Functions of One Complex Variable

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1. Fundamental Concepts

Complex analysis is fundamental in areas as diverse as:

(a) mathematical physics

(b) applied mathematics

(c) number theory;

in addition, it is an interesting area in its own right.

1.1. Elementary properties of the complex numbers

Definition 1.1. A complex number $z \in \mathbb{C}$ is denoted by $x + \mathrm{i}y$, where $x, y \in \mathbb{R}$ and $\mathrm{i}^2 = -1$. One has that

$$\operatorname{Re} z := x, \quad \operatorname{Im} z := y$$

are the real and imaginary parts of z. The complex conjugate of z is given by

$$\bar{z} := x - \mathrm{i}y.$$

Let $z_j = x_j + \mathrm{i}y_j$ for $j = 1, 2$. The algebraic operations are given by

$$z_1 + z_2 := (x_1 + x_2) + \mathrm{i}(y_1 + y_2), \quad z_1 \cdot z_2 := (x_1 x_2 - y_1 y_2) + \mathrm{i}(x_1 y_2 + x_2 y_1).$$

It is easy to check that

(a) $\operatorname{Re} z = (z + \bar{z})/2$

(b) $\operatorname{Im} z = (z - \bar{z})/2\mathrm{i}$

(c) $\overline{z + w} = \bar{z} + \bar{w}$

(d) $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$

Definition 1.2. The modulus, or absolute value, of z is given by

$$|z| := \sqrt{z \cdot \bar{z}} = \sqrt{x^2 + y^2}.$$

Remark 1.3. Note that $|\operatorname{Re} z| \le |z|$ and $|\operatorname{Im} z| \le |z|$.

Concerning division, note that

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2},$$

so that every nonzero complex number has a multiplicative inverse. As such, one can write

$$\frac{z}{w} = \frac{z \cdot \bar{w}}{|w|^2}.$$

1.2. Further properties of the complex numbers

Recall that

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

One can then define for $z \in \mathbb{C}$

$$\mathrm{e}^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Upon using the fact that $i^2 = -1$ it is not difficult to check that

$$e^{iy} = \cos y + i \sin y.$$

Furthermore, one can formally manipulate the power series to show that

$$e^z := e^x(\cos y + i \sin y).$$

From this definition it is then not difficult to show that

$$e^z \cdot e^w = e^{z+w}.$$

Now, note that $|e^{iy}| = 1$. Since

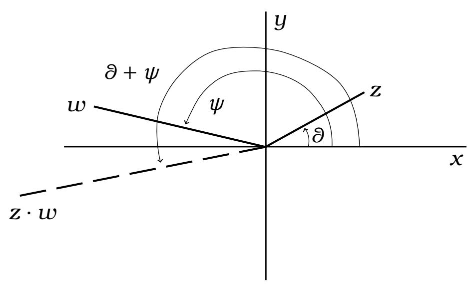
$$z = |z| \cdot \left(\frac{z}{|z|}\right) = z \cdot \xi, \quad |\xi| = 1,$$

there is an angle $\partial := \arg z$ such that $z = re^{i\partial}$, where $r = |z|$. This representation is not unique, as $e^{i\partial} = e^{i(\partial + 2k\pi)}$ for any $k \in \mathbb{Z}$. One typically assumes that $\arg z \in [0, 2\pi)$.

Finally, if $z = re^{i\partial}$ and $w = se^{i\psi}$, then

$$z \cdot w = rse^{i(\partial+\psi)}.$$

Thus, multiplication has the following geometry:



Proposition 1.4 (Triangle inequality). *If $z, w \in \mathbb{C}$, then $|z + w| \leq |z| + |w|$.*

Proof: One can calculate that

$$\begin{aligned} |z+w|^2 &= (z+w) \cdot (\bar{z} + \bar{w}) \\ &= |z|^2 + |w|^2 + z \cdot \bar{w} + \bar{z} \cdot w \\ &= |z|^2 + |w|^2 + 2\,\mathrm{Re}(z \cdot \bar{w}) \\ &\leq |z|^2 + |w|^2 + 2|z| \cdot |w| \\ &= (|z| + |w|)^2. \end{aligned}$$

□

Proposition 1.5 (Cauchy-Schwartz Inequality). *If $z_1, \ldots, z_n, w_1, \ldots, w_n \in \mathbb{C}$, then*

$$\left|\sum_{j=1}^{n} z_j \cdot w_j\right|^2 \leq \sum_{j=1}^{n} |z_j|^2 \sum_{j=1}^{n} |w_j|^2.$$

Proof: See [8, Proposition 1.2.4]. □

Remark 1.6. From now on we will write $zw := z \cdot w$.

1.3. Complex polynomials

A complex polynomial is of the form

$$f(z,\bar{z})=\sum a_{\ell m}z^\ell\bar{z}^m,$$

where $a_{\ell m}\in\mathbb{C}$. A complex polynomial can be written as $f(x,y)=u(x,y)+\mathrm{i}v(x,y)$, where u and v are of the form $\sum b_{\ell m}x^\ell y^m$, with $b_{\ell m}\in\mathbb{R}$.

Definition 1.7. Let $U\subset\mathbb{R}^2$ be open. A continuous function $f\ U\mapsto\mathbb{R}$ is C^1 (or continuously differentiable) on U if $f_x:=\partial f/\partial x$ and $f_y:=\partial f/\partial y$ exist and are continuous on U. In this case we write $f\in C^1(U)$.

Definition 1.8. $f\in C^k(U)$ for $k=1,2,\dots$ if f and all the partial derivatives up to and including order k exist and are continuous on U.

Definition 1.9. $f=u+\mathrm{i}v\,:\,U\mapsto\mathbb{C}$ is $C^k(U)$ if $u,v\in C^k(U)$.

We now wish to define a reasonable derivative for complex polynomials. Set

$$\frac{\partial}{\partial z}:=\frac{1}{2}\left(\frac{\partial}{\partial x}-\mathrm{i}\frac{\partial}{\partial y}\right),\quad \frac{\partial}{\partial\bar{z}}:=\frac{1}{2}\left(\frac{\partial}{\partial x}+\mathrm{i}\frac{\partial}{\partial y}\right).$$

Since $z=x+\mathrm{i}y$ and $\bar{z}=x-\mathrm{i}y$ it can be checked that

$$\frac{\partial}{\partial z}z=1,\ \frac{\partial}{\partial z}\bar{z}=0;\quad \frac{\partial}{\partial\bar{z}}z=0,\ \frac{\partial}{\partial\bar{z}}\bar{z}=1.$$

Note that

$$\frac{\partial}{\partial z}f=\frac{1}{2}(u_x+v_y)+\frac{\mathrm{i}}{2}(-u_y+v_x),\quad \frac{\partial}{\partial\bar{z}}f=\frac{1}{2}(u_x-v_y)+\frac{\mathrm{i}}{2}(u_y+v_x).$$

Proposition 1.10. *The operators*

$$\frac{\partial}{\partial z},\quad \frac{\partial}{\partial\bar{z}}$$

are linear and satisfy the product rule.

Proposition 1.11. *Set*

$$p(z,\bar{z})=\sum a_{\ell m}z^\ell\bar{z}^m.$$

p contains no $\bar{z}$ terms if and only if $\partial p/\partial\bar{z}=0$.

Proof: Suppose that $a_{\ell m}=0$ for all $m>0$, so that

$$p(z,\bar{z})=\sum a_{\ell 0}z^\ell.$$

As a consequence of the product rule one then has that

$$\frac{\partial}{\partial\bar{z}}p=\sum a_{\ell 0}\ell z^{\ell-1}\frac{\partial}{\partial\bar{z}}z=0.$$

Now suppose that $\partial p/\partial\bar{z}=0$. One then has that

$$\frac{\partial^{\ell+m}}{\partial z^\ell\partial\bar{z}^m}p=0$$

for any $m\ge 1$. But,

$$\frac{\partial^{\ell+m}}{\partial z^\ell\partial\bar{z}^m}p(0,0)=\ell!m!a_{\ell m},$$

so that $a_{\ell m}=0$ for any $m\ge 1$. □

1.4. Holomorphic functions, and Cauchy-Riemann equations, and harmonic functions

Definition 1.12. $f \in C^1(U)$ is holomorphic (analytic) if

$$\frac{\partial}{\partial \bar{z}} f = 0$$

at every point of U.

Remark 1.13. A polynomial is holomorphic if and only if it is a function of z alone.

Lemma 1.14. *$f = u + \mathrm{i}v \in C^1(U)$ is holomorphic if and only if it satisfies the Cauchy-Riemann equations*

$$u_x = v_y, \quad u_y = -v_x$$

at every point of U.

Proof: This follows immediately from the fact that

$$\frac{\partial}{\partial \bar{z}} f = \frac{1}{2}(u_x - v_y) + \frac{\mathrm{i}}{2}(u_y + v_x). \qquad \square$$

Corollary 1.15. *If f is holomorphic, then*

$$\frac{\partial}{\partial z} f = f_x = -\mathrm{i} f_y$$

on U.

Proof: By definition

$$\frac{\partial}{\partial z} f = \frac{1}{2}(u_x + v_y) + \frac{\mathrm{i}}{2}(-u_y + v_x).$$

By the Cauchy-Riemann equations the above can be rewritten as

$$\frac{\partial}{\partial z} f = u_x + \mathrm{i}v_x = -\mathrm{i}(u_y + \mathrm{i}v_y). \qquad \square$$

Now suppose that f is holomorphic and satisfies the Cauchy-Riemann equations. If $f \in C^2(U)$, then by applying the appropriate derivatives to the Cauchy-Riemann equations one gets that

$$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0.$$

Definition 1.16. The Laplace operator (Laplacian) is given by

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

If $u \in C^2(U)$, then u is called harmonic if $\Delta u = 0$ on U.

Remark 1.17. It can be checked that

$$\Delta = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}.$$

Lemma 1.18. *Let $u(x, y)$ be a real-valued harmonic polynomial. There is a holomorphic polynomial $Q(z)$ such that $u = \operatorname{Re} Q$.*

Proof: Recall that

$$x = \operatorname{Re} z = \frac{1}{2}(z + \bar{z}), \quad y = \operatorname{Im} z = \frac{1}{2\mathrm{i}}(z - \bar{z}),$$

Set

$$P(z, \bar{z}) := u\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2\mathrm{i}}\right) = \sum a_{\ell m} z^{\ell} \bar{z}^{m}.$$

Since u is harmonic, one has that

$$\frac{\partial^{\ell+m}}{\partial \bar{z}^{\ell} \partial \bar{z}^{m}} P = 0$$

for any $\ell \geq 1$ and $m \geq 1$. Thus, $a_{\ell m} = 0$ for $\ell \geq 1$ and $m \geq 1$, so that

$$P(z, \bar{z}) = a_{00} + \sum_{\ell \geq 1} a_{\ell 0} z^{\ell} + \sum_{m \geq 1} a_{0m} \bar{z}^{m}.$$

Since P is real-valued, one has that $P = \bar{P}$; hence,

$$a_{00} = \overline{a_{00}}, \quad a_{\ell 0} = \overline{a_{0\ell}}.$$

Note that this implies that $a_{00} \in \mathbb{R}$. Thus,

$$\begin{aligned} P(z, \bar{z}) &= a_{00} + \sum_{\ell \geq 1} a_{\ell 0} z^{\ell} + \sum_{\ell \geq 1} \overline{a_{\ell 0}} \bar{z}^{\ell} \\ &= \mathrm{Re}(a_{00} + 2 \sum_{\ell \geq 1} a_{\ell 0} z^{\ell}) \\ &:= \mathrm{Re}\, Q(z). \end{aligned}$$ □

Example. Let us give an explicit description of all real-valued harmonic polynomials of second degree. By the above lemma it is known that $u = \mathrm{Re}\, Q$, where Q is a holomorphic second degree polynomial. Since

$$Q(z) = a_0 + a_1 z + a_2 z^2, \quad a_j \in \mathbb{C},$$

by setting $a_j := a_{j,\mathrm{r}} + \mathrm{i} a_{j,\mathrm{i}}$, and using that fact that $z^2 = x^2 - y^2 + \mathrm{i}2xy$, it is seen that

$$\mathrm{Re}\, Q = a_{0,\mathrm{r}} + a_{1,\mathrm{r}} x - a_{1,\mathrm{i}} y + a_{2,\mathrm{r}}(x^2 - y^2) - 2a_{2,\mathrm{i}} xy,$$

i.e., a holomorphic second degree polynomial is a linear combination of $1, x, y, x^2 - y^2, xy$.

1.5. Real and holomorphic antiderivatives

Lemma 1.19. *Let U be an open convex set, and suppose that $f, g \in C^1(U)$ satisfy $f_y = g_x$. There is a function $h \in C^2(U)$ such that $f = h_x$ and $g = h_y$. Furthermore, if f and g are real-valued, then so is h.*

Proof: A Math 311 problem (see [8, Theorem 1.5.1]). □

Remark 1.20. By setting $f = -u_y$ and $g = u_x$, it is seen from the above that if u is harmonic, then there is a function v such that

$$v_x = -u_y, \quad v_y = u_x.$$

By setting $f = u + \mathrm{i}v$, one then gets that f is holomorphic. Hence, if u is harmonic there is a holomorphic function f such that $u = \mathrm{Re} f$.

Theorem 1.21. *Let U be an open convex set, and suppose that f is holomorphic on U. There is a holomorphic function F such that $f = \partial F/\partial z$.*

Proof: Since f satisfies the Cauchy-Riemann equations, there exist functions $h_1, h_2 \in C^2(U)$ such that

$$u = \frac{\partial}{\partial x} h_1 = \frac{\partial}{\partial y} h_2, \quad v = \frac{\partial}{\partial x} h_2 = -\frac{\partial}{\partial y} h_1.$$

Set $F = h_1 + \mathrm{i}h_2$. It is clear that F satisfies the Cauchy-Riemann equations. Furthermore,

$$\frac{\partial}{\partial z} F = F_x = u + \mathrm{i}v = f.$$ □

2. Complex Line Integrals

2.1. Real and complex line integrals

Definition 2.1. $\phi : [a,b] \mapsto \mathbb{R}$ satisfies $\phi \in C^1([a,b])$ if

(a) ϕ' exists and is continuous on (a,b)

(b) $\lim_{t\to a^+} \phi'(t)$ and $\lim_{t\to b^-} \phi'(t)$ exist.

Definition 2.2. Let $\gamma := \gamma_1 + \mathrm{i}\gamma_2 : [a,b] \mapsto \mathbb{C}$. We write $\gamma \in C^1([a,b])$ if $\gamma_1, \gamma_2 \in C^1([a,b])$. In this case

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\mathrm{d}\gamma_1}{\mathrm{d}t} + \mathrm{i}\frac{\mathrm{d}\gamma_2}{\mathrm{d}t} \quad (= \gamma'(t)).$$

Remark 2.3. Note that if $\gamma \in C^1([a,b])$, then

$$\gamma(b) - \gamma(a) = \int_a^b \gamma'(t)\,\mathrm{d}t,$$

where

$$\int_a^b \gamma'(t)\,\mathrm{d}t = \int_a^b \gamma_1'(t)\,\mathrm{d}t + \mathrm{i}\int_a^b \gamma_2'(t)\,\mathrm{d}t.$$

Let $U \subset \mathbb{C}$ be open, and let $\gamma : [a,b] \mapsto U$ be C^1. Suppose that $f = u + \mathrm{i}v : U \mapsto \mathbb{C}$ and $f \in C^1(U)$ is holomorphic. Upon setting

$$u(\gamma(t)) := u(\gamma_1(t), \gamma_2(t)), \quad v(\gamma(t)) := v(\gamma_1(t), \gamma_2(t)),$$

an application of the chain rule yields that

$$u(\gamma(b)) - u(\gamma(a)) = \int_a^b \{\frac{\partial u}{\partial x}(\gamma(t))\gamma_1'(t) + \frac{\partial u}{\partial y}(\gamma(t))\gamma_2'(t)\}\,\mathrm{d}t$$

(similar statement for v). By using the Cauchy-Riemann equations it is seen that

$$\begin{aligned} f(\gamma(b)) - f(\gamma(a)) &= \int_a^b \frac{\partial f}{\partial x}(\gamma(t))\gamma'(t)\,\mathrm{d}t \\ &= \int_a^b \frac{\partial f}{\partial z}(\gamma(t))\gamma'(t)\,\mathrm{d}t. \end{aligned}$$

Definition 2.4. Let $U \subset \mathbb{C}$ be open, and let $\gamma : [a,b] \mapsto U$ be C^1. If $F : u \mapsto \mathbb{C}$ is continuous, the complex line integral is defined by

$$\oint_\gamma F(z)\,\mathrm{d}z := \int_a^b F(\gamma(t))\gamma'(t)\,\mathrm{d}t.$$

Proposition 2.5. *If f is holomorphic on U, then*

$$f(\gamma(b)) - f(\gamma(a)) = \oint_\gamma \frac{\partial f}{\partial z}(z)\,\mathrm{d}z.$$

Remark 2.6. One has:

(a) The above proposition can be restated to say that holomorphic functions satisfy a version of the Fundamental Theorem of Calculus.

(b) The result is independent of the "speed" at which the path $\gamma(t)$ is traversed.

2.2. Complex differentiability and conformality

Definition 2.7. Let $U \subset \mathbb{C}$ be open, and let $f : U \mapsto \mathbb{C}$. If the limit exists, for $z_0 \in U$ one writes

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Theorem 2.8. *Let $U \subset \mathbb{C}$ be open, and suppose that f is holomorphic on U. Then f' exists at each point of U, and*

$$f'(z) = \frac{\partial f}{\partial z}.$$

Remark 2.9. As a consequence of the previous proposition, holomorphic functions satisfy the Fundamental Theorem of Calculus, i.e.,

$$f(\gamma(b)) - f(\gamma(a)) = \oint_\gamma f'(z)\,\mathrm{d}z.$$

Proof: Let $z_0 \in U$ be given. Since U is open, there is a $\delta > 0$ such that the set

$$B(z_0, \delta) := \{z \in U : |z - z_0| < \delta\} \subset U.$$

Pick $z \in B(z_0, \delta)$, and define

$$\gamma(t) = z_0 + (z - z_0)t;$$

note that $\gamma : [0, 1] \mapsto B(z_0, \delta)$ with $\gamma(0) = z_0$ and $\gamma(1) = z$. Since

$$\begin{aligned} f(z) - f(z_0) &= \oint_\gamma \frac{\partial f}{\partial z}\,\mathrm{d}z \\ &= \int_0^1 \frac{\partial f}{\partial z}(\gamma(t))(z - z_0)\,\mathrm{d}t, \end{aligned}$$

one gets that

$$\begin{aligned} \frac{f(z) - f(z_0)}{z - z_0} &= \int_0^1 \frac{\partial f}{\partial z}(\gamma(t))\,\mathrm{d}t \\ &= \frac{\partial f}{\partial z}(z_0) + \int_0^1 \left[\frac{\partial f}{\partial z}(\gamma(t)) - \frac{\partial f}{\partial z}(z_0)\right]\mathrm{d}t. \end{aligned}$$

Since $f \in C^1(U)$, for each $\epsilon > 0$ there is a $\delta > 0$ such that for $w \in B(z_0, \delta)$ one has

$$\left|\frac{\partial f}{\partial z}(w) - \frac{\partial f}{\partial z}(z_0)\right| < \epsilon.$$

In particular, since $|\gamma(t) - z_0| = t|z - z_0| < \delta$ for $t \in [0, 1]$, one has that

$$\left|\frac{\partial f}{\partial z}(\gamma(t)) - \frac{\partial f}{\partial z}(z_0)\right| < \epsilon.$$

Hence,

$$\begin{aligned} \left|\frac{f(z) - f(z_0)}{z - z_0} - \frac{\partial f}{\partial z}(z_0)\right| &= \left|\int_0^1 \left[\frac{\partial f}{\partial z}(\gamma(t)) - \frac{\partial f}{\partial z}(z_0)\right]\mathrm{d}t\right| \\ &\le \int_0^1 \epsilon\,\mathrm{d}t = \epsilon, \end{aligned}$$

which shows that the limit exists and equals $(\partial f/\partial z)(z_0)$. □

Theorem 2.10. *Suppose that $f \in C(U)$ is such that f' exists at each point of U. Then f is holomorphic on U.*

Remark 2.11. As a consequence, f is holomorphic if and only if f' exists.

