- - \boldsymbol{Y}_+, \boldsymbol{U}_1)(0) \\ &= \frac{1}{U''(0)} \int_{-\infty}^{+\infty} (U'(x))^2 \, \mathrm{d}x \\ &= -\frac{1}{2U''(0)} E'(1). \end{aligned}$$





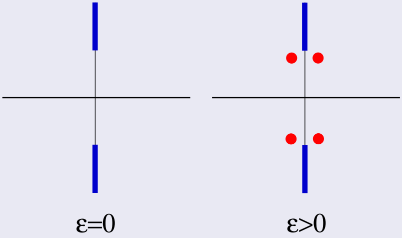
- The manner in which the manifolds intersect encodes information related to the stability of the wave.

This idea of relating the geometry associated with the construction of the wave in phase space with properties of the Evans function has a long and storied history. For example:

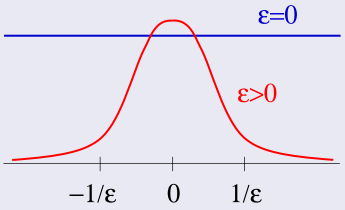
- Evans: propagation of nerve impulses
- Jones: Fitzhugh-Nagumo equation
- Pego/Weinstein: generalized KdV
- Jones, Kopell, Kaper,. . . : Exchange Lemma (singular perturbations)

Outline

When considering perturbations of dispersive systems (or perhaps hyperbolic conservation laws), the spectral stability of the wave cannot be determined without knowing if there are any edge bifurcations.



This is a difficult problem both analytically and numerically, for the unperturbed eigenfunction is not localized, and the perturbed eigenfunction is only weakly localized.



Let us again consider the linear Schrödinger operator

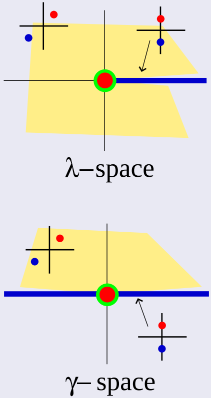
$$\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + U(x)$$

$$U(x) := U_0 \operatorname{sech}^2(x) \; (U_0 > 0),$$

and recall that with $\gamma^2 := \lambda$,

$$\sigma_e(\mathcal{L}) = \{\lambda \in \mathbb{C} \, : \, \operatorname{Im}\gamma = 0\}.$$

For this problem $\sigma_e(\mathcal{L})$ is the absolute spectrum, and $\lambda = 0$ is a branch point.



Recasting the problem as

$$\boldsymbol{W}' = [\boldsymbol{A}_\infty(x,\gamma) + \boldsymbol{R}(x)]\boldsymbol{W},$$

construct the solutions $\boldsymbol{W}_\pm(x,\gamma)$ for $0\,\mathrm{Im}\,\gamma > 0$, analytic in γ for fixed x, which have the asymptotics

$$\lim_{x\to\pm\infty} \left| \boldsymbol{W}_\pm(x,\gamma)\mathrm{e}^{\mp\mathrm{i}\gamma x} - \begin{pmatrix} 1 \\ \pm\mathrm{i}\gamma \end{pmatrix} \right| = 0$$

Recall that for $x \gg 1$,

$$\boldsymbol{W}_-(x,\gamma) \sim a(\gamma)\mathrm{e}^{-\mathrm{i}\gamma x}\begin{pmatrix} 1 \\ -\mathrm{i}\gamma \end{pmatrix} + b(\gamma)\mathrm{e}^{+\mathrm{i}\gamma x}\begin{pmatrix} 1 \\ +\mathrm{i}\gamma \end{pmatrix},$$

where $a(\gamma)$ is the transmission coefficient and $b(\gamma)$ is the reflection coefficient.

