MATH W81: HOMEWORK #5

1. Let

$$\boldsymbol{u}_1 = \frac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad \boldsymbol{u}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{u}_3 = \frac{1}{\sqrt{21}}\begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{w} = \frac{1}{\sqrt{14}}\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}.$$

Let the Hermitian matrix $\boldsymbol{A}$ be given by

$$\boldsymbol{A} = -6\boldsymbol{u}_1\boldsymbol{u}_1^{\mathrm{T}} - 2\boldsymbol{u}_2\boldsymbol{u}_2^{\mathrm{T}} + 7\boldsymbol{u}_3\boldsymbol{u}_3^{\mathrm{T}}.$$

(a) What are the eigenvalues of $\boldsymbol{A}$, and for each eigenvalue what is a corresponding eigenvector?

(b) Set $\boldsymbol{B} = \boldsymbol{A} + \alpha\,\boldsymbol{w}\boldsymbol{w}^{\mathrm{T}}$. Explicitly construct a function, say $f(\lambda)$, which has the properties:

- $f(\lambda) = 0$ if and only if $\lambda \in \sigma(\boldsymbol{B})$
- the graph of $f(\lambda)$ has vertical asymptotes for $\lambda \in \sigma(\boldsymbol{A})$.

(c) Explicitly state how the eigenvalues of $\boldsymbol{B}$ relate to those of $\boldsymbol{A}$ as a function of α.

2. Let

$$\boldsymbol{u}_1 = \frac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad \boldsymbol{u}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad \boldsymbol{u}_3 = \frac{1}{\sqrt{21}}\begin{pmatrix} -4 \\ 1 \\ 2 \end{pmatrix},$$

and let Let

$$\boldsymbol{A} = 3\boldsymbol{u}_1\boldsymbol{u}_1^{\mathrm{T}} + 5\boldsymbol{u}_2\boldsymbol{u}_2^{\mathrm{T}} + 8\boldsymbol{u}_3\boldsymbol{u}_3^{\mathrm{T}}.$$

Let S be the subspace

$$S = \mathrm{Span}\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix}\}.$$

(a) Find a projection matrix $\boldsymbol{P}$ with the properties:

- $\boldsymbol{P} : \mathbb{C}^3 \mapsto S$
- $\boldsymbol{P}^2\boldsymbol{x} = \boldsymbol{P}\boldsymbol{x}$ for any $\boldsymbol{x} \in \mathbb{C}^3$
- $\boldsymbol{P}\boldsymbol{x} = \boldsymbol{0}$ for all $\boldsymbol{x} \in S^{\perp}$.

(b) Find the 2×2 Hermitian matrix representation $\boldsymbol{A}_{\mathrm{rep}}$ for the linear operator $\boldsymbol{PAP} : S \mapsto S$.

(c) How do the eigenvalues for $\boldsymbol{A}_{\mathrm{rep}}$ relate to those for $\boldsymbol{A}$?