Proof: We need to check that the Cauchy-Riemann equations are satisfied. Suppose that $h \in \mathbb{R}$. One then has that

$$\begin{aligned}\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} &= \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}\\ &= \lim_{h\to 0}\frac{u(x_0+h,y_0)-u(x_0,y_0)}{h} + \mathrm{i}\lim_{h\to 0}\frac{v(x_0+h,y_0)-v(x_0,y_0)}{h}\\ &= \frac{\partial f}{\partial x}(x_0,y_0).\end{aligned}$$

Similarly,

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} = -\mathrm{i}\frac{\partial f}{\partial y}(x_0,y_0).$$

Thus, $f_x(x_0,y_0) = -\mathrm{i}f_y(x_0,y_0)$, which implies that the Cauchy-Riemann equations hold. □

2.3. Antiderivatives revisited

Let $U \subset \mathbb{C}$ be an open convex set, let $f : U \mapsto \mathbb{C}$ be holomorphic, and let $z_0 \in U$ be fixed. For $z \in U$ set

$$F(z) := \begin{cases}\dfrac{f(z)-f(z_0)}{z-z_0}, & z \in U\backslash\{z_0\},\\ f'(z_0), & z = z_0.\end{cases}$$

Since f is holomorphic, F is continuous on U; furthermore, it is holomorphic on $U\backslash\{z_0\}$. The appropriate modification of Theorem 1.21 (see [**8**, Theorem 2.3.2]) yields:

Theorem 2.12. *There is a holomorphic H on U such that $H'(z) = F(z)$.*

2.4. The Cauchy integral formula and the Cauchy integral theorem

Definition 2.13. For $P \in \mathbb{C}$ fixed, set

$$D(P,r) := \{z \in \mathbb{C} : |z-P| < r\}, \quad \overline{D}(P,r) := \{z \in \mathbb{C} : |z-P| \le r\},$$

and

$$\partial D(P,r) := \{z \in \mathbb{C} : |z-P| = r\}.$$

Remark 2.14. Note that $\partial D(P,r)$ can be parameterized as the curve $\gamma : [0,1] \mapsto \partial D(P,r)$ by setting

$$\gamma(t) := P + r\mathrm{e}^{2\pi\mathrm{i}t}.$$

This is a counterclockwise orientation for γ, and unless stated explicitly otherwise, this is the orientation that will always be assumed.

Lemma 2.15. *Let $z \in D(z_0,r)$, and let γ be $\partial D(z_0,r)$. Then*

$$\frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{1}{\zeta - z}\,\mathrm{d}\zeta = 1.$$

Proof: Set

$$I(z) := \frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{1}{\zeta - z}\,\mathrm{d}\zeta.$$

Setting $\gamma(t) = z_0 + r\mathrm{e}^{2\pi\mathrm{i}t}$ yields that

$$\begin{aligned}I(z_0) &= \frac{1}{2\pi\mathrm{i}}\int_0^1 \frac{2\pi\mathrm{i}\mathrm{e}^{2\pi\mathrm{i}t}}{\mathrm{e}^{2\pi\mathrm{i}t}}\,\mathrm{d}t\\ &= 1.\end{aligned}$$

Let us now show that $I(z)$ is independent of z. Upon doing so, the lemma will be proved. First, we have that

$$\frac{\partial}{\partial\bar{z}}I(z) = \frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{\partial}{\partial\bar{z}}\left(\frac{1}{\zeta - z}\right)\mathrm{d}\zeta = 0,$$

so that $I(z)$ is holomorphic on $D(z_0, r)$. Similarly, upon using the Fundamental Theorem of Calculus,

$$\begin{aligned}\frac{\partial}{\partial z}I(z) &= \frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{\partial}{\partial z}\left(\frac{1}{\zeta - z}\right)\mathrm{d}\zeta \\ &= -\frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{1}{\zeta - z}\right)\mathrm{d}\zeta \\ &= -\left(\frac{1}{\gamma(1) - z} - \frac{1}{\gamma(0) - z}\right) \\ &= 0.\end{aligned}$$

Hence, $I(z)$ is constant. □

Remark 2.16. One has that:

(a) if the contour is traversed in the clockwise direction, then

$$I(z) = -2\pi\mathrm{i}.$$

(b) the same argument yields

$$\frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{1}{(\zeta - z)^{k+1}}\mathrm{d}\zeta = 0$$

for any $k \in \mathbb{N}$.

Theorem 2.17 (Cauchy Integral Formula)**.** *Let $U \subset \mathbb{C}$ be open, and suppose that $f : U \mapsto \mathbb{C}$ is holomorphic. Let $z_0 \in U$, and let $r > 0$ be such that $\overline{D}(z_0, r) \subset U$. Let $\gamma(t) = z_0 + r\mathrm{e}^{2\pi\mathrm{i}t}$. For each $z \in D(z_0, r)$ one has*

$$f(z) = \frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta.$$

Remark 2.18. One has:

(a) $f(z)$ is determined by the values of f on $\partial D(z_0, r)$

(b) it will later be seen that if f is given by the Cauchy integral formula, then it is holomorphic

Proof: Choose $\epsilon > 0$ so that $D(z_0, r+\epsilon) \subset U$, and fix $z \in D(z_0, r+\epsilon)$. By Theorem 2.12 there is a holomorphic function $H : D(z_0, r+\epsilon) \mapsto \mathbb{C}$ such that

$$H'(\zeta) = \begin{cases} \dfrac{f(\zeta) - f(z)}{\zeta - z}, & \zeta \neq z, \\ f'(z), & \zeta = z. \end{cases}$$

Now choose $z \in D(z_0, r)$. By the Fundamental Theorem of Calculus one has

$$0 = \oint_\gamma H'(\zeta)\,\mathrm{d}\zeta = \oint_\gamma \frac{f(\zeta) - f(z)}{\zeta - z}\mathrm{d}\zeta;$$

hence,

$$\oint_\gamma \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta = \oint_\gamma \frac{f(z)}{\zeta - z}\mathrm{d}\zeta.$$

An evaluation of the second integral yields that

$$\oint_\gamma \frac{f(z)}{\zeta - z}\mathrm{d}\zeta = f(z)\oint_\gamma \frac{1}{\zeta - z}\mathrm{d}\zeta = 2\pi\mathrm{i}f(z).$$ □

Theorem 2.19 (Cauchy Integral Theorem). *Let $U \subset \mathbb{C}$ be an open convex set, let $f : U \mapsto \mathbb{C}$ be holomorphic, and let $\gamma : [a, b] \mapsto U$ be C^1 with $\gamma(a) = \gamma(b)$. Then*

$$\oint_\gamma f(z)\,\mathrm{d}z = 0.$$

Proof: There is a holomorphic function $F : U \mapsto \mathbb{C}$ such that $F' = f$. By the Fundamental Theorem of Calculus,

$$0 = F(\gamma(b)) - F(\gamma(a)) = \oint_\gamma F'(z)\,\mathrm{d}z,$$

which proves the result. □

Example. One has:

(a) **Necessity of holomorphicity:** Consider

$$\oint_\gamma \frac{\bar{\zeta}}{\zeta - 1}\,\mathrm{d}\zeta, \quad \gamma = \partial D(1, 1).$$

It can be checked that this integral is zero. Note, however, that

$$\oint_\gamma \frac{\zeta}{\zeta - 1}\,\mathrm{d}\zeta = 2\pi\mathrm{i}.$$

(b) **Importance of orientation:** In the proof of Theorem 2.17 the orientation of the curve plays a role via Lemma 2.15. If the curve were traversed in a clockwise direction, then as a consequence of Remark 2.16 one would see that

$$f(z) = -\frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

2.6. *An introduction to the Cauchy integral theorem and the Cauchy integral formula for more general curves*

Definition 2.20. A piecewise C^1 curve $\gamma : [a, b] \mapsto \mathbb{C}$ is a continuous curve such that there exists a finite set $a = a_1 < a_2 < \cdots < a_k = b$ with the property that $\gamma|_{[a_j, a_{j+1}]}$ is a C^1 curve.

Definition 2.21. Let $U \subset \mathbb{C}$ be open, and suppose that $\gamma : [a, b] \mapsto U$ is a piecewise C^1 curve. If $f \in C(U)$, then

$$\oint_\gamma f(z)\,\mathrm{d}z = \sum_{j=1}^{k} \oint_{\gamma|_{[a_j, a_{j+1}]}} f(z)\,\mathrm{d}z.$$

Lemma 2.22. *Suppose that $f : U \mapsto \mathbb{C}$ is holomorphic, and suppose that $\gamma : [a, b] \mapsto U$ is a piecewise C^1 curve. Then*

$$f(\gamma(b)) - f(\gamma(a)) = \oint_\gamma f'(z)\,\mathrm{d}z.$$

Proof: The result is true over each segment $\gamma|_{[a_j, a_{j+1}]}$. The definition of the integral yields the final result. □

Let $f : U \mapsto \mathbb{C}$ be holomorphic, and let γ, μ be piecewise C^1 curves contained in U. Suppose that there is a disk $D(z_0, r) \subset U$ such that outside this disk, $\gamma = \mu$. Recall that there is a holomorphic function F such that $F' = f$ on $D(z_0, r)$, so that the Fundamental Theorem of Calculus applies. Inside the disk one must then have that

$$\oint_{\gamma \cap D(z_0, r)} f(z)\,\mathrm{d}z = \oint_{\mu \cap D(z_0, r)} f(z)\,\mathrm{d}z.$$

Since the two curves coincide outside $D(z_0, r)$, one then has that

$$\oint_\gamma f(z)\,\mathrm{d}z = \oint_\mu f(z)\,\mathrm{d}z.$$

Proposition 2.23. *Let $0 < r < R < +\infty$, and define the annulus*

$$\mathscr{A} := \{z \in \mathbb{C} \,:\, r < |z| < R\}.$$

Let $f : \mathscr{A} \mapsto \mathbb{C}$ be holomorphic. If $r < r_1 < r_2 < R$, and if $\gamma_{r_j} := \partial D(0, r_j)$ traversed in the counterclockwise direction, then

$$\oint_{\gamma_{r_1}} f(z)\,\mathrm{d}z = \oint_{\gamma_{r_2}} f(z)\,\mathrm{d}z.$$

Proof: Continuously deform one circle into the other, and use the above argument. □

Remark 2.24. The curves actually only need to be such that they can be continuously deformed to a circle which is entirely contained in the open set U.

Theorem 2.25. *Let $U \subset \mathbb{C}$ be open, and let $f : U \mapsto \mathbb{C}$ be holomorphic. Let $\gamma \subset U$ be a piecewise C^1 closed curve which can be continuously deformed in U to a closed curve lying entirely in a disc contained in U. Then*

$$\oint_\gamma f(z)\,\mathrm{d}z = 0.$$

Suppose that $\bar{D}(z, r) \subset U$, and suppose $\gamma \subset U \backslash \{z\}$ can be continuously deformed in $U \backslash \{z\}$ to $\partial D(z, r)$ equipped with the counterclockwise orientation. Then

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

Example. Let γ be the square centered at the origin with sides of length six, equipped with the counterclockwise orientation. By the above theorem,

$$\oint_\gamma \frac{5}{(\zeta - 1)(\zeta - 2\mathrm{i})}\,\mathrm{d}\zeta = \oint_{\partial D(0,3)} \frac{5}{(\zeta - 1)(\zeta - 2\mathrm{i})}\,\mathrm{d}\zeta.$$

Thus,

$$\begin{aligned}\oint_{\partial D(0,3)} \frac{5}{(\zeta - 1)(\zeta - 2\mathrm{i})}\,\mathrm{d}\zeta &= \oint_{\partial D(0,3)} \frac{1 - 2\mathrm{i}}{\zeta - 1}\,\mathrm{d}\zeta - \oint_{\partial D(0,3)} \frac{1 - 2\mathrm{i}}{\zeta - 2\mathrm{i}}\,\mathrm{d}\zeta \\ &= 0.\end{aligned}$$

3. Applications of the Cauchy Integral

3.1. *Differentiability properties of holomorphic functions*

Theorem 3.1. *Let φ be continuous on $\partial D(P, r)$. The function*

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\partial D(P,r)} \frac{\varphi(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

is defined and holomorphic on $D(P, r)$.

Remark 3.2. First suppose that $P = 0$, $r = 1$, and $\varphi(\zeta) = \bar{\zeta}$. It is easy to check that $f(z) = 0$ for all $z \in D(0, 1)$, and hence has no real relation to $\varphi(z)$. Now suppose that $\varphi(z)$ is holomorphic in a neighborhood of $\overline{D}(P, r)$. As an application of Theorem 2.17 one immediately sees that $f(z) = \varphi(z)$ for all $z \in D(P, r)$.

Proof: Set

$$g(\zeta) := \frac{\varphi(\zeta)}{\zeta - z}.$$

Since $z \in D(P, r)$, it is clear that $g(\zeta)$ is continuous on $\partial D(P, r)$; hence, f is well-defined. Now set

$$h(z) := \frac{\varphi(\zeta)}{\zeta - z}.$$

Since $|\zeta - z| \geq r - |z - P| > 0$ for all $\zeta \in \partial D(P,r)$, one has that $h(w) \to h(z)$ as $w \to z$ uniformly over $\zeta \in \partial D(P,r)$. Thus, one can interchange the order of the limit and integration to get

$$\begin{aligned}\frac{\partial f}{\partial z}(z) &= \frac{1}{2\pi \mathrm{i}} \frac{\partial}{\partial z} \oint_{\partial D(P,r)} \frac{\varphi(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{\partial D(P,r)} \frac{\partial}{\partial z} \frac{\varphi(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{\partial D(P,r)} \frac{\varphi(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta,\end{aligned}$$

and similarly,

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}}(z) &= \frac{1}{2\pi \mathrm{i}} \oint_{\partial D(P,r)} \frac{\partial}{\partial \bar{z}} \frac{\varphi(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \\ &= 0.\end{aligned}$$

Hence, f is holomorphic for all $z \in D(P,r)$, and as a consequence of Theorem 2.8 one has that

$$f'(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\partial D(P,r)} \frac{\varphi(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta. \qquad \square$$

Remark 3.3. This argument shows that f is differentiable to any order, with

$$f^{(k)}(z) = \frac{k!}{2\pi \mathrm{i}} \oint_{\partial D(P,r)} \frac{\varphi(\zeta)}{(\zeta - z)^{k+1}}\,\mathrm{d}\zeta.$$

Corollary 3.4. *Let $U \subset \mathbb{C}$ be an open set, and let $f : U \mapsto \mathbb{C}$ be holomorphic. Then $f \in C^\infty(U)$. Furthermore, if $\overline{D}(P,r) \subset U$ and $z \in D(P,r)$, then*

$$f^{(k)}(z) = \frac{k!}{2\pi \mathrm{i}} \oint_{\partial D(P,r)} \frac{f(\zeta)}{(\zeta - z)^{k+1}}\,\mathrm{d}\zeta.$$

Proof: Follows from the above theorem and the representation of holomorphic functions given in Theorem 2.17. $\square$

Corollary 3.5. *If $f : U \mapsto \mathbb{C}$ is holomorphic, then $f^{(k)} : U \mapsto \mathbb{C}$ is holomorphic.*

Proof: The result follows immediately upon applying the proof of Theorem 3.1 to the representation of $f^{(k)}(z)$ given in Corollary 3.4. $\square$

3.2. Complex power series

For a holomorphic function f, if given $p \in U$ we can now formally write a power series of the form

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!}(z-p)^n.$$

Two questions:

(a) Does the series converge?

(b) If it does converge, does it converge to $f(z)$?

Remark 3.6. The student should review power series from Math 163!

Definition 3.7. Let $P \in \mathbb{C}$ be fixed. A complex power series is of the form

$$\sum_{n=0}^{\infty} a_n(z-P)^n,$$

where the $\{a_k\}_{k=0}^{\infty}$ are complex constants.

Lemma 3.8 (Abel)**.** *If a power series converges at some point z, then it converges for each $w \in D(P, r)$, where $r = |z - P|$.*

Proof: Since the power series converges, one has that $|a_k(z-P)^k| \to 0$ as $k \to \infty$; hence, there is an $M > 0$ such that $|a_k(z-P)^k| \le M$ for all k. This in turn implies that $|a_k| \le Mr^{-k}$ for all k. For each k one has

$$|a_k(w-P)^k| \le |a_k|\,|w-P|^k \le M\left(\frac{|w-P|}{r}\right)^k;$$

hence, for fixed $w \in D(P, r)$ one has that the geometric series $\sum(|w-P|/r)^k$ converges. The series itself then converges, as it converges absolutely. □

Definition 3.9. For a power series, set

$$r := \sup\{|w-P| \,:\, \sum a_n(w-P)^n \text{ converges}\}.$$

Then r is the radius of convergence of the power series, and $D(P, r)$ is the disk of convergence.

Lemma 3.10. *A power series with radius of convergence r converges for each $w \in D(P, r)$, and diverges for each w such that $|w - P| > r$.*

Proof: This is a restatement of Abel's lemma. □

Lemma 3.11 (Ratio test)**.** *The radius of convergence of the power series is given by*

$$\frac{1}{r} = \limsup_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|.$$

Proof: See any textbook in Analysis. □

Remark 3.12. By the root test, the radius of convergence is given by

$$\frac{1}{r} = \limsup_{k\to\infty}|a_k|^{1/k}.$$

The series $\sum_{k=0}^{\infty} f_k(z)$ is uniformly Cauchy on a set U if for each $\epsilon > 0$ there is an N such that if $n \ge m > N$, then

$$\left|\sum_{k=m}^{n} f_k(z)\right| < \epsilon, \quad z \in U.$$

It is known that the a uniformly Cauchy series converges uniformly on U to some limit function. Following the proof of Abel's lemma, it is easy to check that the power series $\sum a_k(z-P)^k$ converges uniformly and absolutely on $\overline{D}(P, R)$, where $0 < R < r$ and r is the radius of convergence. Hence, there is an $f(z)$ such that

$$f(z) = \sum_{k=0}^{\infty} a_k(z-P)^k, \quad |z-P| < r.$$

Lemma 3.13. *Consider*

$$f(z) = \sum_{k=0}^{\infty} a_k(z-P)^k, \quad |z-P| < r,$$

where $r > 0$ is the radius of convergence. Then f is holomorphic on $D(P, r)$. Furthermore, for each $n \ge 1$ one can differentiate termwise to get,

$$f^{(n)}(z) = \sum_{k=n}^{\infty} k(k-1)\ldots(k-n+1)a_k(z-P)^{k-n}, \quad z \in D(P, r).$$

Remark 3.14. Evaluation at $z = P$ reveals $f^{(n)}(P) = n!\,a_n$.