It can be shown that with $U(x) = U_0 \operatorname{sech}^2(x)$

$$a(\gamma) = \frac{\Gamma(-\mathrm{i}\gamma)\Gamma(1-\mathrm{i}\gamma)}{\Gamma(1/2+\sqrt{U_0+1/4}-\mathrm{i}\gamma)\Gamma(1/2-\sqrt{U_0+1/4}-\mathrm{i}\gamma)}$$

$$b(\gamma) = \frac{\Gamma(\mathrm{i}\gamma)\Gamma(1-\mathrm{i}\gamma)}{\Gamma(1/2+\sqrt{U_0+1/4})\Gamma(1/2-\sqrt{U_0+1/4})}.$$

It can be shown that with $U(x) = U_0 \operatorname{sech}^2(x)$

$$a(\gamma) = \frac{\Gamma(-\mathrm{i}\gamma)\Gamma(1-\mathrm{i}\gamma)}{\Gamma(1/2+\sqrt{U_0+1/4}-\mathrm{i}\gamma)\Gamma(1/2-\sqrt{U_0+1/4}-\mathrm{i}\gamma)}$$

$$b(\gamma) = \frac{\Gamma(\mathrm{i}\gamma)\Gamma(1-\mathrm{i}\gamma)}{\Gamma(1/2+\sqrt{U_0+1/4})\Gamma(1/2-\sqrt{U_0+1/4})}.$$

Observations: $b(\gamma)$

One has that $b(\gamma) \equiv 0$ if and only if

$$\gamma = k(k+1), \quad k \in \mathbb{N}.$$

In this case the potential is said to be reflectionless. Otherwise, $b(\gamma)$ has simple poles at $\gamma = \mathrm{i}m$, $m \in \mathbb{Z}$, and is nonzero elsewhere.

It can be shown that with $U(x) = U_0 \operatorname{sech}^2(x)$

$$a(\gamma) = \frac{\Gamma(-\mathrm{i}\gamma)\Gamma(1-\mathrm{i}\gamma)}{\Gamma(1/2+\sqrt{U_0+1/4}-\mathrm{i}\gamma)\Gamma(1/2-\sqrt{U_0+1/4}-\mathrm{i}\gamma)}$$

$$b(\gamma) = \frac{\Gamma(\mathrm{i}\gamma)\Gamma(1-\mathrm{i}\gamma)}{\Gamma(1/2+\sqrt{U_0+1/4})\Gamma(1/2-\sqrt{U_0+1/4})}.$$

Observations: $a(\gamma)$

If the potential is not reflectionless, the zeros of $a(\gamma)$ are given by

$$\gamma = -\mathrm{i}\left(\frac{1}{2} \pm \sqrt{U_0+1/4} + \ell\right), \quad \ell \in \mathbb{N};$$

furthermore, $a(\gamma)$ has a simple pole at $\gamma = 0$, and double poles at $\gamma = -\mathrm{i}m$, $m \in \mathbb{N}$.

It can be shown that with $U(x) = U_0 \operatorname{sech}^2(x)$

$$a(\gamma) = \frac{\Gamma(-\mathrm{i}\gamma)\Gamma(1-\mathrm{i}\gamma)}{\Gamma(1/2+\sqrt{U_0+1/4}-\mathrm{i}\gamma)\Gamma(1/2-\sqrt{U_0+1/4}-\mathrm{i}\gamma)}$$

$$b(\gamma) = \frac{\Gamma(\mathrm{i}\gamma)\Gamma(1-\mathrm{i}\gamma)}{\Gamma(1/2+\sqrt{U_0+1/4})\Gamma(1/2-\sqrt{U_0+1/4})}.$$

Observations: $a(\gamma)$

If the potential is reflectionless, i.e., $U_0 = k(k+1)$, then one has that

$$|a(\gamma)| = \begin{cases} 0, & \gamma = \mathrm{i}\ell,\ \ell = 1,\dots,k \\ +\infty, & \gamma = -\mathrm{i}\ell,\ \ell = 1,\dots,k; \end{cases}$$

otherwise, $a(\gamma)$ is nonzero and analytic.

