Math W81: Homework #5

Group 1: Let X represent the population density for deer, and let Y represent the population density for wolves. The governing equations of interest are:

$$\frac{dX}{dt} = R_X X\left(1 - \frac{X}{K_X}\right) - b_X XY - P_X X$$
$$\frac{dY}{dt} = -R_Y Y + b_Y XY - P_Y Y.$$

The equations are supposed to provide a simple model for *predator-prey system* for which the deer and wolf populations are hunted at some rate that is proportional to their population densities.

(a) Provide a clear explanation as to what each term in the equations describes. For example, what does the term

$$R_X X\left(1 - \frac{X}{K_X}\right)$$

describe?

(b) Rescale $X = \alpha x, Y = \beta y, t = \gamma s$ for appropriate α, β, γ to derive the dimensionless model ($' = d/ds$):

$$x' = x(1-x) - axy - px$$
$$y' = -by + cxy.$$

Clearly state the functional relationship between the new parameters and the old parameters.

(c) Determine and analyze all of qualitatively different phase portraits, as far as long-term behavior is concerned. What are the biological conditions associated with each phase portrait?

(d) Find conditions under which the two species can stably coexist. Describe your mathematical results from a biological perspective.

Group 2: A simple laser model is given by

$$\frac{dn}{dt} = GnN - kn$$
$$\frac{dN}{dt} = -fN - GnN + P.$$

Here N is the number of excited atoms, n is the number of photons in the laser field, $G > 0$ is the gain coefficient for stimulated emission, $k > 0$ is the decay rate due to loss of photons by mirror transmission, scattering, etc., $f > 0$ is the decay rate for spontaneous emission, and $p \in \mathbb{R}$ is the pump strength.

(a) Provide a clear explanation as to what each term in the equations describes. For example, what does the term GnN describe?

(b) Rescale $n = \alpha x, N = \beta y, t = \gamma s$ for appropriate α, β, γ to derive the dimensionless model ($' = \mathrm{d}/\mathrm{d}s$):

$$\begin{aligned} x' &= xy - x \\ y' &= -ay - xy + b. \end{aligned}$$

Clearly state the functional relationship between the new parameters and the old parameters.

(c) Find and classify all of the equilibrium solutions. Sketch all of the qualitatively different phase portraits that occur as the parameters are varied. Pay particular attention to those parameter regimes for which equilibrium solutions either (a) undergo a bifurcation, and/or (b) change their stability type.

Group 3: This is a variant of Exercise 9.8.9 of the text. For a given population, let f represent the population density of females and m represent the population density of males. Under the assumption that the environment has a total carrying capacity K, the model equations can be written as

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} &= f\left(B_f \frac{m}{m+f} - D_f\right)\left(1 - \frac{m+f}{K}\right) \\ \frac{\mathrm{d}m}{\mathrm{d}t} &= m\left(B_m \frac{f}{m+f} - D_m\right)\left(1 - \frac{m+f}{K}\right). \end{aligned}$$

(a) Provide a clear explanation as to what each term in the equations describes. For example, what does the term

$$f\left(B_f \frac{m}{m+f} - D_f\right)$$

describe?

(b) Rescale $f = \alpha x, m = \beta y, t = \gamma s$ for appropriate α, β, γ to derive the dimensionless model ($' = \mathrm{d}/\mathrm{d}s$):

$$\begin{aligned} x' &= x\left(\frac{ay}{x+y} - 1\right)(1 - x - y) \\ y' &= by\left(\frac{cx}{x+y} - 1\right)(1 - x - y). \end{aligned}$$

Clearly state the functional relationship between the new parameters and the old parameters.

(c) Suppose that $a, c \neq 1$. What does this mean biologically? Calculate the equilibrium solutions and their stability. Make sure that you look at each of the (at least) four important cases (you'll see what I mean once you start doing the calculations). Provide a phase portrait for each case. Describe your mathematical results from a biological perspective.

Group 4: Let X represent the population density for deer, and let Y represent the population density for elk. The governing equations of interest are:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = R_X X\left(1 - \frac{X}{K_X}\right) - b_X XY$$
$$\frac{\mathrm{d}Y}{\mathrm{d}t} = R_Y Y\left(1 - \frac{Y}{K_Y}\right) - b_Y XY.$$

The equations are supposed to provide a simple model for two species which are in *competition* with each other.

(a) Provide a clear explanation as to what each term in the equations describes. For example, what does the term

$$R_X X\left(1 - \frac{X}{K_X}\right)$$

describe?

(b) Rescale $X = \alpha x, Y = \beta y, t = \gamma s$ for appropriate α, β, γ to derive the dimensionless model $(' = \mathrm{d}/\mathrm{d}s)$:

$$x' = x(1-x) - axy$$
$$y' = by(1-y) - cxy.$$

Clearly state the functional relationship between the new parameters and the old parameters.

(c) Show that there are four qualitatively different phase portraits, as far as long-term behavior is concerned. What are the biological conditions associated with each phase portrait?

(d) Find conditions under which the two species can stably coexist. Describe your mathematical results from a biological perspective.

Group 5: Let $X, Y \geq 0$ represent beaver population densities in neighboring areas. The governing equations of interest are:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = R_X X\left(1 - \frac{X}{K_X}\right) - M\left(\frac{X}{K_X} - \frac{Y}{K_Y}\right) - PX$$
$$\frac{\mathrm{d}Y}{\mathrm{d}t} = R_Y Y\left(1 - \frac{Y}{K_Y}\right) + M\left(\frac{X}{K_X} - \frac{Y}{K_Y}\right).$$

The equations are supposed to provide a simple model for the *social fence hypothesis*, in which beavers in a given area compete for vital resources, and when the competition reaches a critical level social pressure is exerted on some individual beavers to depart (*within-group aggression*). As individuals attempt to depart, there is territorial pressure exerted against their departure by a neighboring population (*between-group aggression*). If the within-group aggression in area A is stronger than the between-group aggression exerted in B, the social fence is then said to be open for migration from A to B. Finally, it is assumed that population X in area A is controlled by trapping at some rate that is proportional to the population density, whereas population Y in area B is not.

(a) Provide a clear explanation as to what each term in the equations describes. For example, what does the term

$$R_X X\left(1-\frac{X}{K_X}\right)$$

describe?

(b) Rescale $X=\alpha x, Y=\beta y, t=\gamma s$ for appropriate α,β,γ to derive the dimensionless model ($'=\mathrm{d}/\mathrm{d}s$):

$$\begin{aligned} x' &= x(1-x)-a(x-y)-px \\ y' &= by(1-y)+c(x-y). \end{aligned}$$

Clearly state the functional relationship between the new parameters and the old parameters.

(c) Reasonable baseline parameters are given by

$$R_X=0.335,\ R_Y=0.3015,\ K_X=1.107,\ K_Y=0.9963,\ M=0.3473.$$

Assume that there is no trapping. Calculate the equilibrium solutions and their stability. Provide a phase portrait. Describe your mathematical results from a biological perspective.

(d) Now perform a study on the effects of trapping with all other constants as in part (c). As the trapping rate varies, how is the number and type of the equilibrium solutions effected? Provide phase portraits for several different trapping rates. Describe your mathematical results from a biological perspective.