Proof: Without loss of generality suppose that $P = 0$. It will be shown only for $n = 1$, as the rest follows from an induction argument. Let $z \in D(0, r)$ be fixed, and let $|h| \le (r - |z|)/2$. Consider

$$f(z+h) - f(z) = \sum_{k=1}^{\infty} [a_k (z+h)^k - a_k z^k].$$

Since z^k is holomorphic, one can write

$$(z+h)^k - z^k = hk \int_0^1 (z+th)^{k-1}\,\mathrm{d}t;$$

hence,

$$\frac{f(z+h) - f(z)}{h} = \sum_{k=1}^{\infty} k a_k \int_0^1 (z+th)^{k-1}\,\mathrm{d}t. \tag{3.1}$$

Note that

$$\lim_{h\to 0} \int_0^1 (z+th)^{k-1}\,\mathrm{d}t = z^{k-1}.$$

Now,

$$|\int_0^1 (z+th)^{k-1}\,\mathrm{d}t| \le (|z| + |h|)^{k-1},$$

so that

$$\begin{aligned} |k a_k \int_0^1 (z+th)^{k-1}\,\mathrm{d}t| &\le k|a_k|(|z| + |h|)^{k-1} \\ &\le k|a_k| \left(\frac{r+|z|}{2}\right)^{k-1}. \end{aligned}$$

By the ratio test the series $\sum k|a_k|((r + |z|)/2)^{k-1}$ converges, so by the Weierstrass M-test the series in equation (3.1) converges uniformly in h. Hence, upon taking the limit as $h \to 0$, and interchanging the limit and the summation, one gets that

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \sum_{k=1}^{\infty} k a_k z^{k-1}.$$

By the ratio test the series on the right-hand side also converges uniformly for $z \in D(0, r)$. □

Lemma 3.15. *Suppose that $\sum a_k (z - P)^k$ and $\sum b_k (z - P)^k$ converge on $D(P, r)$, and suppose that*

$$\sum a_k (z-P)^k = \sum b_k (z-P)^k, \quad z \in D(P, r).$$

Then $a_k = b_k$ for all k.

Proof: Differentiate term-by-term and evaluate at $z = P$ to get the result. □

Remark 3.16. As a consequence, a power series on $D(P, r)$ is unique.

3.3. *The power series expansion for a holomorphic function*

We now know that a power series defines a holomorphic function. Furthermore, a holomorphic function has a formal power series. *If* the formal power series converges, then by the uniqueness lemma one has that a function is holomorphic on $D(p, r)$ if and only if

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!} (z-p)^n, \quad z \in D(p, r).$$

Theorem 3.17. *Let $U \subset \mathbb{C}$ be open, and let $f : u \mapsto \mathbb{C}$ be holomorphic. Let $P \in U$, and suppose that $D(P, r) \subset U$. Then*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(p)}{n!}(z-p)^n, \quad z \in D(p, r).$$

Remark 3.18. If $\hat{r}$ is the distance from P to $\mathbb{C}\backslash U$, then the power series will converge at least on $D(P, \hat{r})$.

Proof: Assume without loss of generality that $P = 0$. Given $z \in D(0, r)$, choose $|z| < r' < r$, so that $z \in D(0, r') \subset \overline{D}(0, r') \subset D(0, r)$. For $z \in D(0, r')$ one has that

$$\begin{aligned} f(z) &= \frac{1}{2\pi\mathrm{i}} \oint_{|\zeta|=r'} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta \\ &= \frac{1}{2\pi\mathrm{i}} \oint_{|\zeta|=r'} \frac{f(\zeta)}{\zeta} \frac{1}{1 - z/\zeta} \,\mathrm{d}\zeta \\ &= \frac{1}{2\pi\mathrm{i}} \oint_{|\zeta|=r'} \frac{f(\zeta)}{\zeta} \sum_{k=0}^{\infty} (z/\zeta)^k \,\mathrm{d}\zeta. \end{aligned}$$

The sum converges absolutely and uniformly on $\overline{D}(0, r')$, as $|z/\zeta| < 1$. As a consequence, the order of integration and summation can be interchanged, so that

$$\begin{aligned} f(z) &= \frac{1}{2\pi\mathrm{i}} \sum_{k=0}^{\infty} \oint_{|\zeta|=r'} \frac{f(\zeta)}{\zeta} (z/\zeta)^k \,\mathrm{d}\zeta \\ &= \frac{1}{2\pi\mathrm{i}} \sum_{k=0}^{\infty} z^k \oint_{|\zeta|=r'} \frac{f(\zeta)}{\zeta^{k+1}} \,\mathrm{d}\zeta \\ &= \sum_{k=0}^{\infty} z^k \frac{f^{(k)}(0)}{k!}. \end{aligned}$$

□

Example. Consider $f(z) = 1/(z - 3\mathrm{i})$. The power series expansion about $z = 0$ is given by

$$f(z) = \frac{\mathrm{i}}{3} \sum_{n=0}^{\infty} \left(\frac{z}{3\mathrm{i}}\right)^n, \quad z \in D(0, 3).$$

The expansion about $z = -\mathrm{i}$ is given by

$$f(z) = \frac{\mathrm{i}}{4} \sum_{n=0}^{\infty} \left(\frac{z + \mathrm{i}}{4\mathrm{i}}\right)^n, \quad z \in D(-\mathrm{i}, 4).$$

3.4. The Cauchy estimates and Liouville's theorem

Remark 3.19. In all that follows, unless explicitly stated otherwise, it will be assumed that $U \subset \mathbb{C}$ is open, and that $f : U \mapsto \mathbb{C}$ is holomorphic.

Theorem 3.20 (Cauchy estimates)**.** *Suppose that $\overline{D}(P, r) \subset U$ for a given $P \in U$. Set*

$$M := \sup_{z \in \overline{D}(P,r)} |f(z)|.$$

Then for each $k \in \mathbb{N}$ one has

$$|f^{(k)}(P)| \leq \frac{Mk!}{r^k}.$$

Proof: Since f is continuous, the bound $M < \infty$ exists. Now

$$\begin{aligned}|f^{(k)}(P)| &= \left|\frac{k!}{2\pi \mathrm{i}}\oint_{|\zeta-P|=r}\frac{f(\zeta)}{(\zeta-P)^{k+1}}\,\mathrm{d}\zeta\right| \\ &\leq \frac{k!}{2\pi}\cdot 2\pi r\cdot\frac{M}{r^{k+1}} \\ &= \frac{Mk!}{r^k}.\end{aligned}$$

□

Remark 3.21. The fact that f is holomorphic is crucial in this estimate.

Proposition 3.22. *If $f'(z) = 0$ on U, then f is constant on U.*

Proof: Since f is holomorphic,

$$f' = f_x = -\mathrm{i}f_y.$$

Hence, $f_x = f_y = 0$, so that f is constant. □

Definition 3.23. A holomorphic function is entire if it is holomorphic on all of $\mathbb{C}$.

Example. One has that

$$\mathrm{e}^z := \sum_{n=0}^{\infty}\frac{z^n}{n!},\quad \cos z := \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!},\quad \sin z := \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}$$

are entire (use the ratio test). However, $1/(z-1)$ is not, as it has a singularity at $z = 1$.

Theorem 3.24 (Liouville's theorem)**.** *A bounded entire function is constant.*

Proof: Let $P \in \mathbb{C}$ be given. By the Cauchy estimates one has

$$|f^{(k)}(P)| \leq \frac{Mk!}{r^k},\quad r > 0.$$

Since $r > 0$ is arbitrary, and the bound M is uniform, this implies that $f^{(k)}(P) = 0$ for each $k \in \mathbb{N}$. The power series for f then contains only $f(P)$. Since this series converges for all $z \in \mathbb{C}$, f is constant. □

Corollary 3.25. *If f is entire, and if for some fixed $j > 0$*

$$|f(z)| \leq C|z|^j,\quad |z| > 1,$$

then f is a polynomial in z of degree at most j.

Proof: Let $r > 1$ be given. By the above argument, for $k > j$ one has that $f^{(k)}(0) = 0$. Hence, the power series centered at $z = 0$ terminates at $n = j$. □

Theorem 3.26 (Fundamental Theorem of Algebra)**.** *Let $p(z)$ be a nonconstant holomorphic polynomial. There is an $a \in \mathbb{C}$ such that $p(a) = 0$.*

Proof: Suppose not, so that $g(z) := 1/p(z)$ is entire. By the above corollary, one further has that $|g(z)| \to 0$ as $|z| \to \infty$; hence, g is bounded. Thus, g is a constant, which is a contradiction. □

Corollary 3.27. *Let $p(z)$ be a nonconstant holomorphic polynomial of degree k. There are numbers $a_1, \ldots, a_k$ (not necessarily distinct) and a nonzero $C \in \mathbb{C}$ such that*

$$p(z) = C(z-a_1)\cdots(z-a_k).$$

Proof: As a consequence of the Fundamental Theorem of Algebra, upon using the Euclidean algorithm one has

$$p(z) = (z - a_1)p_1(z),$$

where $p_1(z)$ is a holomorphic polynomial of degree $k-1$. If $k \geq 2$, then one has that

$$p_1(z) = (z - a_2)p_2(z),$$

where $p_2(z)$ is a holomorphic polynomial of degree $k-2$. Hence,

$$p(z) = (z - a_1)(z - a_2)p_2(z).$$

Proceeding until $p_k(z)$, a constant, yields the result. □

3.5. Uniform limits of holomorphic functions

Set

$$f_j(z) := \sum_{n=0}^{j} a_n z^n.$$

When considering the sequence of holomorphic functions $\{f_j(z)\}$, it was seen that if the sequence converges uniformly on $\overline{D}(P,r)$, then the limit function $f(z)$ is also holomorphic. Furthermore, $f_j^{(k)} \to f^{(k)}$ for each $k \geq 1$. This result can be generalized:

Theorem 3.28. *Let $f_j : U \mapsto \mathbb{C}$ be a sequence of holomorphic functions. Suppose that there is a function $f : U \mapsto \mathbb{C}$ such that on each compact set $E \subset U$, $f_j \to f$ uniformly. Then f is holomorphic on U.*

Remark 3.29. Note, however, that this is not true in general. Consider $g(x) = |x|$ for $|x| \leq 1$. By the Weierstrass approximation theorem there is a sequence of polynomials which converges uniformly to $g(x)$. Hence, while the limit is continuous, it is not even differentiable.

Proof: Let $P \in U$ be given, and let $r > 0$ be such that $\overline{D}(P,r) \subset U$. Since $\{f_j\}$ is a continuous family on $\overline{D}(P,r)$ which converges uniformly, f is also continuous. Thus, upon using the fact that each f_j is holomorphic, for any $z \in D(P,r)$,

$$\begin{aligned} f(z) &= \lim_{j\to\infty} f_j(z) \\ &= \lim_{j\to\infty} \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta-P|=r} \frac{f_j(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta-P|=r} \lim_{j\to\infty} \frac{f_j(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{|\zeta-P|=r} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta. \end{aligned}$$

The interchange of the limit and integral is justified by the fact that for fixed z

$$\frac{f_j(\zeta)}{\zeta - z} \to \frac{f(\zeta)}{\zeta - z}$$

uniformly in ζ on $|\zeta - P| = r$. Thus, f is holomorphic on $D(P,r)$, which implies that f is holomorphic on U. □

Corollary 3.30. *For each $k \geq 1$ one has that $f_j^{(k)} \to f^{(k)}$ uniformly on compact sets.*

Proof: For each $k \geq 1$ one has

$$f_j^{(k)}(z) = \frac{k!}{2\pi \mathrm{i}} \oint_{|\zeta-P|=r} \frac{f_j(\zeta)}{(\zeta - z)^{k+1}}\,\mathrm{d}\zeta.$$

Using the same argument as above yields that

$$\begin{aligned}\lim_{j\to\infty} f_j^{(k)}(z) &= \frac{k!}{2\pi\mathrm{i}}\oint_{|\zeta-P|=r}\lim_{j\to\infty}\frac{f_j(\zeta)}{(\zeta-z)^{k+1}}\,\mathrm{d}\zeta\\ &= \frac{k!}{2\pi\mathrm{i}}\oint_{|\zeta-P|=r}\frac{f(\zeta)}{(\zeta-z)^{k+1}}\,\mathrm{d}\zeta\\ &= f^{(k)}(z).\end{aligned}$$

□

3.6. The zeros of holomorphic functions

Definition 3.31. A point $z_0 \in \mathbf{Z}$ is an accumulation point if there is a sequence $\{z_j\} \in \mathbf{Z}\backslash\{z_0\}$ such that $\lim_{j\to\infty} z_j = z_0$.

Theorem 3.32. *Let $f : U \mapsto \mathbb{C}$ be a holomorphic function, where $U \subset \mathbb{C}$ is open and connected. Let $\mathbf{Z} := \{z \in U : f(z) = 0\}$. If $\mathbf{Z}$ has an accumulation point in U, then $f(z) = 0$ for all $z \in U$.*

Remark 3.33. Consider $f(z) = \sin(1/(1-z))$, which is holomorphic on $D(0,1)$. One has $\mathbf{Z} = \{1 - 1/n\pi : n = 1, 2, \dots\}$. The accumulation point is $z = 1 \in \partial D(0,1)$; hence, there is no contradiction.

Proof: Let $z_0 \in U$ be an accumulation point in $\mathbf{Z}$. The first claim is that $f^{(k)}(z_0) = 0$ for each $k \ge 0$. Suppose not, and let N be the least integer such that $f^{(N)}(z_0) \neq 0$. There is an $r_0 > 0$ such that

$$f(z) = \sum_{n=N}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n, \quad z \in D(z_0, r_0).$$

Upon setting

$$g(z) := \sum_{n=N}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^{n-N}, \quad z \in D(z_0, r_0),$$

one has that g is holomorphic with $g(z_0) \neq 0$. But, since $g(z) = f(z)/(z-z_0)^N$, there is a sequence $\{z_j\} \in \mathbf{Z}$ such that $g(z_j) = 0$. By continuity this implies that $g(z_0) = 0$, which is a contradiction.

Thus, $f(z) = 0$ for all $z \in D(z_0, r_0)$, which implies that $D(z_0, r) \subset \mathbf{Z}$. Pick another point $z_1 \neq z_0 \in D(z_0, r_0)$. Applying the same argument yields that there is an $r_1 > 0$ such that $f(z) = 0$ for all $z \in D(z_1, r_1)$. Repeating the procedure and using the fact that U is connected yields the result. □

Corollary 3.34. *If f, g are holomorphic, and if $\{z \in U : f(z) = g(z)\}$ has an accumulation point in U, then $f(z) = g(z)$ for all $z \in U$.*

Proof: Consider $h(z) := f(z) - g(z)$, and apply the above result. □

Corollary 3.35. *If there is a $P \in U$ such that $f^{(k)}(P) = 0$ for all $k \ge 0$, then $f(z) = 0$ for all $z \in U$.*

Proof: There is an $r > 0$ such that on $D(P, r) \subset U$ one has

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(P)}{n!}(z-P)^n = 0.$$

Now apply the above result. □

Example. One has:

(a) Recall the power series expansions for $\sin z$ and $\cos z$. Furthermore, recall that both of these functions are entire. One has that for $x \in \mathbb{R}$, $\sin^2 x + \cos^2 x = 1$. Set $f(z) := \sin^2 z + \cos^2 z - 1$. We have that $\mathbb{R} \subset \mathbf{Z}$; hence, by the above theorem $f(z) = 0$ for all $z \in \mathbb{C}$.

(b) Recall Euler's formula for $x \in \mathbb{R}$:
$$e^{ix} = \cos x + i \sin x,$$
from which one gets $\cos x = (e^{ix} + e^{-ix})/2$. Applying a similar argument as in the previous example, one has that the identity holds for all $z \in \mathbb{C}$.

(c) The above idea can also be used to derive the trigonometric identities.

4. Meromorphic Functions and Residues

4.1. *The behavior of a holomorphic function near an isolated singularity*

Definition 4.1. Suppose that $f : U\backslash\{P\} \mapsto \mathbb{C}$ is holomorphic. Then f has an isolated singularity at P.

There are three possibilities for f near an isolated singularity:

(a) there is an $r > 0$ and $M > 0$ such that $|f(z)| \le M$ for all $z \in D(P, r)\backslash\{P\}$

(b) $\lim_{z\to P} |f(z)| = +\infty$

(c) neither (a) nor (b) applies.

Definition 4.2. In case (a) f is said to have a removable singularity at P, in case (b) f is said to have a pole at P, and in case (c) f is said to have an essential singularity at P.

Theorem 4.3. *Suppose that $f : D(P, r)\backslash\{P\} \mapsto \mathbb{C}$ is holomorphic and bounded. Then*

(a) $\lim_{z\to P} f(z)$ exists

(b) the function $\hat{f} : D(P, r) \mapsto \mathbb{C}$ defined by
$$\hat{f}(z) := \begin{cases} f(z), & z \neq P \\ \lim\limits_{\zeta\to P} f(\zeta), & z = P \end{cases}$$
is holomorphic.

Remark 4.4. The assumption that f is holomorphic is crucial. For example, the uniformly bounded function $f(z) = \sin(1/|z|) \in C^\infty(\mathbb{C}\backslash\{0\})$ has no limit at $z = 0$.

Proof: Consider the function
$$g(z) = \begin{cases} (z-P)^2 f(z), & z \neq P \\ 0 & z = P. \end{cases}$$
It is clear that $g \in C^1(D(P, r)\backslash\{P\})$ Note that if $g \in C^1(D(P, r))$, then by the product rule
$$\frac{\partial g}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}(z-P)^2 f(z) + (z-P)^2 \frac{\partial f}{\partial \bar{z}} = 0$$
on $D(P, r)\backslash\{P\}$, so by continuity the result holds for all $z \in D(P, r)$. Hence, g is holomorphic on $D(P, r)$.

Now let us show that $\hat{f}$ exists. Since there is an $M > 0$ such that $|f(z)| \le M$ for all $z \in D(P, r)\backslash\{P\}$, one has that $|g(z)| \le M|z-P|^2$ on $D(P, r)$. Since g is holomorphic, this then implies that the power series expansion has the form
$$g(z) = \sum_{j=2}^{\infty} a_j (z-P)^j, \quad |z| < r.$$
Setting $H(z) := g(z)/(z-P)^2$ yields a holomorphic function which satisfies $H(z) = f(z)$ for $z \neq P$. Since $\lim_{z\to P} H(z) = a_2$, the function H is desired holomorphic extension $\hat{f}$.