Finally, for $\mathrm{Im}\,\gamma > 0$ recall that the Evans function is given by

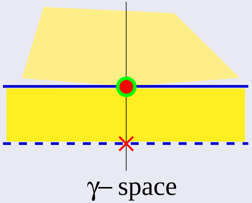
$$\begin{aligned} E(\gamma) &= \det(\boldsymbol{W}_-, \boldsymbol{W}_+)(x, \gamma) \\ &= 2\mathrm{i}\gamma a(\gamma). \end{aligned}$$

Finally, for $\operatorname{Im}\gamma > 0$ recall that the Evans function is given by

$$\begin{aligned} E(\gamma) &= \det(\boldsymbol{W}_-, \boldsymbol{W}_+)(x,\gamma) \\ &= 2\mathrm{i}\gamma a(\gamma). \end{aligned}$$

Observation

Recalling that $\gamma^2 = \lambda$, one has that the Evans function is analytic only for $\operatorname{Im}\gamma > 0$. By a direct calculation, however, it is analytic for $\operatorname{Im}\gamma > -1$ (extension onto the appropriate Riemann surface).

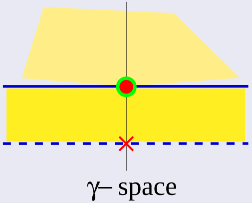


Finally, for $\operatorname{Im}\gamma > 0$ recall that the Evans function is given by

$$E(\gamma) = \det(\boldsymbol{W}_-, \boldsymbol{W}_+)(x, \gamma) = 2\mathrm{i}\gamma a(\gamma).$$

Question

Under what condition(s) on the potential can one guarantee that the Evans function for the linear Schrödinger operator can be extended across the boundary of the absolute spectrum?



Answer: Gap Lemma

The Evans function for the linear Schrödinger operator

$$\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + U(x), \quad |U(x)| \leq C\mathrm{e}^{-2\rho_0|x|},$$

can be analytically extended from $\mathrm{Im}\,\gamma > 0$ to $\mathrm{Im}\,\gamma > -\rho_0$.

Observations

Answer: Gap Lemma

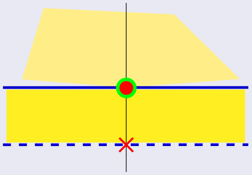
The Evans function for the linear Schrödinger operator

$$\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + U(x), \quad |U(x)| \leq C\mathrm{e}^{-2\rho_0|x|},$$

can be analytically extended from $\mathrm{Im}\,\gamma > 0$ to $\mathrm{Im}\,\gamma > -\rho_0$.

Observations

In particular, the Evans function is analytic at the branch point $\gamma = 0$. The exponential decay of the potential is crucial here.



Answer: Gap Lemma

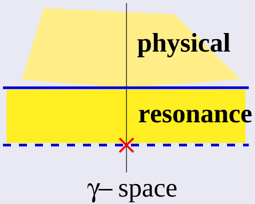
The Evans function for the linear Schrödinger operator

$$\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + U(x), \quad |U(x)| \le C\mathrm{e}^{-2\rho_0|x|},$$

can be analytically extended from $\mathrm{Im}\,\gamma > 0$ to $\mathrm{Im}\,\gamma > -\rho_0$.

Observations

The transformation $\lambda = \gamma^2$ defines a two-sheeted Riemann surface. The principal sheet, $\mathrm{Im}\,\gamma > 0$, is the one on which one finds eigenvalues. The resonant sheet, $\mathrm{Im}\,\gamma < 0$, is the one on which one finds resonance poles.



Answer: Gap Lemma

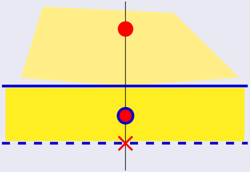
The Evans function for the linear Schrödinger operator

$$\mathcal{L} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + U(x), \quad |U(x)| \leq C\mathrm{e}^{-2\rho_0|x|},$$

can be analytically extended from $\mathrm{Im}\,\gamma > 0$ to $\mathrm{Im}\,\gamma > -\rho_0$.

