Math W81: Homework #5

1. Let

\[ u_1 = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad u_3 = \frac{1}{\sqrt{21}} \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix}, \quad w = \frac{1}{\sqrt{14}} \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}. \]

Let the Hermitian matrix \( A \) be given by

\[ A = -6u_1u_1^T - 2u_2u_2^T + 7u_3u_3^T. \]

(a) What are the eigenvalues of \( A \), and for each eigenvalue what is a corresponding eigenvector?

(b) Set \( B = A + \alpha ww^T \). Explicitly construct a function, say \( f(\lambda) \), which has the properties:

- \( f(\lambda) = 0 \) if and only if \( \lambda \in \sigma(B) \)
- the graph of \( f(\lambda) \) has vertical asymptotes for \( \lambda \in \sigma(A) \).

(c) Explicitly state how the eigenvalues of \( B \) relate to those of \( A \) as a function of \( \alpha \).

2. Let

\[ u_1 = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad u_3 = \frac{1}{\sqrt{21}} \begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix}, \]

and let

\[ A = 3u_1u_1^T + 5u_2u_2^T + 8u_3u_3^T. \]

Let \( S \) be the subspace

\[ S = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix} \right\}. \]

(a) Find a projection matrix \( P \) with the properties:

- \( P : \mathbb{C}^3 \rightarrow S \)
- \( P^2 x = Px \) for any \( x \in \mathbb{C}^3 \)
- \( Px = 0 \) for all \( x \in S^\perp \).

(b) Find the \( 2 \times 2 \) Hermitian matrix representation \( A_{\text{rep}} \) for the linear operator \( PAP : S \mapsto S \).

(c) How do the eigenvalues for \( A_{\text{rep}} \) relate to those for \( A \)?