Now it must be shown that $g \in C^1$ at $z = P$. First note that for $h \in \mathbb{R}$,

$$\lim_{h\to 0} \frac{g(P+h) - g(P)}{h} = \lim_{h\to 0} hf(P+h) = 0$$

(the second equality follows from the fact that f is bounded). Thus, $g_x(P) = 0$. Similarly, $g_y(P) = 0$.

In order that $g \in C^1$, it must then be shown that $\lim_{z\to P} g_x(P) = \lim_{z\to P} g_y(P) = 0$. Let $z_0 \in D(P, r/2)\backslash\{P\}$. The Cauchy estimate applied on $D(z_0, |z_0 - P|) \subset D(P, r)$ yields that

$$|f'(z_0)| \le \frac{M}{|z_0 - P|},$$

and since f is holomorphic at z_0,

$$|f_x(z_0)| \le \frac{M}{|z_0 - P|}.$$

Thus, since g is holomorphic at z_0 one has that

$$\begin{aligned} |g_x(z_0)| &= |2(z_0 - P)f(z_0) + (z_0 - P)^2 f_x(z_0)| \\ &\le 2M|z_0 - P| + |z_0 - P|^2 \frac{M}{|z_0 - P|}, \end{aligned}$$

so that $g_x(z_0) \to 0$ as $z_0 \to P$. A similar result holds for g_y, which proves the theorem. □

Example. Let us show that the entire function e^z has the range $\mathbb{C}\backslash\{0\}$. Let $a = a_r + ia_i \in \mathbb{C}\backslash\{0\}$ be given. Since for $w = w_r + iw_i$ one has that $e^w = e^{w_r}(\cos w_i + i \sin w_i)$, solving $e^w = a$ is equivalent to solving

$$e^{w_r} \cos w_i = a_r, \quad e^{w_r} \sin w_i = a_i.$$

In other words,

$$w_r = \frac{1}{2}\ln(a_r^2 + a_i^2), \quad \tan(w_i) = \frac{a_i}{a_r}.$$

Now consider case (c) with the example

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n},$$

which is clearly holomorphic on $D(0, 1)\backslash\{0\}$. Let $a = a_r + ia_i \in \mathbb{C}\backslash\{0\}$ be given, and choose $w \in \mathbb{C}\backslash\{0\}$ so that $e^w = a$. Now, $e^{w+i2k\pi} = a$ for any $k \in \mathbb{Z}$. Let $K > 0$ be sufficiently large so that $w + i2k\pi \neq 0$ for $k \ge K$. Upon setting $z_k := 1/(w + i2k\pi)$ for $k \ge K$, one has that $e^{1/z_k} = a$ with $z_k \to 0$ as $k \to +\infty$. Hence, for any $\epsilon > 0$ there is a z with $0 < |z| < \epsilon$ such that $e^{1/z} = a$, i.e., the range of $e^{1/z} : D(0, \epsilon)\backslash\{0\} \mapsto \mathbb{C}$ is dense in $\mathbb{C}$. In particular, it is not bounded.

Theorem 4.5 (Casorati-Weierstrass)**.** *If $f : D(P, r_0)\backslash\{P\} \mapsto \mathbb{C}$ is holomorphic and if P is an essential singularity of f, then $f(D(P, r)\backslash\{P\})$ is dense in $\mathbb{C}$ for any $0 < r < r_0$.*

Proof: Suppose that the statement fails. There is then a $\beta \in \mathbb{C}$ and an $\epsilon > 0$ such that $|f(z) - \beta| \ge \epsilon$ for all $z \in D(P, r_0)\backslash\{P\}$. Since $f(z) - \beta$ is nonvanishing on $D(P, r_0)\backslash\{P\}$, the function

$$g(z) := \frac{1}{f(z) - \beta}$$

is holomorphic on $D(P, r_0)\backslash\{P\}$. Furthermore, $|g(z)| \le 1/\epsilon$ for all $z \in D(P, r_0)\backslash\{P\}$. Thus, by Theorem 4.3 g has a holomorphic extension $\hat{g}$, and

$$f(z) = \beta + \frac{1}{\hat{g}(z)}.$$

If $\hat{g}(P) \neq 0$, then f is holomorphic on $D(P, r_0)$, i.e., case (a) applies. If $\hat{g}(P) = 0$, then case (b) applies. Thus, f does not have an essential singularity at P, which is a contradiction. □

Example. Which of the functions, if any, have essential singularities at $z = 0$:

$$\sin(1/z), \quad \frac{\sum_{n=0}^{\infty} nz^n}{z^4}.$$

4.2. Expansion around singular points

Definition 4.6. A Laurent series is of the form

$$\sum_{n=-\infty}^{+\infty} a_n(z-P)^n.$$

The series converges if both

$$\sum_{n=-\infty}^{-1} a_n(z-P)^n,\quad \sum_{n=0}^{+\infty} a_n(z-P)^n$$

converge in the usual sense.

Lemma 4.7. *Suppose that the Laurent series converges at $z_1, z_2 \neq P$, and suppose that $|z_1-P|<|z_2-P|$. The series then converges on the annulus $D(P,|z_2-P|)\backslash\overline{D}(P,|z_1-P|)$.*

Proof: By Abel's lemma the series $\sum_{n=0}^{+\infty} a_n(z-P)^n$ converges for $|z-P| < \max\{|z_1-P|,|z_2-P|\} = |z_2-P|$. Setting $w=(z-P)^{-1}$ and again using Abel's lemma yields that the series $\sum_{n=0}^{+\infty} a_{-n}w^n$ converges for $|w| < \max\{1/|z_1-P|, 1/|z_2-P|\} = 1/|z_1-P|$, i.e., $|z-P|>|z_1-P|$. □

Remark 4.8. Unless stated otherwise, henceforth an annulus with $r_1<r_2$ will be represented by

$$\mathcal{A} := D(P,r_2)\backslash\overline{D}(P,r_1).$$

Lemma 4.9. *Suppose that the Laurent series converges at minimally one point. There are then unique $r_1 \le r_2$ such that the series converges on the annulus $\mathcal{A}$. Furthermore, the convergence is uniform in* int$(\mathcal{A})$.

Remark 4.10. If $r_1<r_2$, then the Laurent series is holomorphic on the annulus.

Example. One has:

(a) The Laurent series

$$\mathrm{e}^{1/z} = \sum_{n=-\infty}^{0} \frac{z^n}{|n|!}$$

converges on $\mathbb{C}\backslash\{0\}$. Recall that $z=0$ is an essential singularity.

(b) The Laurent series

$$\sum_{n=-\infty}^{+\infty} \frac{z^n}{n^4}$$

converges only on the circle $|z|=1$. The convergence is absolute.

Assuming that $r_1<r_2$, set

$$f(z) := \sum_{n=-\infty}^{+\infty} a_n(z-P)^n.$$

Since the convergence is uniform in $\mathcal{A}$, for any $r_1<r<r_2$ one has that

$$\begin{aligned}\oint_{|\zeta-P|=r} \frac{f(\zeta)}{(\zeta-P)^{j+1}}\,\mathrm{d}\zeta &= \oint_{|\zeta-P|=r}\sum_{n=-\infty}^{+\infty} a_n(\zeta-P)^{n-j-1}\,\mathrm{d}\zeta\\ &= \sum_{n=-\infty}^{+\infty}\oint_{|\zeta-P|=r} a_n(\zeta-P)^{n-j-1}\,\mathrm{d}\zeta.\end{aligned}$$

An explicit calculation yields that

$$\oint_{|\zeta-P|=r} (\zeta-P)^{n-j-1}\,\mathrm{d}\zeta = \begin{cases}0, & n\neq j\\ 2\pi\mathrm{i}, & n=j.\end{cases}$$

Hence, one has that

$$a_j = \frac{1}{2\pi\mathrm{i}}\oint_{|\zeta-P|=r}\frac{f(\zeta)}{(\zeta-P)^{j+1}}\,\mathrm{d}\zeta,$$

so that the coefficients of the Laurent series are uniquely determined by f.

4.3. Existence of Laurent expansions

We know that a Laurent expansion on $\mathcal{A}$ defines a holomorphic function f. It must now be shown that a function holomorphic on $\mathcal{A}$ can be represented by a Laurent series. It is important to keep in mind that if one considers a circle contained in $\mathcal{A}$, then the holomorphic function can be represented by a Taylor series, i.e., there are no powers of z^{-j} in the expansion. This is due to the fact that the circle is a simply connected domain. An annulus is not simply connected, and hence there are as of yet no theorems giving the existence of a series representation for a holomorphic function which is valid on the entire annulus.

Theorem 4.11 (Cauchy integral formula for an annulus)**.** *Suppose that there exist $0 \le r_1 < r_2 \le +\infty$ such that $f : \mathcal{A} \mapsto \mathbb{C}$ is holomorphic. Then for each $r_1 < s_1 < s_2 < r_2$ and each $z \in D(P, s_2)\backslash\overline{D}(P, s_1)$ it holds that*

$$f(z) = \frac{1}{2\pi i}\oint_{|\zeta-P|=s_2}\frac{f(\zeta)}{\zeta-z}\,d\zeta - \frac{1}{2\pi i}\oint_{|\zeta-P|=s_1}\frac{f(\zeta)}{\zeta-z}\,d\zeta.$$

Proof: Fix $z \in D(P, s_2)\backslash\overline{D}(P, s_1)$, and for each $\zeta \in \mathcal{A}$ set

$$g(\zeta) := \begin{cases} \dfrac{f(\zeta)-f(z)}{\zeta-z}, & \zeta \neq z \\ f'(z), & \zeta = z. \end{cases}$$

By the Riemann removable singularities theorem Theorem 4.3 g is holomorphic on $\mathcal{A}$. Thus, one has that

$$\oint_{|\zeta-P|=s_2} g(\zeta)\,d\zeta = \oint_{|\zeta-P|=s_1} g(\zeta)\,d\zeta,$$

which upon using the definition of g and the fact that neither curve contains the point z yields that

$$\oint_{|\zeta-P|=s_2}\frac{f(\zeta)}{\zeta-z}\,d\zeta - \oint_{|\zeta-P|=s_2}\frac{f(z)}{\zeta-z}\,d\zeta = \oint_{|\zeta-P|=s_1}\frac{f(\zeta)}{\zeta-z}\,d\zeta - \oint_{|\zeta-P|=s_1}\frac{f(z)}{\zeta-z}\,d\zeta.$$

Upon rearranging, and using the Cauchy integral formula to get that

$$\oint_{|\zeta-P|=s_2}\frac{f(z)}{\zeta-z}\,d\zeta = 2\pi i f(z), \quad \oint_{|\zeta-P|=s_1}\frac{f(z)}{\zeta-z}\,d\zeta = 0,$$

one gets the desired result. □

Theorem 4.12. *If $0 \le r_1 < r_2$, and if $f : \mathcal{A} \mapsto \mathbb{C}$ is holomorphic, then f has a Laurent series which converges on $\mathcal{A}$ to f, and which converges absolutely and uniformly on $D(P, s_2)\backslash\overline{D}(P, s_1)$ for any $r_1 < s_1 < s_2 < r_2$.*

Proof: Fix $z \in D(P, s_2)\backslash\overline{D}(P, s_1)$. Since $z \in D(P, s_2)$, the geometric series

$$\frac{\zeta-P}{\zeta-z} = \frac{1}{1-\frac{z-P}{\zeta-P}} = \sum_{n=0}^{+\infty}\frac{(z-P)^n}{(\zeta-P)^n}$$

converges uniformly, so that

$$\begin{aligned}\oint_{|\zeta-P|=s_2}\frac{f(\zeta)}{\zeta-z}\,d\zeta &= \sum_{n=0}^{+\infty}\left(\oint_{|\zeta-P|=s_2}\frac{f(\zeta)}{(\zeta-P)^{n+1}}\,d\zeta\right)(z-P)^n \\ &= 2\pi i\sum_{n=0}^{+\infty}a_n(z-P)^n.\end{aligned}$$

A similar argument yields that for $|z-P| < s_1$ the geometric series

$$\frac{z-P}{\zeta-z} = -\frac{1}{1-\frac{\zeta-P}{z-P}} = -\sum_{n=0}^{+\infty}\frac{(\zeta-P)^n}{(z-P)^n}$$

converges uniformly, so that

$$\begin{aligned}
\oint_{|\zeta-P|=s_1}\frac{f(\zeta)}{\zeta-z}\,\mathrm{d}\zeta &= -\sum_{n=0}^{+\infty}\left(\oint_{|\zeta-P|=s_1}\frac{f(\zeta)}{(\zeta-P)^{-n}}\,\mathrm{d}\zeta\right)(z-P)^{-(n+1)}\\
&= -\sum_{n=-\infty}^{-1}\left(\oint_{|\zeta-P|=s_1}\frac{f(\zeta)}{(\zeta-P)^{n+1}}\,\mathrm{d}\zeta\right)(z-P)^{n}\\
&= -2\pi\mathrm{i}\sum_{n=-\infty}^{-1}a_n(z-P)^n.
\end{aligned}$$

Thus, the Cauchy integral formula yields that f has the desired Laurent series. □

Corollary 4.13. *If $f : D(P,r)\backslash\{P\}$ is holomorphic, then f has a unique Laurent series given by*

$$f(z) = \sum_{n=-\infty}^{+\infty}a_n(z-P)^n,$$

where

$$a_n = \frac{1}{2\pi\mathrm{i}}\oint_{|\zeta-P|=s}\frac{f(\zeta)}{(\zeta-P)^{n+1}}\,\mathrm{d}\zeta$$

for any $0 < s < r$. The sum converges uniformly on compact subsets of $D(P,r)\backslash\{P\}$, and absolutely for all $z \in D(P,r)\backslash\{P\}$.

We can now use the Laurent series to classify the singularity at the point P:

(a) Removable singularity if and only if $a_n = 0$ for all $n \le -1$

(b) Pole of order k if $a_n = 0$ for $n < -k$ with $a_{-k} \neq 0$, i.e., $|f(z)| \ge C|z-P|^{-k}$ for some nonzero C and $|z-P|$ sufficiently small. To see this, note that

$$f(z) = (z-P)^{-k}\left(a_{-k} + \sum_{n=-k+1}^{+\infty}a_n(z-P)^{n+k}\right),$$

and that the sum defines a holomorphic function on $D(P,r)$.

(c) Essential singularity otherwise.

4.4. Examples of Laurent expansions

Definition 4.14. If f has a pole of order k at P, the principal part of f at P is given by

$$\sum_{n=-k}^{-1}a_n(z-P)^n.$$

How does one compute the coefficients directly without using the contour integration? It is clear that the order of the pole can be determined by finding the unique integer k such that

$$\lim_{z\to P}(z-P)^k f(z) \neq 0, \quad \lim_{z\to P}|(z-P)^j f(z)| = +\infty,\ j < k.$$

Supposing that f has a pole of order k at $z = P$, one quickly sees that

$$a_{-k} = \lim_{z\to P}(z-P)^k f(z).$$

Expanding upon this idea yields the following:

Lemma 4.15. *Let $f : D(P, r)\backslash\{P\}$ be holomorphic, and assume that f has a pole of order k at $z = P$. Then for $n \geq -k$,*

$$a_n = \frac{1}{(k+n)!}\frac{\mathrm{d}^{k+n}}{\mathrm{d}z^{k+n}}((z-P)^k f(z))\bigg|_{z=P}.$$

Example. Consider $f(z) := \mathrm{e}^z/\sin z$ on the strip $U := \{z : |\operatorname{Im} z| < \pi\}$. Since

$$\sin z = \frac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}},$$

one has that $\sin z = 0$ if and only if $\mathrm{e}^{\mathrm{i}x} = \pm\mathrm{e}^y$, i.e.,

$$\cos x = \pm\mathrm{e}^y, \quad \sin x = 0.$$

This implies that $y = 0$ and $x = n\pi$. Thus, on U, f has a singularity only at $z = 0$. Using the Taylor expansions for each function yields that

$$\begin{aligned}\lim_{z\to 0} zf(z) &= \lim_{z\to 0} z\frac{1 + z + \cdots}{z - z^3/3! + \cdots} \\ &= 1,\end{aligned}$$

so that

$$f(z) = \frac{1}{z} + \sum_{n=0}^{+\infty} a_n z^n.$$

This sum converges on $D(0, \pi)\backslash\{0\}$.

4.5. The calculus of residues

Definition 4.16. A set $U \subset \mathbb{C}$ is simply connected if every closed curve is continuously deformable to a point.

Example. $D(0, 1)$ is simply connected, whereas $D(0, 1)\backslash\{0\}$ is not.

Definition 4.17. Let $\gamma : [a, b] \mapsto \mathbb{C}$ be a piecewise C^1 closed curve. Suppose that $P \notin \gamma([a, b])$. The index of γ with respect to P, written $\operatorname{Ind}_\gamma(P)$, is given by

$$\operatorname{Ind}_\gamma(P) := \frac{1}{2\pi\mathrm{i}}\oint_\gamma \frac{1}{\zeta - P}\,\mathrm{d}\zeta.$$

The index of γ is also known as the winding number of the curve γ about the point P.

Remark 4.18. It is important to note that the definition does not require that γ be a simple closed curve.

Lemma 4.19. $\operatorname{Ind}_\gamma(P)$ *is an integer.*

Proof: Set

$$I(t) := \int_a^t \frac{\gamma'(s)}{\gamma(s) - P}\,\mathrm{d}s,$$

and set

$$g(t) := (\gamma(t) - P)\mathrm{e}^{-I(t)}.$$

Note that $\gamma \in C^1([a, b])$ implies that $g \in C^1([a, b])$. A routine calculation shows that $g'(t) = 0$ for all $t \in [a, b]$; hence, g is a constant. Now, $g(a) = \gamma(a) - P$, and using the fact that γ is a closed curve yields that

$$\begin{aligned}g(b) &= (\gamma(b) - P)\mathrm{e}^{-I(b)} \\ &= (\gamma(a) - P)\mathrm{e}^{-I(b)}.\end{aligned}$$

Since $g(a) = g(b)$, this then implies that $I(b) = 2k\pi\mathrm{i}$ for some $k \in \mathbb{Z}$. □

Remark 4.20. One has:

(a) If $P \notin \mathrm{int}(\gamma)$, then $\mathrm{Ind}_\gamma(P) = 0$.

(b) If γ is a circle which runs around P k times in a counterclockwise direction, then $\mathrm{Ind}_\gamma(P) = k$, while if the direction is clockwise then $\mathrm{Ind}_\gamma(P) = -k$. This result generalizes to arbitrary C^1 closed curves.

Definition 4.21. Let f be a holomorphic function with a pole of order k at P. The residue of f at P is given by

$$\mathrm{Res}_f(P) := \frac{1}{(k-1)!}\frac{\mathrm{d}^{k-1}}{\mathrm{d}z^{k-1}}((z-P)^k f(z))\Big|_{z=P}.$$

Remark 4.22. As a consequence of Lemma 4.15, note that $a_{-1} = \mathrm{Res}_f(P)$.