Observations

The eigenfunction corresponding to an eigenvalue decays exponentially as $|x| \to \infty$, while the eigenfunction corresponding to a resonance pole grows exponentially as $|x| \to \infty$.



γ– space

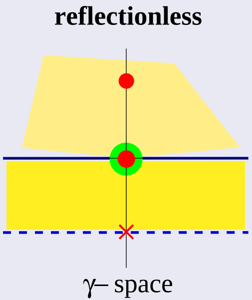
One has that

$$U(x) = U_0 \operatorname{sech}^2(x) \implies E(\gamma) = 2\mathrm{i}\gamma a(\gamma).$$

Recalling the properties of the transmission coefficient yields that

- $E(\gamma) \neq 0$ for $\gamma \in \mathbb{R}\backslash\{0\}$
- a reflectionless potential iff $E(0) = 0,\ E'(0) \neq 0$
- otherwise, $E(0) \neq 0$.

Consequently, under a small perturbation of the potential a resonance pole can pass through the origin and become an eigenvalue if and only if the potential is reflectionless.

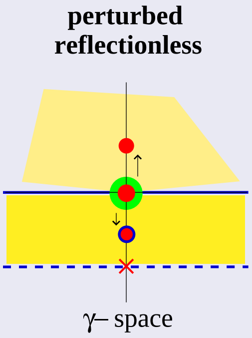


Simple example: perturbation calculation at the branch point

Consider the perturbed reflectionless potential

$$U(x) = k(k+1)\,\mathrm{sech}^2(x) + \epsilon V(x)$$
$$|V(x)| \leq C e^{-\mu|x|}.$$

When $\epsilon = 0$, one has that $\gamma = 0$ is a simple zero of the Evans function. This zero will generically become either an eigenvalue or resonance pole for $\epsilon > 0$.



Simple example: perturbation calculation at the branch point

The Evans function will have the Taylor expansion

$$E(\gamma, \epsilon) = \partial_\epsilon E(0,0)\epsilon + \partial_\gamma E(0,0)\gamma + \mathcal{O}(2);$$

thus, the perturbed zero $\gamma = \gamma_\epsilon$ will have the expansion

$$\gamma_\epsilon := -\frac{\partial_\epsilon E(0,0)}{\partial_\gamma E(0,0)}\epsilon + \mathcal{O}(\epsilon^2).$$

The perturbed zero will be:

- an eigenvalue if $\mathrm{Im}\,\gamma_\epsilon > 0$
- a resonance pole if $\mathrm{Im}\,\gamma_\epsilon < 0$.

Note that the perturbed eigenvalue will have the expansion

$$\lambda = \gamma_\epsilon^2 \epsilon^2 + \mathcal{O}(\epsilon^3).$$

Recall that the Evans function is given by

$$E(\gamma, \epsilon) = \det(\boldsymbol{W}_-, \boldsymbol{W}_+)(x, \gamma, \epsilon).$$

For the reflectionless potential one has

$$\boldsymbol{W}_-(x, 0, 0) = \begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix}, \quad \psi(x) := F(1 + k, -k, 1, (1 + T)/2),$$

where F is the hypergeometric function and $T := \tanh(x)$. As in the section on the orientation index, under the assumption that $\boldsymbol{W}_-(x, 0, 0) = \boldsymbol{W}_+(x, 0, 0)$ (recall that $E(0) = 0$) it can be shown that

$$\begin{aligned} \partial_\epsilon E(0, 0) &= \det(\partial_\epsilon[\boldsymbol{W}_- - \boldsymbol{W}_+], \boldsymbol{W}_+)(x, 0, 0) \\ &= -\int_{-\infty}^{+\infty} V(x)\psi^2(x)\,\mathrm{d}x. \end{aligned}$$

