MATH W81: HOMEWORK #1

Group 1: Consider the model given by

$$f(x) = a\cos(\pi x).$$

(a) For each $a > 0$ find an interval $I \subset \mathbb{R}$ such that $f : I \mapsto I$.

(b) Show that there is an increasing sequence $0 < a_1 < a_2 < \cdots$ with $\lim_{n\to\infty} a_n = +\infty$ such that a saddle-node bifurcation occurs when when $a = a_j$. Further show that the sequence is alternating in the sense that the new fixed points are negative for a_j and positive for a_{j+1}.

(c) Show that for each fixed point there is an a^* such that a flip bifurcation occurs when $a = a^*$.

(d) For $0.1 \le a \le 6$ numerically generate a bifurcation diagram, as well as a plot of the Lyapunov exponent. Compare and contrast your results when compared to those for the logistic map. Do not be afraid to do a thorough exploration! Explain the discrepancies between what you see here and what you see for the logistic map.

Group 2: Consider the tent map given by

$$f(x) = \begin{cases} ax, & 0 \le x < 1/2 \\ a(1-x), & 1/2 < x \le 1. \end{cases}$$

(a) Find a^* such that $f : [0,1] \mapsto [0,1]$ for $0 \le a \le a^*$.

(b) Find all of the fixed points for $0 \le a < 1$. Determine their stability.

(c) Find all of the fixed points for $1 < a \le a^*$. Determine their stability.

(d) Find all of the period-2 points for $1 < a \le a^*$. Determine their stability.

(e) Suppose that for $1 < a \le a^*$ you have a period-N point for some N. Determine the stability (first try this for $a = 2$).

(f) Numerically generate the bifurcation diagram, as well as a plot of the Lyapunov exponent, for $0 < a \le a^*$. For which values of a do you suspect that the solutions are chaotic?

(g) Analytically compute the Lyapunov exponent. Does this computation agree with the numerics?

Group 3: When modeling the dynamics of a blood cell population the mapping is given by

$$f(x) = ax + \tilde{b}x^s \mathrm{e}^{-\tilde{r}x},$$

where x_n represents the population at time n, $0 < a < 1$ is the destruction coefficient, and $b, s, r > 0$. The term $p(x) = \tilde{b}x^s\mathrm{e}^{-\tilde{r}x}$ is called the production function.

(a) Show that the dynamical system

$$x_{n+1} = f(x_n)$$

is equivalent to

$$y_{n+1} = g(y_n), \quad g(y) = ay + by^s\mathrm{e}^{-y}.$$

Explicitly determine b in terms of $\tilde{b}, \tilde{r}$. Henceforth only consider the system $y_{n+1} = g(y_n)$.

(b) For each $0 < a < 1$ and $b, s > 0$ show that there is an interval $I \subset \mathbb{R}^+$ such that $g : I \mapsto I$.

(c) If $s = 1$, find all of the fixed points and determine their stability.

(d) If $s = 1$, find the parameter values for which a flip bifurcation occurs.

(e) If $s = 1$, numerically generate a bifurcation diagram, as well as a plot of the Lyapunov exponent. Compare and contrast your results when compared to those for the logistic map. Do not be afraid to do a thorough exploration!

(f) Suppose that $s = 4$, and set

$$p(y) = by^4 \mathrm{e}^{-y}.$$

- Show that there is a $b^* > 0$ such that if $b > b^*$, then $p(y)$ has two inflection points which lie above the line $z = y$.
- Show that if $b > b^*$ then there exist two positive fixed points, say $0 < y_1 < y_2$.
- As $a \to 0^+$ the fixed point $y_2 \to y_2^*$. Derive a condition which ensures that $p'(y_2^*) < -1$.
- Suppose that b is chosen so that the production function satisfies:
  (a) $p(y)$ has two inflection points which lie above the line $z = y$
  (b) $p'(y_2^*) < -1$.