Theorem 4.23 (Residue theorem)**.** *Suppose that $U \subset \mathbb{C}$ is a simply connected set, and that $P_1, \dots, P_n \in U$ are distinct points. Suppose that $f : U \backslash \{P_1, \dots, P_n\} \mapsto \mathbb{C}$ is holomorphic and $\gamma \subset U \backslash \{P_1, \dots, P_n\}$ is a closed piecewise C^1 curve. Then*

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{i=1}^n \mathrm{Ind}_\gamma(P_i)\,\mathrm{Res}_f(P_i).$$

Proof: For each $j = 1, \dots, n$ expand f in a Laurent series about P_j. Denote the principle part at P_j by

$$s_j(z) := \sum_{k=-\infty}^{-1} a_k^j (z-P_j)^k,$$

and set $S(z) := \sum_{j=1}^n s_j(z)$. Since each P_j is an isolated singularity, each $s_j(z)$ is holomorphic on $\mathbb{C}\backslash\{P_j\}$, so that $S(z)$ is holomorphic on $\mathbb{C}\backslash\{P_1, \dots, P_n\}$. Now, the function $g(z) := f(z) - S(z)$ has a removable singularity at each point P_j as $f(z) - s_j(z)$ has a Laurent expansion at P_j with no negative powers, and each $s_k(z)$ for $k \neq j$ is holomorphic at P_j. Since U is simply connected, this then implies that

$$\oint_\gamma g(z)\,\mathrm{d}z = 0,$$

i.e.,

$$\oint_\gamma f(z)\,\mathrm{d}z = \oint_\gamma S(z)\,\mathrm{d}z = \sum_{j=1}^n \oint_\gamma s_j(z)\,\mathrm{d}z.$$

Now, $\gamma([a,b])$ is a compact set and $P_j \notin \gamma([a,b])$; hence, $s_j(z)$ converges uniformly on γ. Thus, one can interchange the summation and integration to get

$$\begin{aligned}\oint_\gamma s_j(z)\,\mathrm{d}z &= \sum_{k=-\infty}^{-1} a_k^j \oint_\gamma (z-P_j)^k\,\mathrm{d}z\\ &= a_{-1}^j \oint_\gamma (z-P_j)^{-1}\,\mathrm{d}z\\ &= 2\pi\mathrm{i}\,\mathrm{Ind}_\gamma(P_j)\,\mathrm{Res}_f(P_j).\end{aligned}$$

The second equality follows from the fact that $(z-P_j)^{-k}$ has a holomorphic antiderivative for $k \geq 2$. The result now follows. □

4.6. Applications of the calculus of residues

Before we look at some explicit problems, we need the following preliminary results. The first allows one to easily compute path integrals along large circular arcs, and the second one allows us to easily compute residues in the case of simple poles.

Lemma 4.24. *Let C_R be a circular arc of radius R centered at $z = 0$. If $zf(z) \to 0$ uniformly as $R \to \infty$, then*

$$\lim_{R\to\infty}\oint_{C_R} f(z)\,\mathrm{d}z = 0.$$

Proof: Let $\partial > 0$ be the angle enclosed by C_R. Then

$$\left|\oint_{C_R} f(z)\,\mathrm{d}z\right| \le \int_0^{\partial} |f(z)|R\,\mathrm{d}\phi \le R \sup_{z\in C_R} |f(z)|\partial.$$

Since $zf(z) \to 0$ uniformly as $R \to \infty$, the result now follows. □

Lemma 4.25. *Consider $h(z) = f(z)/g(z)$, and suppose that $g(z)$ has simple zeros at $P_1, \ldots, P_n$ with $f(P_j) \neq 0$ for $j = 1, \ldots, n$. Then*

$$\mathrm{Res}_h(P_j) = \frac{f(P_j)}{g'(P_j)}.$$

Proof: Since $g(z)$ has a simple zero at P_j, it has the Taylor expansion

$$g(z) = g'(P_j)(z - P_j) + \sum_{k=2}^{+\infty} \frac{g^{(k)}(P_j)}{k!}(z - P_j)^k.$$

Since

$$\mathrm{Res}_h(P_j) = \lim_{z\to P_j}(z - P_j)h(z),$$

the result immediately follows. □

Example. Let us evaluate

$$\int_{-\infty}^{+\infty} \frac{1}{64 + x^6}\,\mathrm{d}x. \tag{4.1}$$

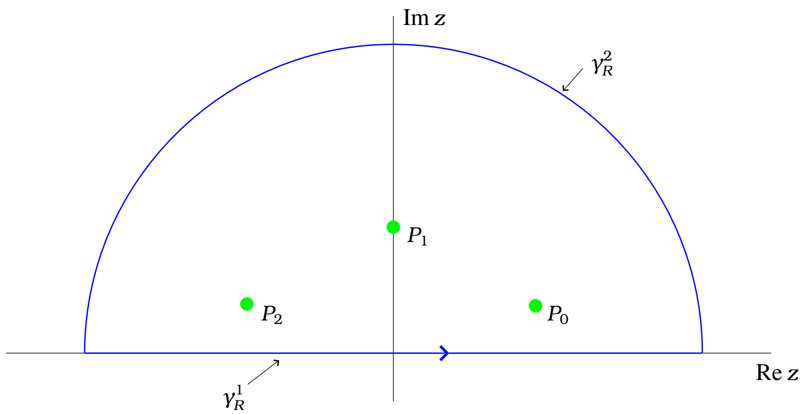


Figure 1: The contour of integration associated with equation (4.1).

This will be done by looking at

$$\oint_{\gamma_R} \frac{1}{64 + z^6}\,\mathrm{d}z,$$

where for $R > 2$ one defines $\gamma_R := \gamma_R^1 \cup \gamma_R^2$, where

$$\gamma_R^1(t) := t + \mathrm{i}0, \quad -R \le t \le R$$
$$\gamma_R^2(t) := R\mathrm{e}^{\mathrm{i}t}, \quad 0 \le t \le \pi$$

(see Figure 1). Set $U := \mathbb{C}$, and set $P_j := 2\mathrm{e}^{\mathrm{i}(1+2j)\pi/6}$ for $j = 0, \ldots, 5$. Then $f(z) := 1/(64 + z^6)$ is holomorphic on $U\backslash\{P_0, \ldots, P_5\}$, and the residue theorem applies. By the choice of γ_R one then has that

$$\oint_{\gamma_R} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{j=0}^{2} \mathrm{Ind}_{\gamma_R}(P_j)\,\mathrm{Res}_f(P_j).$$

Since each pole is simple, upon applying Lemma 4.25 one sees that

$$\mathrm{Res}_f(P_0) = \frac{1}{384}(-\sqrt{3}-\mathrm{i}), \quad \mathrm{Res}_f(P_1) = -\frac{1}{192}\mathrm{i}, \quad \mathrm{Res}_f(P_2) = \frac{1}{384}(\sqrt{3}-\mathrm{i});$$

furthermore, the choice of γ_R yields $\mathrm{Ind}_{\gamma_R}(P_j) = 1$ for $j = 0, \dots, 2$. Thus,

$$\oint_{\gamma_R} f(z)\,\mathrm{d}z = \frac{\pi}{48}.$$

Now,

$$\oint_{\gamma_R} f(z)\,\mathrm{d}z = \oint_{\gamma_R^1} f(z)\,\mathrm{d}z + \oint_{\gamma_R^2} f(z)\,\mathrm{d}z.$$

It is easy to see that

$$\lim_{R\to+\infty} \oint_{\gamma_R^1} f(z)\,\mathrm{d}z = \lim_{R\to+\infty} \int_{-R}^{+R} f(t)\,\mathrm{d}t = \int_{-\infty}^{+\infty} \frac{1}{64+t^6}\,\mathrm{d}t.$$

Furthermore, by Lemma 4.24 one has that

$$\lim_{R\to+\infty} \left| \oint_{\gamma_R^2} f(z)\,\mathrm{d}z \right| = 0.$$

Hence, one finally has that

$$\int_{-\infty}^{+\infty} \frac{1}{64+x^6}\,\mathrm{d}x = \frac{\pi}{48}.$$

Example. Consider

$$\int_{-\infty}^{+\infty} \mathrm{sech}^2(x)\cos(bx)\,\mathrm{d}x. \tag{4.2}$$

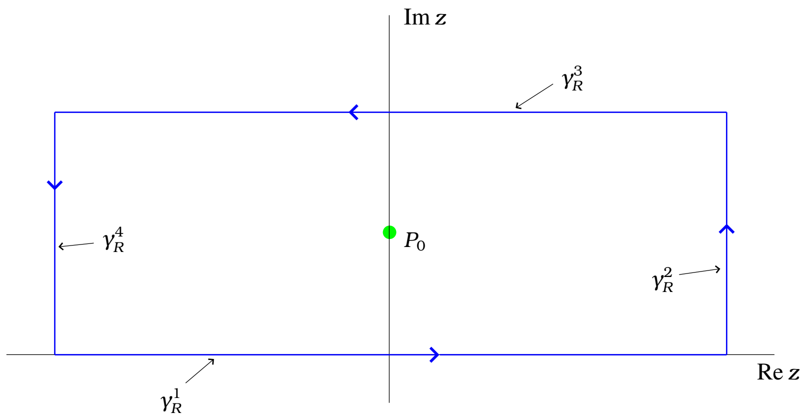


Figure 2: The contour of integration associated with equation (4.2).

This will be done by looking at

$$\oint_{\gamma_R} \mathrm{sech}^2(z)\mathrm{e}^{\mathrm{i}bz}\,\mathrm{d}z$$

for a suitably chosen contour γ_R. The poles of $\mathrm{sech}(z)$ are located at $z = \mathrm{i}(2k+1)\pi/2$, $k \in \mathbb{Z}$. Hence, a contour as in the previous example will not work. For $R > 1$ set $\gamma_R := \cup_{j=1}^4 \gamma_R^j$, where

$$\begin{aligned}
\gamma_R^1(t) &:= t + \mathrm{i}0, \quad -R \le t \le R \\
\gamma_R^2(t) &:= R + \mathrm{i}t, \quad 0 \le t \le \pi \\
\gamma_R^3(t) &:= t + \mathrm{i}\pi, \quad R \le t \le -R \\
\gamma_R^4(t) &:= -R + \mathrm{i}t, \quad \pi \le t \le 0
\end{aligned}$$

(see Figure 2). A routine calculation shows that $\operatorname{sech}(z+\mathrm{i}\pi) = -\operatorname{sech}(z)$, and that $\mathrm{e}^{\mathrm{i}b(z+\mathrm{i}\pi)} = \mathrm{e}^{-b\pi}\mathrm{e}^{\mathrm{i}bz}$. After a bit of algebra this yields that

$$\oint_{\gamma_R^1} \operatorname{sech}^2(z)\mathrm{e}^{\mathrm{i}bz}\,\mathrm{d}z + \oint_{\gamma_R^3} \operatorname{sech}^2(z)\mathrm{e}^{\mathrm{i}bz}\,\mathrm{d}z = \mathrm{e}^{-\pi b/2}(\mathrm{e}^{\pi b/2} - \mathrm{e}^{-\pi b/2}) \int_{-R}^{+R} \operatorname{sech}^2(t)\mathrm{e}^{\mathrm{i}bt}\,\mathrm{d}t.$$

When considering γ_R^2 note that for $z = R + \mathrm{i}t$,

$$\operatorname{sech}(z) = 2\mathrm{e}^{-R}\left(\frac{1}{\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-2R-\mathrm{i}t}}\right).$$

Using a similar argument as in the proof of Lemma 4.24 then yields that

$$\left|\oint_{\gamma_R^2} \operatorname{sech}^2(z)\mathrm{e}^{\mathrm{i}bz}\,\mathrm{d}z\right| \le C\mathrm{e}^{-2R}$$

for some positive constant C. A similar estimate holds when considering γ_R^4. The pole at $z = \mathrm{i}\pi/2$ is of order two. Recalling the calculation presented in Definition 4.21, one sees via a routine Maple calculation that $\operatorname{Res}_f(\mathrm{i}\pi/2) = -\mathrm{i}b\mathrm{e}^{-\pi b/2}$, so by the residue theorem

$$\oint_{\gamma_R} \operatorname{sech}^2(z)\mathrm{e}^{\mathrm{i}bz}\,\mathrm{d}z = 2\pi b\mathrm{e}^{-\pi b/2}.$$

Thus, upon letting $R \to +\infty$ one has

$$\int_{-\infty}^{+\infty} \operatorname{sech}^2(t)\mathrm{e}^{\mathrm{i}bt}\,\mathrm{d}t = \frac{\pi b}{\sinh(\pi b/2)}.$$

Taking the real and imaginary parts then gives

$$\int_{-\infty}^{+\infty} \operatorname{sech}^2(t)\cos(bt)\,\mathrm{d}t = \frac{\pi b}{\sinh(\pi b/2)}$$

and

$$\int_{-\infty}^{+\infty} \operatorname{sech}^2(t)\sin(bt)\,\mathrm{d}t = 0.$$

Example. Let us evaluate

$$f(x) \coloneqq \sum_{n=-\infty}^{+\infty} \operatorname{sech}^2(n+x).$$

Note that

$$f(x+1) = \sum_{n=-\infty}^{+\infty} \operatorname{sech}^2((n+1)+x) = f(x),$$

so that $f(x)$ can be written as a Fourier series. Since

$$f(-x) = \sum_{n=-\infty}^{+\infty} \operatorname{sech}^2(n-x) = \sum_{n=+\infty}^{-\infty} \operatorname{sech}^2(-n-x) = \sum_{n=+\infty}^{-\infty} \operatorname{sech}^2(n+x) = f(x),$$

one has that

$$f(x) = \sum_{n=0}^{+\infty} \hat{f}_n \cos(2\pi nx),$$

where

$$\hat{f}_0 = \int_{-\infty}^{+\infty} \operatorname{sech}^2(x)\,\mathrm{d}x = 2$$

and

$$\hat{f}_n = 2\int_{-\infty}^{+\infty} \operatorname{sech}^2(x)\cos(2\pi nx)\,\mathrm{d}x.$$

From the previous example one immediately gets that

$$\hat{f}_n = \frac{4\pi^2 n}{\sinh(\pi^2 n)};$$

hence,

$$f(x) = 2 + \sum_{n=1}^{+\infty} \frac{4\pi^2 n}{\sinh(\pi^2 n)} \cos(2\pi n x).$$

A quick numerical calculation yields that

$$\left| f(x) - 2 - \frac{4\pi^2}{\sinh(\pi^2)} \cos(2\pi x) \right| = \mathcal{O}(10^{-7}),$$

so that

$$f(x) \sim 2 + \frac{4\pi^2}{\sinh(\pi^2)} \cos(2\pi x).$$

4.7. Meromorphic functions and singularities at infinity

Definition 4.26. A meromorphic function $f : U\backslash S \mapsto \mathbb{C}$ satisfies

(a) $S \subset U$ is closed and discrete (no accumulation points in U)

(b) f is holomorphic on $U\backslash S$

(c) each $P \in S$ is a pole of finite order.

Lemma 4.27. *Let $f : U \mapsto \mathbb{C}$ be holomorphic, and let $U \subset \mathbb{C}$ be open and connected. Assuming that $f(z) \not\equiv 0$, set $S := \{z \in U : f(z) = 0\}$. Setting $F(z) := 1/f(z)$, one has that $F : U\backslash S \mapsto \mathbb{C}$ is meromorphic.*

Proof: The set $S \subset U$ is discrete; otherwise, $f(z) \equiv 0$ on U. Each zero of f is of finite order; otherwise, $f(z) \equiv 0$ on U. Hence, each point $P \in S$ is a pole of finite order for F. □

Definition 4.28. Suppose that f is meromorphic on U, and that $\{z : |z| > R\} \subset U$ for some $R > 0$. Set

$$U_\infty := \{z : 0 < |z| < 1/R\},$$

and define $G : U_\infty \mapsto \mathbb{C}$ by $G(z) := f(1/z)$. Then

(a) f has a removable singularity at ∞ if G has a removable singularity at 0

(b) f has a pole at ∞ if G has a pole at 0

(c) f has an essential singularity at ∞ if G has an essential singularity at 0.

Example. $e^z, \sin z, \cos z$ all have essential singularities at ∞.

Assuming that

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$$

converges for $|z| > R$, one has that on U_∞ the function $G(z)$ has the Laurent expansion

$$G(z) = \sum_{n=-\infty}^{+\infty} a_{-n} z^n.$$

One then immediately sees that for the behavior of $G(z)$ near zero,

(a) removable singularity: $a_{-n} = 0$ for $n \le -1$

(b) pole of order N: $a_{-n} = 0$ for $n \leq -(N+1)$

(c) essential singularity: otherwise.

Hence, for the function f one gets the following lemma.

Lemma 4.29. *Suppose that $f : \mathbb{C} \mapsto \mathbb{C}$ is an entire function. Then f has a pole at infinity if and only if it is a nonconstant polynomial.*

Proof: The idea is similar to that presented in Problem 4.18. The detailed proof is left to the student. □

Definition 4.30. f is meromorphic at ∞ if G is meromorphic on U_∞.

Remark 4.31. Note that this definition implies that f has a pole at ∞.

Lemma 4.32. *If f is meromorphic on $\mathbb{C}$ and is also meromorphic at ∞, then f is a rational function. Conversely, every rational function is meromorphic on $\mathbb{C}$ and at ∞.*

Proof: Suppose that $f(z) = P(z)/Q(z)$, where both P and Q are polynomials, i.e.,

$$P(z) := a_n z^z + \cdots + a_0, \quad Q(z) := b_m z^m + \cdots + b_0.$$

It is clear that f is meromorphic on C. Set

$$G(z) = f(1/z) = \frac{a_n z^{-n} + \cdots + a_0}{b_m z^{-m} + \cdots + b_0} = \frac{a_n z^{m-n} + \cdots + a_0 z^m}{b_m + \cdots + b_0 z^m}.$$

If $m \geq n$, then G has a removable singularity at zero, and if $m < n$ then G has a pole at zero of order $n - m$. In either case, G is meromorphic on U_∞, and hence f is meromorphic at ∞.

Now suppose that f is meromorphic on $\mathbb{C}$ and at ∞. If f has a pole at ∞, then by definition there is an $R > 0$ such that no poles exist in the set $\{z \in \mathbb{C} : |z| > R\}$. Furthermore, since the set of poles forms a discrete set, there can be only a finite number of poles in $\overline{D}(0, R)$. If there is an $a \in \mathbb{C}$ such that $f(\infty) = a$, then there is an $R > 0$ such that $|f(z) - a| < 1$ in the set $\{z \in \mathbb{C} : |z| > R\}$. Again, there can be only a finite number of poles in $\overline{D}(0, R)$. Finally, f cannot have an essential singularity at ∞ since then $f(1/z)$ would have an essential singularity at $z = 0$, and hence would not be meromorphic on U_∞.