The calculation of $\partial_\epsilon E(0,0)$ required that $\boldsymbol{W}_-(x,0,0) = \boldsymbol{W}_+(x,0,0)$. This condition will be satisfied via the rescaling $\boldsymbol{W}_+(x,\gamma,0) \mapsto -a(\gamma)\boldsymbol{W}_+(x,\gamma,0)$, which in turn yields that the Evans function is given by

$$E(\gamma,0) = -\mathrm{i}2\gamma a^2(\gamma).$$

This in turn yields that

$$\partial_\gamma E(0,0) = -\mathrm{i}2a^2(0) = -\mathrm{i}2[(-1)^k]^2.$$

Conclusion

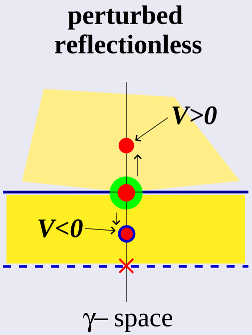
The Evans function has the expansion

$$E(\gamma,\epsilon) = -\mathrm{i}2\gamma - \left(\int_{-\infty}^{+\infty} V(x)\psi^2(x)\,\mathrm{d}x\right)\epsilon + \mathcal{O}(2).$$

The perturbed zero is given by

$$\gamma_\epsilon = \left(\mathrm{i}\frac{1}{2}\int_{-\infty}^{+\infty} V(x)\psi^2(x)\,\mathrm{d}x\right)\epsilon + \mathcal{O}(\epsilon^2).$$

Thus, one recovers the classical result for perturbations of reflectionless potentials.



For a more complicated example, consider the nonlinear Schrödinger equation (NLS), which is given by

$$\mathrm{i}q_t + \frac{1}{2}q_{xx} - q + |q|^2 q = 0 \quad (q \in L^2(\mathbb{R}, \mathbb{C}^2)).$$

Setting $\boldsymbol{u} := (q^*, q)^{\mathrm{T}} \in L^2(\mathbb{R}, \mathbb{C}^2)$, the NLS can be rewritten as

$$\boldsymbol{u}_t + 2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\Omega(\mathcal{L}^{\mathrm{A}}(\boldsymbol{u}))(\boldsymbol{u}) = \boldsymbol{0}.$$

Here $\mathcal{L}^{\mathrm{A}}$ is a certain integro-differential operator, and $\Omega(k) = 1/2 + k^2$ is the dispersion relation. This system is an example of an integrable PDE.

For a more complicated example, consider the nonlinear Schrödinger equation (NLS), which is given by

$$\mathrm{i}q_t + \frac{1}{2}q_{xx} - q + |q|^2 q = 0 \quad (q \in L^2(\mathbb{R}, \mathbb{C}^2)).$$

Setting $\boldsymbol{u} := (q^*, q)^{\mathrm{T}} \in L^2(\mathbb{R}, \mathbb{C}^2)$, the NLS can be rewritten as

$$\boldsymbol{u}_t + 2\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Omega(\mathcal{L}^{\mathrm{A}}(\boldsymbol{u}))(\boldsymbol{u}) = \boldsymbol{0}.$$

Here $\mathcal{L}^{\mathrm{A}}$ is a certain integro-differential operator, and $\Omega(k) = 1/2 + k^2$ is the dispersion relation. This system is an example of an integrable PDE.

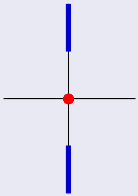
Note the symmetries

$$q(x,t) \mapsto q(x+\xi, t); \quad q(x,t) \mapsto q(x,t)\mathrm{e}^{\mathrm{i}\theta}.$$

The stationary soliton solution is given by $q(x) = Q(x) := \mathrm{sech}(x)$. Upon linearizing about this solution, one has that the associated linear operator has $\lambda = 0$ is an eigenvalue of $m_\mathrm{g}(0) = 2$ (symmetries) and $m_\mathrm{a}(0) = 4$ (Hamiltonian system). Furthermore, with $\mathcal{L}$ representing the linearization about the soliton, one has

$$\sigma_\mathrm{e}(\mathcal{L}) = \{\lambda \in \mathbb{C} : \lambda = \pm \mathrm{i}2\Omega(k),\ k \in \mathbb{R}\}.$$

Any other eigenvalues?