  Show that there is a unique $0 < a^* < 1$ such that the fixed point y_2 undergoes a flip bifurcation when $a = a^*$.
- For $a < a^*$ numerically generate a bifurcation diagram, as well as a plot of the Lyapunov exponent. Compare and contrast your results when compared to those for the logistic map. Do not be afraid to do a thorough exploration!

Group 4: Consider the Gaussian map given by

$$f(x) = \tilde{a} + \mathrm{e}^{-\tilde{b}x^2}.$$

(a) Show that the dynamical system

$$x_{n+1} = f(x_n)$$

is equivalent to

$$y_{n+1} = g(y_n), \quad g(y) = a + b\mathrm{e}^{-y^2}.$$

Explicitly determine a, b in terms of $\tilde{a}, \tilde{b}$. Henceforth only consider the system $y_{n+1} = g(y_n)$.

(b) For each $a \in \mathbb{R}$ and $b > 0$ find an interval $I \subset \mathbb{R}$ such that $g : I \mapsto I$.

(c) If $a > 0$ is fixed, show that:

- there is a unique fixed point $x^* > 0$
- there is a $b_1 > 0$ such that x^* is unstable for $b > b_1$
- there is a $0 < b_2 \leq b_1$ such that a flip bifurcation occurs at $b = b_2$.

(d) If $a < 0$ is fixed, show that:

- there is a $0 < b_1 < b_2$ such that saddle-node bifurcations occur at $b = b_1$ and $b = b_2$
- if $b_1 < b < b_2$, then there are two negative fixed points, say $x_2 < x_1 < 0$
- no flip bifurcation can occur at $x = x_1, x_2$ for any value of b.

(e) For each fixed $b \in \{1, 2, \ldots, 9\}$ and $-3 \leq a \leq 2$ numerically generate a bifurcation diagram, as well as a plot of the Lyapunov exponent. Compare and contrast your results when compared to those for the logistic map. Discuss the manner in which the various values of b effect your results. Do not be afraid to do a thorough exploration!

Group 5: The shift map is given by

$$f(x) = Nx \pmod 1, \quad N \in \mathbb{N}.$$

(a) For any $N \in \mathbb{N}$ show that

$$f^n(x) = N^n x \pmod 1.$$

(b) For any $N \in \mathbb{N}$ show that periodic points are dense in $[0, 1)$. If possible, give them explicitly for $N = 2$.

(c) For any $N \in \mathbb{N}$ show that there is sensitive dependence upon initial conditions, i.e., there is a $\beta > 0$ such that for any $x_0 \in (0, 1)$ and any open interval $I \subset (0, 1)$ containing x_0 there is a $y_0 \in I$ and $n \in \mathbb{N}$ such that

$$|f^n(x_0) - f^n(y_0)| > \beta.$$

(d) For any $N \in \mathbb{N}$ show that f is transitive on $[0, 1]$, i.e., show that for any intervals $I_1, I_2 \subset [0, 1]$ there is a point $x_0 \in I_1$ and $n \in \mathbb{N}$ such that $f^n(x_0) \in I_2$. In conclusion, as a consequence of (b)-(d) one knows that the dynamics of the shift map are chaotic.

(e) If possible, analytically compute the Lyapunov exponent associated with the dynamics. If it is not possible, compute this exponent numerically.

(f) For a given $x \in [0, 1)$, represent x in its base N form as $.a_0 a_1 a_2 \ldots$, where $a_j \in \{0, 1, \ldots, N - 1\}$. Using this representation of x, give a formula for $f(x)$.

(g) Suppose that $N = 2$. If possible, verify that non-periodic orbits generated by a computer eventually end up fixed at 0. Give an explanation for this unexpected phenomena.

(h) Suppose that $N = 2$. Set

$$y_n = \sin^2(\pi x_n).$$

Show that

$$y_{n+1} = 4y_n(1 - y_n);$$

hence, solutions to the logistic map with $a = 4$ exhibit chaotic behavior.