Let $P_1, \ldots, P_k \in \overline{D}(0, R)$ be the poles. There are then integers $n_1, \ldots, n_k$ such that

$$F(z) := (z - P_1)^{n_1} \cdots (z - P_k)^{n_k} f(z)$$

is holomorphic. Clearly, F is rational if and only if f is rational. If F has a removable singularity at ∞, then F is bounded, and the proof is complete. If F has a pole at ∞, then F is a polynomial, and the proof is complete. □

4.8. Multiple-valued functions

Much of the material in this section can be found in [11] and [13, Chapter 4]. Here we will consider entire functions which are not one-to-one, so that the inverse function is multiple-valued. We will need the following result:

Lemma 4.33. *Suppose that $f(z) : G \mapsto \mathbb{C}$ is one-to-one and holomorphic, and suppose that $f^{-1}(z)$ is continuous on the range. If $z_0 \in G$ is such that $f'(z_0) \neq 0$, then f^{-1} is holomorphic at $f(z_0)$, and*

$$\frac{\mathrm{d}}{\mathrm{d}z} f^{-1}(f(z_0)) = \frac{1}{f'(z_0)}.$$

4.8.1. The mapping $w = z^{1/n}$

Write $z = |z|e^{i\vartheta}$. So that the function is one-to-one, the restriction must be made that $\vartheta \in [\vartheta_0 + 2k\pi, \vartheta_0 + 2(k+1)\pi)$ for some $k \in \mathbb{Z}$, where $\vartheta_0 \in \mathbb{R}$ is arbitrary. One has that

$$w = z^{1/n} = |z|^{1/n} \exp\left(\frac{i(\vartheta_0 + 2k\pi)}{n}\right),$$

where $k \in \mathbb{N}$ is arbitrary. Note that for $k = \ell$ one has that $z^{1/n} = a$, where $a := |z|^{1/n} e^{i(\vartheta_0 + 2\ell\pi)/n}$, while for $k = \ell + 1$ one has that $z^{1/n} = a e^{i2\pi/n}$. Thus, w is not continuous on the ray $\arg(z) = \vartheta_0$.

For each $k = 0, \dots, n-1$ let

$$\mathcal{G}_k := \{z \in \mathbb{C} : \arg(z) \in [\vartheta_0 + 2k\pi, \vartheta_0 + 2(k+1)\pi)\}.$$

Then $w : \mathcal{G}_k \mapsto \mathbb{C}$ is one-to-one for each k. The function $w|_{\mathcal{G}_k}$ is a branch of the multi-valued function, and the rays $\arg(z) = \vartheta_0 + 2k\pi$ are branch cuts. It is clear that $w|_{\mathcal{G}_k}$ is continuous. As a consequence of Lemma 4.33 one then has that $w|_{\mathcal{G}_k}$ is holomorphic on $\mathcal{G}_k \backslash \{0\}$, with

$$\frac{d}{dz} w|_{\mathcal{G}_k}(z) = \frac{1}{n w|_{\mathcal{G}_k}^{n-1}(z)}.$$

Definition 4.34. The point $\zeta \in \mathbb{C}$ is a branch point of $f(z)$ if there exists an $r_0 > 0$ such that one complete circuit around any Jordan curve $\gamma \subset D(\zeta, r_0)$ with $\zeta \in \text{int}(\gamma)$ yields that each branch of $f(z)$ is carried into another branch. If a finite number of circuits, say n, carries every branch into itself, then ζ is a branch point of order $n-1$. In this case, if $f(z)$ has a limit at ζ, then the branch point is an algebraic branch point.

Note that in the above discussion one has that n circuits around $z = 0$ carries $\mathcal{G}_k$ back to itself. It is then clear that $z = 0$ is an algebraic branch point of order $n-1$ for the function $z^{1/n}$. Further note that upon setting $s = 1/z$ the function $z^{1/n}$ becomes $s^{-1/n}$. This function has an algebraic branch point of order $n-1$ at $s = 0$. Hence, the point $z = \infty$ can also be regarded as an algebraic branch point of order $n-1$ for $z^{1/n}$.

Remark 4.35. It is easy to see that the function $(z-a)^{1/n}$ has an algebraic branch point at $z = a, \infty$ of order $n-1$. The function

$$\left(\frac{z-a}{z-b}\right)^{1/n}$$

has algebraic branch points of order $n-1$ at $z = a, b$. The point ∞ is no longer a branch point, as setting $s = 1/z$ yields

$$\left(\frac{z-a}{z-b}\right)^{1/n} \mapsto \left(\frac{1-as}{1-bs}\right)^{1/n},$$

which is holomorphic at $s = 0$.

4.8.2. The mapping $w = P(z)^{1/n}$

Now consider

$$w = [(z-a_1)^{\alpha_1} \cdots (z-a_k)^{\alpha_k}]^{1/n}.$$

From the previous section it is clear that the only potential finite branch points are $z = a_1, \dots, a_k$. When considering the function $(z-a_j)^{\alpha_j/n}$, suppose that $\alpha_j = \delta_j \alpha_j'$ and $n = \delta_j n_j$, where δ_j is the greatest common divisor of α_j and n. One then has that

$$(z-a_j)^{\alpha_j/n} = (z-a_j)^{\alpha_j'/n_j}.$$

Assuming that $\alpha_j'/n_j \notin \mathbb{Z}$, from the above discussion it follows that a_j is an algebraic branch point of order $n_j - 1$. If $\alpha_j'/n_j \in \mathbb{Z}$, the a_j is either a regular point or pole. Since for $z \sim a_j$ one has that

$$w = A_j(z)(z-a_j)^{\alpha_j/n},$$

where $A_j(z)$ is holomorphic, one then can classify each potential branch point. Letting $s = 1/z$ yields

$$w = s^{-N/n}[(1-a_1 s)^{a_1}\cdots(1-a_k s)^{a_k}]^{1/n}, \quad N := a_1 + \cdots + a_k.$$

If $N/n \notin \mathbb{Z}$, and if δ_∞ is the greatest common divisor of N and n with $n = \delta_\infty n_\infty$, then ∞ is an algebraic branch point of order $n_\infty - 1$.

Example. Consider $f(z) := [(z-a)(z-b)]^{1/2}$, where $a < 0 < b \in \mathbb{R}$. From the above discussion $z = a, b$ are each algebraic branch points of order one. Furthermore, since $N = n = 2$, ∞ is not a branch point. Setting

$$z - a = r_1 e^{i\vartheta_1}, \quad z - b = r_2 e^{i\vartheta_2}$$

yields

$$f(z) = \sqrt{r_1 r_2} e^{i(\vartheta_1+\vartheta_2)/2}.$$

If one takes $0 \le \vartheta_1, \vartheta_2 < 2\pi$, then the branch cut is $[a, b]$. If one takes $0 \le \vartheta_1 < 2\pi$ and $-\pi \le \vartheta_2 < \pi$, then the branch cut is $(-\infty, a] \cup [b, +\infty)$. In this case $f(0) = i\sqrt{|ab|}$. Finally, the same branch cut occurs if one takes $-\pi \le \vartheta_1 < \pi$ and $0 \le \vartheta_2 < 2\pi$, again with $f(0) = i\sqrt{|ab|}$. For a more complete discussion, see [1, Section 2.3].

4.8.3. The logarithm

Recall that $e^z = e^{z+i2k\pi}$ for any $k \in \mathbb{Z}$. Hence, the inverse can only be on

$$\mathcal{G}_k := \{z \in \mathbb{C} : \arg(z_0) + 2k\pi \le \arg(z) < \arg(z_0) + 2(k+1)\pi, \ k \in \mathbb{Z}\}.$$

On $\mathcal{G}_k$ the inverse is given by

$$\ln z := \ln|z| + i\arg z, \quad \arg(z_0) + 2k\pi \le \arg(z) < \arg(z_0) + 2(k+1)\pi;$$

furthermore, one has that

$$\frac{d}{dz}\ln z = \frac{1}{z}, \quad \arg(z_0) + 2k\pi \le \arg(z) < \arg(z_0) + 2(k+1)\pi.$$

Note that $z = 0$ and $z = \infty$ are branch points; however, they are branch points of infinite order (logarithmic branch points).

The function z^a for $a \in \mathbb{C}$ can now be defined as

$$z^a := e^{a\ln z}.$$

If $a \in \mathbb{R}$ is irrational, then $z = 0$ and $z = \infty$ are branch points of infinite order.

Example. Consider the following:

(a) $1^{\sqrt{3}} = e^{\sqrt{3}\ln 1} = e^{2\sqrt{3}k\pi i}$, $k \in \mathbb{Z}$

(b) $i^i = e^{i\ln i} = e^{i(4k+1)\pi i/2} = e^{-(4k+1)\pi/2}$, $k \in \mathbb{Z}$.

4.8.4. Computational examples

Lemma 4.36. *Let C_ϵ be a circular arc of radius ϵ centered at z_0.*

(a) If $(z - z_0)f(z) \to 0$ uniformly as $\epsilon \to 0$, then

$$\lim_{\epsilon\to 0}\oint_{C_\epsilon} f(z)\,dz = 0.$$

(b) *If $f(z)$ has a simple pole at z_0, then*

$$\lim_{\epsilon\to 0}\oint_{C_\epsilon} f(z)\,\mathrm{d}z = \mathrm{i}\vartheta\,\mathrm{Res}_f(z_0),$$

where the integration is carried out in the counterclockwise direction.

Remark 4.37. If the direction is carried out in a clockwise direction, then in part (b) one has that

$$\lim_{\epsilon\to 0}\oint_{C_\epsilon} f(z)\,\mathrm{d}z = -\mathrm{i}\vartheta\,\mathrm{Res}_f(z_0),$$

Proof: For part (a), one has that $|(z-z_0)f(z)| \le \delta_\epsilon$, where $\delta_\epsilon \to 0$ as $\epsilon \to 0$. In other words, $|f(z)| \le \delta_\epsilon/\epsilon$ on C_ϵ. One then has that

$$\left|\oint_{C_\epsilon} f(z)\,\mathrm{d}z\right| \le \int_0^\vartheta |f(z)|\epsilon\,\mathrm{d}\varphi = \delta_\epsilon\vartheta,$$

from which follows the result.

Now consider part (b). Since $f(z)$ has a simple pole at z_0, $f(z)$ has the Laurent expansion

$$f(z) = \frac{a_{-1}}{z-z_0} + g(z),$$

where $g(z)$ is holomorphic in a neighborhood of z_0. By part (a),

$$\lim_{\epsilon\to 0}\oint_{C_\epsilon} g(z)\,\mathrm{d}z = 0.$$

Setting $z = z_0 + \epsilon\mathrm{e}^{\mathrm{i}\varphi}$ then yields that

$$\oint_{C_\epsilon}\frac{a_{-1}}{z-z_0}\,\mathrm{d}z = a_{-1}\int_0^\vartheta \mathrm{i}\,\mathrm{d}\varphi = \mathrm{i}\vartheta a_{-1},$$

from which follows the result. □

Example. Consider

$$\int_{-\infty}^{+\infty}\frac{\cos x - \cos a}{x^2-a^2}\,\mathrm{d}x, \quad a \in \mathbb{R}. \tag{4.3}$$

Note that the integrand is well-defined at $x = \pm a$, and that the integral converges. This will be done by computing

$$\oint_\gamma \frac{\mathrm{e}^{\mathrm{i}z} - \cos a}{z^2-a^2}\,\mathrm{d}z,$$

where γ is composed of the pieces:

$$\begin{aligned}
\gamma^- &:= -R + t, \quad 0 \le t \le R - a - \epsilon\\
\gamma^0 &:= -a + \epsilon + t, \quad 0 \le t \le 2a - 2\epsilon\\
\gamma^+ &:= a + \epsilon + t, \quad 0 \le t \le R - a - \epsilon\\
\gamma_\epsilon^\pm &:= z = \pm a + \epsilon\mathrm{e}^{\mathrm{i}\varphi}, \quad 0 \le \varphi \le \pi\\
\gamma_R &:= R\mathrm{e}^{\mathrm{i}\varphi}, \quad 0 \le \varphi \le \pi
\end{aligned}$$

(see Figure 3).

As a consequence of Lemma 4.24 it is known that

$$\lim_{R\to\infty}\oint_{\gamma_R}\frac{\mathrm{e}^{\mathrm{i}z} - \cos a}{z^2-a^2}\,\mathrm{d}z = 0.$$

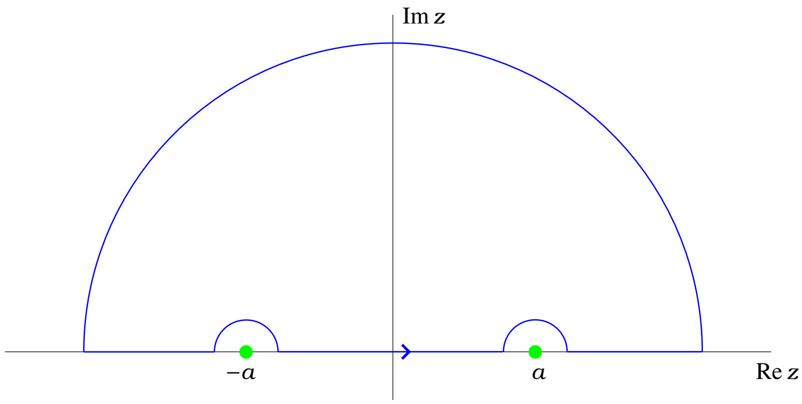


Figure 3: The contour of integration associated with equation (4.3).

Furthermore, by Lemma 4.36 one has that

$$\lim_{\epsilon\to 0}\oint_{\gamma_\epsilon^\pm}\frac{\mathrm{e}^{\mathrm{i}z}-\cos a}{z^2-a^2}\,\mathrm{d}z=-\mathrm{i}\pi\,\mathrm{Res}_f(\pm a)=-\mathrm{i}\pi\left(\pm\frac{\mathrm{e}^{\pm\mathrm{i}a}}{2a}\mp\frac{\cos a}{2a}\right)$$

(note that the direction is clockwise). One then gets that

$$0=\lim_{\epsilon\to 0,R\to\infty}\oint_\gamma\frac{\mathrm{e}^{\mathrm{i}z}-\cos a}{z^2-a^2}\,\mathrm{d}z=\int_{-\infty}^{+\infty}\frac{\mathrm{e}^{\mathrm{i}x}-\cos a}{x^2-a^2}\,\mathrm{d}x+\frac{\pi\sin a}{a},$$

from which one gets by taking the real part that

$$\int_{-\infty}^{+\infty}\frac{\cos x-\cos a}{x^2-a^2}\,\mathrm{d}x=-\frac{\pi\sin a}{a}.$$

Example. Consider

$$\int_0^{+\infty}\frac{\ln^2 x}{x^2+a^2}\,\mathrm{d}x,\quad a\in\mathbb{R}^+. \tag{4.4}$$

Note that the improper integral converges at both $x=0$ and $x=+\infty$. This will be evaluated by computing

$$\oint_\gamma\frac{\ln^2 z}{z^2+a^2}\,\mathrm{d}z,$$

where γ is composed of the pieces:

$$\begin{aligned}\gamma^- &:= \epsilon+t,\quad 0\le t\le R-\epsilon\\ \gamma^+ &:= -R+t,\quad 0\le t\le R-\epsilon\\ \gamma_\epsilon &:= z=\epsilon\mathrm{e}^{\mathrm{i}\varphi},\quad 0\le\varphi\le\pi\\ \gamma_R &:= R\mathrm{e}^{\mathrm{i}\varphi},\quad 0\le\varphi\le\pi\end{aligned}$$

(see Figure 4). In the above $0<\epsilon<a<R$. Furthermore, $\ln z$ will be defined on the branch $-\pi/2\le\arg(z)<3\pi/2$.

Note that upon applying Lemma 4.24,

$$\lim_{R\to\infty}\oint_{\gamma_R}\frac{\ln^2 z}{z^2+a^2}\,\mathrm{d}z=0,$$

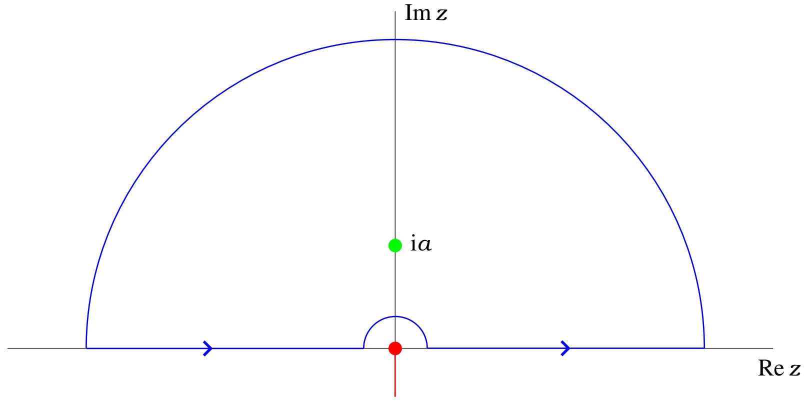


Figure 4: The contour of integration associated with equation (4.4).

and that since $z(\ln z)^2 \to 0$ as $z \to 0$, one has that upon applying Lemma 4.36

$$\lim_{\epsilon\to\infty} \oint_{\gamma_\epsilon} \frac{\ln^2 z}{z^2+a^2}\,\mathrm{d}z = 0.$$

Now consider the curve γ^-. Here one has that

$$\begin{aligned}\oint_{\gamma^-} \frac{\ln^2 z}{z^2+a^2}\,\mathrm{d}z &= \int_R^\epsilon \frac{\ln^2(r\mathrm{e}^{\mathrm{i}\pi})}{(r\mathrm{e}^{\mathrm{i}\pi})^2+a^2}\mathrm{e}^{\mathrm{i}\pi}\,\mathrm{d}r \\ &= \int_\epsilon^R \frac{\ln^2 x + 2\mathrm{i}\pi\ln x - \pi^2}{x^2+a^2}\,\mathrm{d}x.\end{aligned}$$

Thus, upon letting $\epsilon \to 0$ and $R \to \infty$ one has that

$$\begin{aligned}\oint_\gamma \frac{\ln^2 z}{z^2+a^2}\,\mathrm{d}z &= 2\int_0^{+\infty} \frac{\ln^2 x}{x^2+a^2}\,\mathrm{d}x + 2\mathrm{i}\pi\int_0^{+\infty}\frac{\ln x}{x^2+a^2}\,\mathrm{d}x - \pi^2\int_0^{+\infty}\frac{1}{x^2+a^2}\,\mathrm{d}x \\ &= 2\int_0^{+\infty} \frac{\ln^2 x}{x^2+a^2}\,\mathrm{d}x + 2\mathrm{i}\pi\int_0^{+\infty}\frac{\ln x}{x^2+a^2}\,\mathrm{d}x - \frac{\pi^3}{2a}.\end{aligned}$$