The key to answering this question is the underlying linear scattering problem associated with the NLS: the Zakharov-Shabat (ZS) problem

$$\boldsymbol{v}_x = \begin{pmatrix} -\mathrm{i}k & q(x,t) \\ -q^*(x,t) & \mathrm{i}k \end{pmatrix} \boldsymbol{v}, \quad k \in \mathbb{C}.$$

As with the linear Schrödinger problem, one can define transmission and reflection coefficients for the ZS problem:

- $a(k)$ and $b(k)$ for $\mathrm{Im}\,k > 0$
- $\bar{a}(k)$ and $\bar{b}(k)$ for $\mathrm{Im}\,k < 0$

(the essential spectrum is $\mathrm{Im}\,k = 0$).

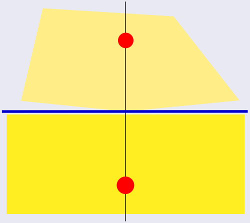
The soliton is a reflectionless potential, with

$$a(k) = \frac{\sqrt{2}\,k - \mathrm{i}}{\sqrt{2}\,k + \mathrm{i}}, \quad \bar{a}(k) = \frac{\sqrt{2}\,k + \mathrm{i}}{\sqrt{2}\,k - \mathrm{i}}; \qquad b(k) = \bar{b}(k) \equiv 0.$$

Note that

$$a(\mathrm{i}/\sqrt{2}) = \bar{a}(-\mathrm{i}/\sqrt{2}) = 0,$$

that these zeros are simple, and that the transmission coefficients are otherwise nonzero.



k-space

Invert the dispersion relations

$$\lambda = \mathrm{i}2\Omega(k), \quad \lambda = -\mathrm{i}2\Omega(\bar{k}),$$

via

$$k(\lambda) = \frac{1}{\sqrt{2}} e^{\mathrm{i}3\pi/4}\sqrt{\lambda - \mathrm{i}}, \quad \arg(\lambda - \mathrm{i}) \in (-3\pi/2, \pi/2]$$
$$\bar{k}(\lambda) = \frac{1}{\sqrt{2}} e^{-\mathrm{i}3\pi/4}\sqrt{\lambda + \mathrm{i}}, \quad \arg(\lambda - \mathrm{i}) \in (-\pi/2, 3\pi/2].$$

The branch cuts have been chosen so that $\mathrm{Im}\, k > 0$ and $\mathrm{Im}\, \bar{k} < 0$, and that k and $\bar{k}$ are analytic for $\lambda \notin \sigma_{\mathrm{e}}(\mathcal{L})$. Note that

$$k(0) = \frac{\mathrm{i}}{\sqrt{2}}, \quad \bar{k}(0) = -\frac{\mathrm{i}}{\sqrt{2}};$$

in particular,

$$a \circ k(0) = \bar{a} \circ \bar{k}(0) = 0.$$

Evans function for NLS

It can (eventually) be shown that

$$E(\lambda) = 8[a \circ k(\lambda)]^2[\bar{a} \circ \bar{k}(\lambda)]^2\sqrt{\lambda - \mathrm{i}}\sqrt{\lambda + \mathrm{i}}.$$

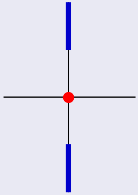
Evans function for NLS

It can (eventually) be shown that

$$E(\lambda) = 8[a \circ k(\lambda)]^2[\bar{a} \circ \bar{k}(\lambda)]^2\sqrt{\lambda - \mathrm{i}}\sqrt{\lambda + \mathrm{i}}.$$

Previous observations and an analysis of the above yields:

- $\lambda = 0$ is a zero or order 4
- there are no other zeros in $\mathbb{C}\backslash\sigma_{\mathrm{e}}(\mathcal{L})$
- the other zeros (simple on the Riemann surface $\gamma^2 = \lambda^2 + 1$) are the branch points $\lambda = \pm\mathrm{i}$



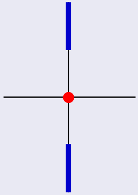
Evans function for NLS

It can (eventually) be shown that

$$E(\lambda) = 8[a \circ k(\lambda)]^2[\bar{a} \circ \bar{k}(\lambda)]^2\sqrt{\lambda - \mathrm{i}}\sqrt{\lambda + \mathrm{i}}.$$

Previous observations and an analysis of the above yields:

- $\lambda = 0$ is a zero or order 4
- there are no other zeros in $\mathbb{C}\backslash\sigma_{\mathrm{e}}(\mathcal{L})$
- the other zeros (simple on the Riemann surface $\gamma^2 = \lambda^2 + 1$) are the branch points $\lambda = \pm\mathrm{i}$



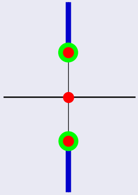
Evans function for NLS

It can (eventually) be shown that

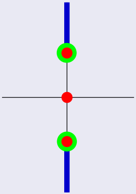
$$E(\lambda) = 8[a \circ k(\lambda)]^2 [\bar{a} \circ \bar{k}(\lambda)]^2 \sqrt{\lambda - \mathrm{i}}\sqrt{\lambda + \mathrm{i}}.$$

Previous observations and an analysis of the above yields:

- $\lambda = 0$ is a zero or order 4
- there are no other zeros in $\mathbb{C} \backslash \sigma_{\mathrm{e}}(\mathcal{L})$
- the other zeros (simple on the Riemann surface $\gamma^2 = \lambda^2 + 1$) are the branch points $\lambda = \pm\mathrm{i}$



When discussing perturbations of the NLS, one can then conclude that unstable eigenvalues arise either from $\lambda = 0$ (regular perturbation theory) or the branch points $\lambda = \pm \mathrm{i}$.

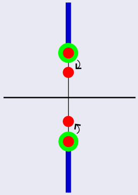


It is known regarding the branch point zeros that:

- Hamiltonian perturbations will not give an unstable eigenvalue (eigenvalues come in quartets $\{\pm\lambda, \pm\lambda^*\}$)
- if

$$\sigma_{\mathrm{e}}(\mathcal{L}) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \leq C\epsilon\},$$

then no instability arises from the branch point (recall that the zero is $\mathcal{O}(\epsilon^2)$ close to the branch point).

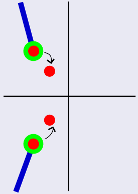


It is known regarding the branch point zeros that:

- Hamiltonian perturbations will not give an unstable eigenvalue (eigenvalues come in quartets $\{\pm\lambda, \pm\lambda^*\}$)
- if

$$\sigma_{\mathrm{e}}(\mathcal{L}) \subset \{\lambda \in \mathbb{C} \,:\, \mathrm{Re}\,\lambda \le C\epsilon\},$$

then no instability arises from the branch point (recall that the zero is $\mathcal{O}(\epsilon^2)$ close to the branch point).



Outline

1. Construction of the Evans function
   - Simple example: the linear Schrödinger equation
   - General construction
2. The orientation index
   - Simple example: direct calculation
   - Simple example: connection to phase space geometry
3. Edge bifurcations
   - Simple example: explicit computation
   - Simple example: perturbation calculation at the branch point
   - Extension: nonlinear Schrödinger equation
4. **Singular perturbations**

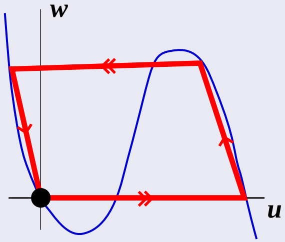
Singular perturbation problems have the feature that the wave has different time scales associated with different pieces of it. When considering the stability of such waves, this property can often be exploited in order to simply the calculation of the Evans function. For example, consider the Fitzhugh-Nagumo equation

$$\begin{aligned} u_t &= u_{yy} + cu_y + f(u) - w \\ w_t &= cw_y + \epsilon(u - \gamma w) \end{aligned} \qquad (y := x - ct),$$

where $0 < \epsilon \ll 1$ and $c = \mathcal{O}(1)$.