Since

$$\mathrm{Res}_f(\mathrm{i}a) = \frac{\ln^2(\mathrm{i}a)}{2\mathrm{i}a} = \frac{\pi}{2a}\ln a - \mathrm{i}\frac{\ln^2 a - \pi^2/4}{2a},$$

by the residue theorem one has that

$$\oint_\gamma \frac{\ln^2 z}{z^2+a^2}\,\mathrm{d}z = \frac{\pi}{a}(\ln^2 a - \frac{\pi^2}{4}) + \mathrm{i}\frac{\pi^2\ln a}{a}.$$

Equating real and imaginary parts then yields that

$$\int_0^{+\infty} \frac{\ln^2 x}{x^2+a^2}\,\mathrm{d}x = \frac{\pi^3}{8a} + \frac{\pi\ln^2 a}{2a},$$

and that

$$\int_0^{+\infty} \frac{\ln x}{x^2+a^2}\,\mathrm{d}x = \frac{\pi\ln a}{2a}.$$

Example. Consider

$$\int_0^\infty \frac{x^a}{x^2+1}\,\mathrm{d}x, \quad |a| < 1.$$

Note that the improper integral converges at both $x = 0$ and $x = \infty$. This will be computed by evaluating

$$\oint_\gamma \frac{z^a}{z^2+1}\,\mathrm{d}z$$

where γ is the same contour as in the previous example, with $0 < \epsilon < 1 < R$. Furthermore, z^a will be defined on the branch $-\pi/2 \le \arg(z) < 3\pi/2$. Note that

$$\lim_{R\to\infty} \oint_{\gamma_R} \frac{z^a}{z^2+1}\,\mathrm{d}z = 0,$$

and that since $z^{1+a} \to 0$ as $z \to 0$,

$$\lim_{\epsilon\to\infty} \oint_{\gamma_\epsilon} \frac{z^a}{z^2+1}\,\mathrm{d}z = 0.$$

Now consider the curve γ^-. Here one has that

$$\begin{aligned}\oint_{\gamma^-} \frac{z^a}{z^2+1}\,\mathrm{d}z &= \int_R^\epsilon \frac{(r\mathrm{e}^{\mathrm{i}\pi})^a}{(r\mathrm{e}^{\mathrm{i}\pi})^2+1}\mathrm{e}^{\mathrm{i}\pi}\,\mathrm{d}r \\ &= \mathrm{e}^{\mathrm{i}a\pi}\int_\epsilon^R \frac{x^a}{x^2+1}\,\mathrm{d}x.\end{aligned}$$

Thus, upon letting $\epsilon \to 0$ and $R \to \infty$ one has that

$$\oint_\gamma \frac{z^a}{z^2+1}\,\mathrm{d}z = \mathrm{e}^{\mathrm{i}a\pi/2}(\mathrm{e}^{\mathrm{i}a\pi/2} + \mathrm{e}^{-\mathrm{i}a\pi/2})\int_0^\infty \frac{x^a}{x^2+1}\,\mathrm{d}x.$$

Since

$$\mathrm{Res}_f(\mathrm{i}) = \frac{\mathrm{e}^{\mathrm{i}a\pi/2}}{2\mathrm{i}},$$

by the residue theorem one has that

$$\oint_\gamma \frac{z^a}{z^2+1}\,\mathrm{d}z = \pi\mathrm{e}^{\mathrm{i}a\pi/2}.$$

Equating real and imaginary parts then yields that

$$\int_0^\infty \frac{x^a}{x^2+1}\,\mathrm{d}x = \frac{\pi}{2\cos(a\pi/2)}.$$

4.9. The Cauchy Principal Value

In Problem **4.55** you were asked to compute

$$\int_{-\infty}^{+\infty} \frac{x}{\sinh(x)-1}\,\mathrm{d}x.$$

However, this integral does not converge, as there is a pole of order one at $x^+ := \ln(\sqrt{2}+1)$. A careful look at the calculation shows, however, that you actually computed

$$\lim_{\epsilon\to 0^+}\Big(\int_{-\infty}^{x^+-\epsilon} + \int_{x^++\epsilon}^{+\infty}\Big)\frac{x}{\sinh(x)-1}\,\mathrm{d}x,$$

which did converge.

Proposition 4.38. *Suppose that $f \in C^1([-1,+1])$. Then*

$$\lim_{\epsilon\to 0^+}\Big(\int_{-1}^{-\epsilon} + \int_{+\epsilon}^{+1}\Big)\frac{f(x)}{x}\,\mathrm{d}x$$

exists.

Proof: By Taylor's theorem one has that

$$f(x) = f(0) + x\int_0^1 f'(sx)\,\mathrm{d}s,$$

so that

$$\frac{f(x)}{x} = \frac{f(0)}{x} + \int_0^1 f'(sx)\,\mathrm{d}s.$$

It is clear that

$$\lim_{\epsilon\to0^+}\Big(\int_{-1}^{-\epsilon} + \int_{+\epsilon}^{+1}\Big)\int_0^1 f'(sx)\,\mathrm{d}s\,\mathrm{d}x < \infty.$$

Now, it can be quickly checked that

$$\lim_{\epsilon\to0^+}\Big(\int_{-1}^{-\epsilon} + \int_{+\epsilon}^{+1}\Big)\frac{f(0)}{x}\,\mathrm{d}x = 0;$$

hence, the limit exists. □

Definition 4.39. Suppose that $f(x)$ has a pole at $x = x_0$. If $a < x_0 < b$, the Cauchy Principal Value integral is given by

$$\mathrm{P.V.}\int_a^b f(x)\,\mathrm{d}x := \lim_{\epsilon\to0^+}\left(\int_a^{x_0-\epsilon} f(x)\,\mathrm{d}x + \int_{x_0+\epsilon}^b f(x)\,\mathrm{d}x\right).$$

Example. One has that

$$\mathrm{P.V.}\int_{-1}^2 \frac{\mathrm{d}x}{x} = \ln 2,$$

while in general the integral does not exist.

Example. Consider

$$\mathrm{P.V.}\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{px}}{1-\mathrm{e}^x}\,\mathrm{d}x, \quad p \in (0,1). \tag{4.5}$$

Note that the improper integral converges at $x = \pm\infty$; however, it diverges at the simple pole $x = 0$. For this reason one must compute the Cauchy Principal Value of the integral. Upon setting

$$f(z) := \frac{\mathrm{e}^{pz}}{1-\mathrm{e}^z},$$

one sees that $f(z)$ has simple poles at $z = \mathrm{i}2k\pi$, $k \in \mathbb{Z}$; furthermore, one sees that $f(z+\mathrm{i}2\pi) = \mathrm{e}^{\mathrm{i}2\pi p}f(z)$. This second observation justifies evaluating

$$\oint_\gamma f(z)\,\mathrm{d}z,$$

where γ is composed of the pieces (see Figure 5):

$$\begin{aligned}
\gamma_0^- &:= \epsilon + t, \quad 0 \le t \le R-\epsilon\\
\gamma_0^+ &:= -R + t, \quad 0 \le t \le R-\epsilon\\
\gamma_0^\epsilon &:= \epsilon\mathrm{e}^{\mathrm{i}\phi}, \quad 0 \le \phi \le \pi\\
\gamma_1^- &:= \epsilon + t + \mathrm{i}2\pi, \quad 0 \le t \le R-\epsilon\\
\gamma_1^+ &:= -R + t + \mathrm{i}2\pi, \quad 0 \le t \le R-\epsilon\\
\gamma_1^\epsilon &:= \mathrm{i}2\pi + \epsilon\mathrm{e}^{\mathrm{i}\phi}, \quad \pi \le \phi \le 2\pi\\
\gamma_\ell &:= -R + \mathrm{i}t, \quad 0 \le t \le 2\pi\\
\gamma_\mathrm{r} &:= R + \mathrm{i}t, \quad 0 \le t \le 2\pi.
\end{aligned}$$

An application of the residue theorem shows that

$$\oint_\gamma f(z)\,\mathrm{d}z = 0.$$

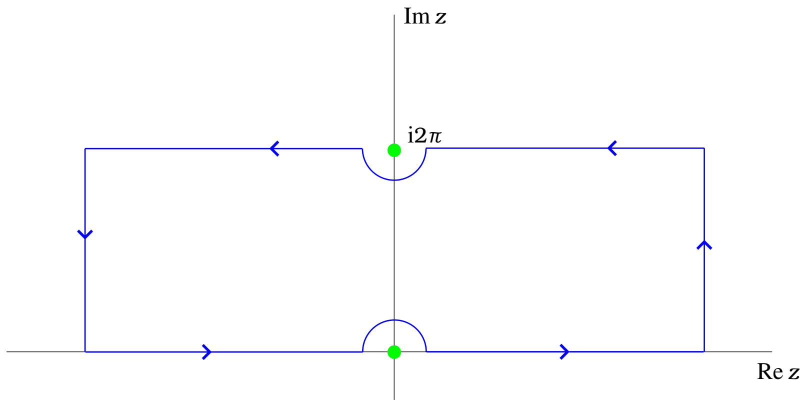


Figure 5: The contour of integration associated with equation (4.5).

It is not difficult to show that

$$\lim_{R\to+\infty}\oint_{\gamma_{t,r}} f(z)\,\mathrm{d}z = 0.$$

Furthermore, upon applying Lemma 4.36 one has that

$$\lim_{\epsilon\to0^+}\oint_{\gamma_0^\epsilon} f(z)\,\mathrm{d}z = \mathrm{i}\pi,\quad \lim_{\epsilon\to0^+}\oint_{\gamma_1^\epsilon} f(z)\,\mathrm{d}z = \mathrm{i}\pi\mathrm{e}^{\mathrm{i}2\pi p}.$$

One then sees that upon letting $R\to+\infty$, $\epsilon\to0^+$,

$$(1-\mathrm{e}^{\mathrm{i}2\pi p})\,\mathrm{P.V.}\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x + \mathrm{i}\pi(1+\mathrm{e}^{\mathrm{i}2\pi p}) = 0,$$

i.e.,

$$\mathrm{P.V.}\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = \pi\cot(\pi p).$$

An important application of complex variables is to solve equations for functions analytic in a certain region, given a relationship on a boundary (the Riemann-Hilbert problem). A simple example is the following. Suppose that $\psi^+(z)$ is holomorphic for $\operatorname{Im} z > 0$, and $\psi^-(z)$ is holomorphic for $\operatorname{Im} z < 0$. Further suppose that there is an absolutely integrable function $f(z)$ which is holomorphic on $\operatorname{Im} z = 0$. We wish to find $\psi^\pm(z)$ such that $\psi^\pm(z)\to0$ as $z\to\infty$ and

$$\psi^+(x)-\psi^-(x) = f(x),\quad x\in\mathbb{R}.$$

Set

$$F^\pm(x) := \lim_{\epsilon\to0^+}\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\frac{f(z)}{z-(x\pm\mathrm{i}\epsilon)}\,\mathrm{d}z,$$

and consider $F^+(x)$. Set γ to be the contour composed of the pieces:

$$\begin{aligned}
\gamma^- &:= -R+t,\quad 0\le t\le R-x-\delta\\
\gamma^+ &:= x+\delta+t,\quad 0\le t\le R-\delta\\
\gamma_\delta &:= z = x+\delta\mathrm{e}^{\mathrm{i}\varphi},\quad \pi\le\varphi\le2\pi.
\end{aligned}$$

If one sets $h(z)$ to be the integrand, one clearly has that

$$\mathrm{Res}_h(x+\mathrm{i}\epsilon) = f(x+\mathrm{i}\epsilon).$$

Hence, one gets that

$$\begin{aligned} F^+(x) &= \lim_{R\to\infty}\lim_{\delta\to 0^+}\lim_{\epsilon\to 0^+}\oint_\gamma h(z)\,\mathrm{d}z \\ &= \text{P.V.}\ \frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{+\infty}\frac{f(z)}{z-x}\,\mathrm{d}z + \frac{1}{2}f(x). \end{aligned}$$

Similarly,

$$F^-(x) = \text{P.V.}\ \frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{+\infty}\frac{f(z)}{z-x}\,\mathrm{d}z - \frac{1}{2}f(x).$$

Note that this implies that

$$F^+(x) - F^-(x) = f(x).$$

Thus, the problem is solved by setting $\psi^\pm(x) = F^\pm(x)$.

Definition 4.40. The Hilbert transform is given by

$$H(f)(x) := \text{P.V.}\ \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{f(y)}{y-x}\,\mathrm{d}y.$$

Remark 4.41. One has that:

(a) The solution to the above problem in the case that $f(z)$ is holomorphic on $\operatorname{Im} z = 0$ is then given by

$$\psi^\pm(x) = \frac{1}{2}(\pm 1 - \mathrm{i}H)f(x).$$

(b) It can be shown that the above solution is still valid even in the case that $f(x)$ is only Hölder continuous for $x \in \mathbb{R}$ [1, Lemma 7.2.1].

Example. Let us compute $H(f)$, where $f(x) := 1/(1+x^2)$. Set $g(z) := f(z)/(z-x)$. We will evaluate

$$\oint_\gamma g(z)\,\mathrm{d}z,$$

where we consider the contour γ which is composed of the pieces:

$$\begin{aligned} \gamma^- &:= -R + t, \quad 0 \le t \le R - x - \delta \\ \gamma^+ &:= x + \delta + t, \quad 0 \le t \le R - \delta \\ \gamma_\epsilon &:= z = x + \epsilon \mathrm{e}^{\mathrm{i}\varphi}, \quad 0 \le \varphi \le \pi \\ \gamma_R &:= z = R\mathrm{e}^{\mathrm{i}\varphi}, \quad 0 \le \varphi \le \pi. \end{aligned}$$

It is clear that

$$\lim_{R\to\infty}\oint_{\gamma_R} g(z)\,\mathrm{d}z = 0, \qquad \lim_{\epsilon\to 0^+}\oint_{\gamma_\epsilon} g(z)\,\mathrm{d}z = -\mathrm{i}\pi f(x).$$

Finally, as an application of the residue theorem one has that

$$\oint_\gamma g(z)\,\mathrm{d}z = -\pi\left(\frac{x+\mathrm{i}}{1+x^2}\right).$$

Thus, in the limit one has that

$$\text{P.V.}\ \int_{-\infty}^{+\infty}\frac{f(z)}{z-x}\,\mathrm{d}z - \mathrm{i}\pi f(x) = -\pi\left(\frac{x+\mathrm{i}}{1+x^2}\right)$$

i.e.,

$$H(f)(x) = -\frac{x}{1+x^2}.$$

5. The Zeros of a Holomorphic Function

Throughout this chapter it will be assumed that $f : U \mapsto \mathbb{C}$ is holomorphic, where $U \subset \mathbb{C}$ is open and connected. It will further be assumed that $f \not\equiv 0$ on U.

5.1. Counting zeros and poles

Suppose that $f(z_0) = 0$ for some $z_0 \in U$. There exists an $N \in \mathbb{N}$ such that $f^{(N)}(z_0) \neq 0$, so that f has the Taylor expansion

$$f(z) = \sum_{n=N}^{+\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \quad z \in \overline{D}(z_0, r).$$

Definition 5.1. f has a zero or order (multiplicity) N. If $N = 1$, the point z_0 is a simple zero.

Now set

$$H(z) := \frac{f(z)}{(z - z_0)^N}.$$

One has that $H(z)$ is holomorphic on $\overline{D}(z_0, r)$ with

$$H(z_0) = \frac{f^{(N)}(z_0)}{N!} \neq 0.$$

Now, for $\zeta \in \overline{D}(z_0, r)\backslash\{z_0\}$ one has that

$$\begin{aligned} \frac{f'(\zeta)}{f(\zeta)} &= \frac{(\zeta - z_0)^N H'(\zeta) + N(\zeta - z_0)^{N-1}H(\zeta)}{(\zeta - z_0)^N H(\zeta)} \\ &= \frac{H'(\zeta)}{H(\zeta)} + \frac{N}{\zeta - z_0}. \end{aligned}$$

By construction $H'(\zeta)/H(\zeta)$ is holomorphic on $\overline{D}(z_0, r)$; hence,

$$\oint_{\partial D(z_0,r)} \frac{H'(\zeta)}{H(\zeta)}\, \mathrm{d}\zeta = 0.$$

Thus,

$$\oint_{\partial D(z_0,r)} \frac{f'(\zeta)}{f(\zeta)}\, \mathrm{d}\zeta = N \oint_{\partial D(z_0,r)} \frac{1}{\zeta - z_0}\, \mathrm{d}\zeta = 2\pi \mathrm{i} N.$$

The following proposition has now been proven.

Proposition 5.2. *If f has a zero of order N at z_0 and no other zeros in $\overline{D}(z_0, r)$, then*

$$\frac{1}{2\pi \mathrm{i}} \oint_{\partial D(z_0,r)} \frac{f'(\zeta)}{f(\zeta)}\, \mathrm{d}\zeta = N.$$

Now suppose that f has k zeros $z_1, \dots, z_k \in D(P, r)$ with multiplicities n_ℓ at z_ℓ, and further suppose that f is nonvanishing on $\partial D(P, r)$. Setting

$$H(z) := \frac{f(z)}{(z - z_1)^{n_1} \cdots (z - z_k)^{n_k}}$$

yields that $H(z)$ is nonzero and holomorphic on $\overline{D}(P, r)$. Calculating as above yields that

$$\frac{f'(\zeta)}{f(\zeta)} = \frac{H'(\zeta)}{H(\zeta)} + \sum_{\ell=1}^{k} \frac{n_\ell}{\zeta - z_\ell},$$

which further yields that

$$\oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)}\, \mathrm{d}\zeta = 2\pi \mathrm{i} \sum_{\ell=1}^{k} n_\ell.$$

We have now proven:

Lemma 5.3 (Argument Principle). *If f has zeros of order n_ℓ at the points $z_\ell \in D(P,r)$ and is nonzero on $\partial D(P,r)$, then*

$$\frac{1}{2\pi\mathrm{i}}\oint_{\partial D(P,r)}\frac{f'(\zeta)}{f(\zeta)}\,\mathrm{d}\zeta=\sum_{\ell=1}^{k}n_\ell.$$

Remark 5.4. If one writes

$$f(z)=|f(z)|\mathrm{e}^{\mathrm{i}\arg f(z)},$$

then one has that

$$\frac{f'(\zeta)}{f(\zeta)}=\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln(f(\zeta)),$$

so that

$$\frac{1}{2\pi\mathrm{i}}\oint_{\partial D(P,r)}\frac{f'(\zeta)}{f(\zeta)}\,\mathrm{d}\zeta=\frac{1}{2\pi}(\arg f(z))|_{\partial D(P,r)}.$$

The quantity

$$\frac{1}{2\pi}(\arg f(z))|_{\partial D(P,r)}$$

is the winding number.