The steady-state ODE is given by

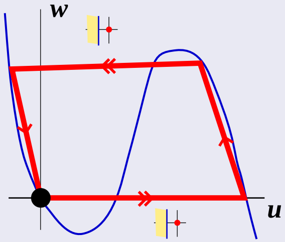
$$
\begin{aligned}
u' &= v \\
v' &= -cv - f(u) + w \qquad \left(' := \frac{\mathrm{d}}{\mathrm{d}y} \right) \\
w' &= -\frac{\epsilon}{c}(u - \gamma w)
\end{aligned}
$$



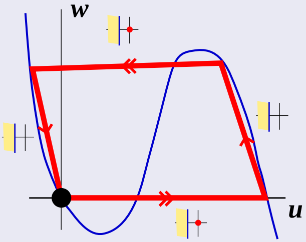
Along the fast pieces, the associated PDE is given by

$$u_t = u_{yy} + cu_y + f(u) - w$$
$$w_t = 0.$$

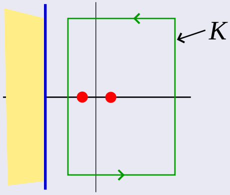
By Sturm-Liouville theory one has that $\lambda = 0$ is a simple zero; furthermore, there are no eigenvalues in $\mathrm{Re}\,\lambda > 0$.

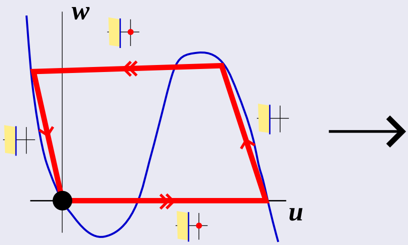


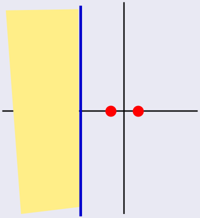
Along the slow pieces, it can be shown that the Evans function is nonzero for all λ in the domain of definition.



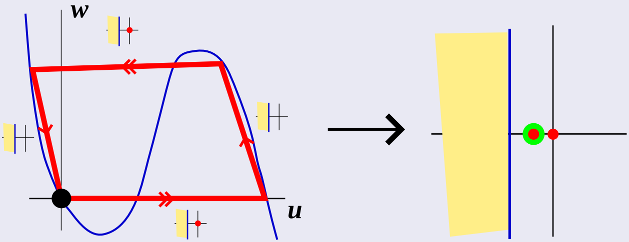
Upon using the bundle construction of the Evans function, it can be shown that $W(E(K)) = 2$; furthermore, both of these eigenvalues are $\mathcal{O}(\epsilon)$.



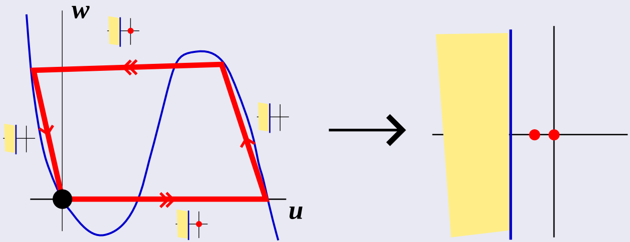




The spatial translation invariance implies that one of these eigenvalues lies precisely at $\lambda = 0$; hence, there is only one eigenvalue which can lead to an instability.



The location of the $\mathcal{O}(\epsilon)$ eigenvalue can be determined via the orientation index. The calculation is subtle due to the differing time-scales associated with the wave (Exchange Lemma). Recall that $E'(0)$ is intimately related to the manner in which the relevant manifolds intersect in the ODE phase space. The singular nature of the wave requires that the Exchange Lemma be used to make this calculation. In conclusion, one finds that the wave is indeed stable.



As a final comment, the calculation here implicitly uses the fact that the fast piece and slow piece “ignore” each other, i.e., the coupling between these pieces is trivial. There are problems in which this coupling is nontrivial, which leads to pole/zero cancelations when constructing the Evans function (NLEP).