Example. Let us determine the number of zeros (counting multiplicity) of the function $f(z):=z^3+1$ located in the first quadrant. Define

$$\arg f(z)=\phi,\quad \tan\phi=\frac{\operatorname{Im}f(z)}{\operatorname{Re}f(z)}.$$

Set γ to be the curve composed of the pieces:

$$\begin{aligned}\gamma_{\mathrm{r}}&:=t,\quad 0\le t\le R\\ \gamma_{\mathrm{i}}&:=\mathrm{i}t,\quad 0\le t\le R\\ \gamma_R&:=R\mathrm{e}^{\mathrm{i}\phi},\quad 0\le\phi\le\pi/2.\end{aligned}$$

On γ_{r} it is clear that $\arg f(z)=0$. Upon setting $z:=R\mathrm{e}^{\mathrm{i}\vartheta}$, one has that for $R\gg1$ that on γ_R, $f(z)\sim R^3\mathrm{e}^{\mathrm{i}3\vartheta}$, so that as $R\to+\infty$,

$$0\le\arg f(z)\le\frac{3\pi}{2}.$$

Finally, consider γ_{i}. On this curve $f(z)=-\mathrm{i}y^3+1$, so that for $\tan\phi$ runs from $3\pi/2$ to 2π as y descends from $+\infty$ to 0. Here we use the fact that $\operatorname{Im}f<0$ and $\operatorname{Re}f>0$ on γ_{i}. Thus, from the Argument Principle we have that

$$\lim_{R\to\infty}\oint_\gamma\frac{f'(\zeta)}{f(\zeta)}\,\mathrm{d}\zeta=1.$$

Example. When considering the stability of nonlinear waves, it is important to find those eigenvalues which satisfy $\operatorname{Re}\lambda>0$, as these eigenvalues correspond to instabilities. In some circumstances there is a function, $E(\lambda)$, holomorphic in $U:=\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>0\}$ such that $E(\lambda)=0$ for $\lambda\in U$ if and only if λ is an eigenvalue. This function $E(\lambda)$ is the generalization of the characteristic equation used to calculate the eigenvalues of matrices. The interested student should consult my web page for more details.

Now suppose that f has a pole of order N at $z=P$, and is nowhere zero on $\overline{D}(P,r)\backslash\{P\}$. Upon setting

$$H(z):=(z-P)^Nf(z),$$

one has that $H(z)$ is holomorphic and nonzero on $\overline{D}(P,r)$. Arguing as above yields that

$$\frac{H'(\zeta)}{H(\zeta)}=\frac{f'(\zeta)}{f(\zeta)}+\frac{N}{\zeta-P},$$

so that

$$\oint_{\partial D(P,r)}\frac{f'(\zeta)}{f(\zeta)}\,\mathrm{d}\zeta=-2\pi\mathrm{i}N.$$

Clearly this formula generalizes to multiple poles. Combining this result along with the Argument Principle yields the following.

Lemma 5.5 (Argument Principle for Meromorphic Functions). *Suppose that f is meromorphic on $\overline{D}(P,r)$ and has neither zeros nor poles on $\partial D(P,r)$. Let $z_1,\dots,z_p \in D(P,r)$ be the zeros of order n_ℓ, and let $w_1,\dots,w_q \in D(P,r)$ be the poles of order m_ℓ. Then*

$$\frac{1}{2\pi\mathrm{i}}\oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)}\,\mathrm{d}\zeta = \sum_{\ell=1}^{p} n_\ell - \sum_{\ell=1}^{q} m_\ell.$$

5.2. The local geometry of holomorphic functions

Theorem 5.6 (Open mapping theorem). *Let $f : U \mapsto \mathbb{C}$ be a nonconstant holomorphic function on the open connected set U. Then $f(U) \subset \mathbb{C}$ is open.*

Proof: Given $Q \in f(U)$, we need to show that there is an $\epsilon > 0$ such that $D(Q,\epsilon) \subset f(U)$.

Select $P \in U$ such that $f(P) = Q$, and set $g(z) := f(z) - Q$. Since $g(P) = 0$ and $g(z)$ is nonconstant, there is an $r > 0$ such that $\overline{D}(P,r) \subset U$ and $g(z) \neq 0$ for $z \in \overline{D}(P,r)\backslash\{P\}$. The argument principle implies that there is an $n \in \mathbb{N}$ such that

$$\frac{1}{2\pi\mathrm{i}}\oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta) - Q}\,\mathrm{d}\zeta = n.$$

By the continuity of $g(z)$ and the fact that $\partial D(P,r)$ is compact, one has that there is an $\epsilon > 0$ such that $|g(z)| > \epsilon$ for $z \in \partial D(P,r)$. The claim is that for this ϵ, $D(Q,\epsilon) \subset f(U)$.

Set

$$N(z) := \frac{1}{2\pi\mathrm{i}}\oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta) - z}\,\mathrm{d}\zeta, \quad z \in D(Q,\epsilon),$$

and note that $N(Q) = n$. Further note that for $z \in D(Q,\epsilon)$,

$$|f(\zeta) - z| \geq |f(\zeta) - Q| - |z - Q| > \epsilon - |z - Q| > 0;$$

hence, a standard argument yields that $N(z)$ is holomorphic with

$$N'(z) = \frac{1}{2\pi\mathrm{i}}\oint_{\partial D(P,r)} \frac{f'(\zeta)}{(f(\zeta) - z)^2}\,\mathrm{d}\zeta.$$

Setting

$$H(\zeta) := -\frac{1}{f(\zeta) - z},$$

one clearly has that

$$H'(\zeta) = \frac{f'(\zeta)}{(f(\zeta) - z)^2}.$$

Hence, $N'(z) = 0$ for all $z \in D(Q,\epsilon)$, so that $N(z)$ is a constant. From the above one then has that $N(z) = n$.

Since $n \geq 1$, for each fixed $z \in D(Q,\epsilon)$ there is a $\zeta \in D(P,r)$ such that $f(\zeta) = z$. Hence, $D(Q,\epsilon) \subset f(U)$, which proves the claim. □

Remark 5.7. The fact that f is holomorphic is crucial. For a counterexample, consider $f(z) := |z|^2$, for which $f(\mathbb{C}) = \mathbb{R}$, and $\mathbb{R} \subset \mathbb{C}$ is not open.

Suppose that $f(z) - Q$ has a zero of order k for some $k \in \mathbb{N}$. Then to lowest order one has that $f(z) \sim Q + a_k(z-P)^k$, so that every $w \in D(Q,\epsilon)$ has k distinct preimages, i.e.,

$$z \sim \left(\frac{w - Q}{a_k}\right)^{1/k}.$$

Note that in this scenario, $f'(P) = 0$. The point P is called a multiple point of order k. The question to be answered: can this heuristic argument be made rigorous?

Lemma 5.8. *Let $f : U \mapsto \mathbb{C}$ be a nonconstant holomorphic function on the open connected set U. The multiple points are isolated.*

Proof: Since f is nonconstant, f' is not identically zero. Since f' is holomorphic, the zeros of f' are isolated. □

Remark 5.9. Alternatively, one has that branch points of f^{-1} are isolated.

Theorem 5.10. *Let $f : U \mapsto \mathbb{C}$ be a nonconstant holomorphic function on the open connected set U. Let $P \in U$ be such that $f(P) = Q$ with order k. There exists a $\delta, \epsilon > 0$ such that each $q \in D(Q,\epsilon)\backslash\{Q\}$ has k distinct preimages in $D(P,\delta)$.*

Proof: From the previous lemma there exists a $\delta_1 > 0$ such that each $z \in D(P,\delta_1)\backslash\{P\}$ is a simple point of f. Take $0 < \delta < \delta_1$, and choose $\epsilon > 0$ such that $D(Q,\epsilon) \subset f(D(P,\delta))$ with $D(Q,\epsilon) \cap f(\partial D(P,\delta)) = \emptyset$. As in the proof of the Open Mapping Theorem, for each $q \in D(Q,\epsilon)\backslash\{Q\}$ one has that

$$\frac{1}{2\pi \mathrm{i}} \oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta) - q}\,\mathrm{d}\zeta = k.$$

If $p_1, \dots, p_\ell \in D(P,\delta)$ are the zeros of $f(z) - q$ with $n_1, \dots, n_\ell$ as their orders, then $\sum n_j = k$. However, the choice of δ yields that $n_j = 1$ for each j, so that $\ell = k$. Hence, the point q has precisely k preimages $p_1, \dots, p_k$, each of which is a simple point of f. □

5.3. Further results on the zeros of holomorphic functions

Let $f, g : U \mapsto \mathbb{C}$ be holomorphic. Suppose that $\overline{D}(P,r) \subset U$, and that for each $\zeta \in \partial D(P,r)$ one has that

$$|f(\zeta) - g(\zeta)| < |g(\zeta)|. \tag{5.1}$$

If one further assumes that both f and g are nonzero for $\zeta \in \overline{D}(P,r)$, the number of zeros of f and g within $D(P,r)$ can be determined by a winding number calculation. If one defines

$$f_t(\zeta) := tf(\zeta) + (1-t)g(\zeta) = g(\zeta) + t(f(\zeta) - g(\zeta)), \quad t \in [0,1]$$

one has that as a consequence of equation (5.1) that $f_t(\zeta) \neq 0$ for each $t \in [0,1]$. Now set

$$I_t := \frac{1}{2\pi \mathrm{i}} \oint_{\partial D(P,r)} \frac{f_t'(\zeta)}{f_t(\zeta)}\,\mathrm{d}\zeta.$$

Since $I_t \in \mathbb{N}$ is continuous, there exists an integer k such that $I_t \equiv k$. In particular, $I_0 = I_1$. This argument yields:

Theorem 5.11 (Rouché's theorem)**.** *Let $f, g : U \mapsto \mathbb{C}$ be holomorphic. Suppose that $\overline{D}(P,r) \subset U$, and that for each $\zeta \in \partial D(P,r)$ one has that*

$$|f(\zeta) - g(\zeta)| < |g(\zeta)|.$$

If f and g are nonzero for $\zeta \in \partial D(P,r)$, then

$$\frac{1}{2\pi \mathrm{i}} \oint_{\partial D(P,r)} \frac{f'(\zeta)}{f(\zeta)}\,\mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \oint_{\partial D(P,r)} \frac{g'(\zeta)}{g(\zeta)}\,\mathrm{d}\zeta.$$

If one maps $f \mapsto f + g$ in Theorem 5.11, then one gets the following version of Rouché's theorem.

Corollary 5.12. *Let $f, g : U \mapsto \mathbb{C}$ be holomorphic. Suppose that $\overline{D}(P,r) \subset U$, and that for each $\zeta \in \partial D(P,r)$ one has that*

$$|f(\zeta)| < |g(\zeta)|.$$

Set $h(z) := f(z) + g(z)$. If h and g are nonzero for $\zeta \in \partial D(P,r)$, then

$$\frac{1}{2\pi \mathrm{i}} \oint_{\partial D(P,r)} \frac{g'(\zeta)}{g(\zeta)}\,\mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \oint_{\partial D(P,r)} \frac{h'(\zeta)}{h(\zeta)}\,\mathrm{d}\zeta.$$

Example. Consider $h(z) := e^z - 4z - 1$. We will show that $h(z) = 0$ has exactly one solution in $D(0, 1)$. Set

$$g(z) := -4z, \quad f(z) := e^z - 1.$$

For $z \in \partial D(0, 1)$ one has that

$$|g(z)| = 4, \quad |f(z)| \le |e^z| + 1 < e + 1,$$

so that $|f(z)| < |g(z)|$ for $z \in \partial D(0, 1)$. Since $g(z) = 0$ has only one solution in $D(0, 1)$, by Rouché's theorem $h(z) = 0$ has only one solution.

Theorem 5.13 (Hurwitz's theorem)**.** *Suppose that $U \subset \mathbb{C}$ is open and connected, and that $f_j : U \mapsto \mathbb{C}$ is holomorphic and nowhere vanishing for each $j \in \mathbb{N}$. If $f_j \to f_0$ uniformly on compact subsets of U, then either $f_0 \equiv 0$ or $f_0(z) = 0$ has no solution.*

Remark 5.14. By hypothesis, f_0 is holomorphic.

Proof: Suppose that there exists a $P \in U$ such that $f_0(P) = 0$. If $f_0 \not\equiv 0$, then there exists an $r > 0$ such that $\overline{D}(P, r) \subset U$ with $f_0(z) \neq 0$ for $z \in \overline{D}(P, r) \backslash \{P\}$. Hence,

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f_0'(\zeta)}{f_0(\zeta)} \, d\zeta = k, \quad k \in \mathbb{N}.$$

However, by hypothesis for each j one has that

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f_j'(\zeta)}{f_j(\zeta)} \, d\zeta = 0.$$

Since $f_j \to f_0$ uniformly on $\overline{D}(P, r)$, this yields a contradiction. □

Remark 5.15. Hurwitz's theorem can be relaxed to state that if for some N and some $r > 0$,

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f_j'(\zeta)}{f_j(\zeta)} \, d\zeta = k, \quad j \ge N,$$

with $f_0(\zeta) \neq 0$ for $\zeta \in \partial D(P, R)$, then

$$\frac{1}{2\pi i} \oint_{\partial D(P,r)} \frac{f_0'(\zeta)}{f_0(\zeta)} \, d\zeta = k.$$

Example. Let

$$0 < a_0 < a_1 < \cdots < a_n,$$

and consider

$$g(\vartheta) := a_0 + a_1 \cos \vartheta + \cdots + a_n \cos n\vartheta.$$

We will show that $g(\vartheta) = 0$ has exactly $2n$ distinct solutions in $(0, 2\pi)$, and no imaginary solutions. Consider

$$p(z) := a_0 + a_1 z + \cdots + a_n z^n,$$

and note that $g(\vartheta) = \operatorname{Re} p(e^{i\vartheta})$. Clearly, $p(z) > 0$ for $z \in \mathbb{R}^+$. Assume that $z \notin \mathbb{R}^+$. Then

$$\begin{aligned} |(z-1)p(z)| &\ge |a_n z^{n+1}| - |a_0 + (a_1 - a_0)z + \cdots + (a_n - a_{n-1})z^n| \\ &> |a_n z^{n+1}| - [a_0 + (a_1 - a_0)|z| + \cdots + (a_n - a_{n-1})|z^n|]. \end{aligned}$$

In the second inequality the facts were used that $a_j - a_{j-1} > 0$ and for $z \notin \mathbb{R}^+$, $a_0, (a_1 - a_0)z, \ldots, (a_n - a_{n-1})z^n$ do not all have the same direction. Now assume that $|z| \ge 1$. Then one has that

$$\begin{aligned} a_0 + (a_1 - a_0)|z| + \cdots + (a_n - a_{n-1})|z^n| &\le [a_0 + (a_1 - a_0) + \cdots + (a_n - a_{n-1})]|z|^{n+1} \\ &\le a_n |z|^{n+1}. \end{aligned}$$

Thus, one has that

$$|(z-1)p(z)| > a_n|z|^{n+1} - a_n|z|^{n+1} = 0,$$

i.e., $(z-1)p(z) \neq 0$ for $|z| \geq 1$ and $z \neq 1$. Hence, the n zeros of $p(z)$ lie in $D(0,1)$. Set $\gamma := \partial D(0,1)$. If the curve γ is traversed once in the counterclockwise direction, then by the argument principle the image of γ under the mapping $p(z)$ must wind around the origin n times. In particular, this implies that there are at least $2n$ points (n in the upper-half plane, and n in the lower-half plane) at which the image intersects $\operatorname{Re} z = 0$. Each intersection point corresponds to a particular value $\vartheta = \arg z$; furthermore, each point satisfies $\operatorname{Re} p(e^{i\vartheta}) = 0$. Hence, there exist at least $2n$ distinct solutions to $g(\vartheta) = 0$. Now, to see that there are exactly $2n$ roots, write

$$\cos k\vartheta = \frac{1}{2}(\zeta^k + \zeta^{-k}), \quad \zeta := e^{i\vartheta}.$$

One then has that

$$g(\vartheta) = \frac{1}{2}\zeta^{-n}h(\zeta),$$

where

$$h(\zeta) := a_n + a_{n-1}\zeta + \cdots + a_1\zeta^{n-1} + 2a_0\zeta^n + a_1\zeta^{n+1} + \cdots + a_n\zeta^{2n}.$$

Clearly, $h(\zeta) = 0$ has exactly $2n$ roots, and each of these roots satisfies $\zeta = e^{i\vartheta}$.

Example. Let $\varphi(t) > 0$ be a strictly increasing continuous function on $[0,1]$, and set

$$f(z) := \int_0^1 \varphi(t)\cos(zt)\,dt.$$

Define the appropriate Riemann sums by

$$f_n(z) := \frac{1}{n}\sum_{k=1}^{n} \varphi\left(\frac{k}{n}\right)\cos\frac{kz}{n}.$$

It is clear that $f_n(z) \to f(z)$ uniformly on compact sets; hence, $f(z)$ is entire. Furthermore, since $f(0) > 0$, one has that $f(z) \not\equiv 0$. Now, if one sets $\vartheta := z/n$, then $f_n(\vartheta)$ is of the form prescribed in the previous example, with

$$a_k := \frac{1}{n}\varphi\left(\frac{k}{n}\right).$$

Thus, for each n, $f_n(\vartheta) = 0$ has exactly $2n$ roots in $(0, 2\pi)$, i.e., $f_n(z) = 0$ has $2n$ real roots of the form $z = n\vartheta$ for $\vartheta \in (0, 2\pi)$. As an application of Hurwitz's theorem, all the zeros (if any) of $f(z)$ are real and positive.

5.4. The maximum modulus principle

Definition 5.16. $U \subset \mathbb{C}$ is a domain if it is a connected open set. U is a bounded domain if there is an $R > 0$ such that $U \subset D(0,R)$.

If U is a domain, and $f : U \mapsto \mathbb{C}$ is holomorphic and nonconstant, then $f(U)$ is open. Suppose that there is a point $P \in U$ such that $|f(P)| \geq |f(z)|$ for all $z \in U$. Since f is nonconstant, as an application of the open mapping principle there is an $\epsilon > 0$ such that $D(f(P),\epsilon) \subset f(U)$. In particular, there exist points $\zeta \in D(f(P),\epsilon)$ such that $|\zeta| > |f(P)|$. Hence, f must be constant. This argument yields:

Theorem 5.17 (Maximum modulus principle)**.** *Let $f : U \mapsto \mathbb{C}$ be holomorphic and nonconstant on the domain U. There is no point $P \in U$ such that $|f(P)| \geq |f(z)|$ for all $z \in U$.*

Now suppose that U is a bounded domain. Clearly $|f(z)|$ has a maximum on $\overline{U}$. From the Maximum Modulus Principle this maximum must occur on ∂U.

Corollary 5.18 (Maximum modulus theorem)**.** *Let $f : U \mapsto \mathbb{C}$ be holomorphic and nonconstant on the bounded domain U. The maximum of $|f(z)|$ on $\overline{U}$ occurs on ∂U.*

Now suppose that $f(z)$ never vanishes on U. Upon setting $g(z) := 1/f(z)$ and applying the Maximum Modulus Principle to $g(z)$, one gets the following corollary.

Corollary 5.19. *Let $f : U \mapsto \mathbb{C}$ be holomorphic, nonzero, and nonconstant on the domain U. There is no point $P \in U$ such that $|f(P)| \leq |f(z)|$ for all $z \in U$.*

An important application of these results is the following. Recall that if $f(x+iy) := u(x,y) + iv(x,y)$ is holomorphic, then the harmonic functions $u(x,y)$ and $v(x,y)$ both satisfy Laplace's equation $\Delta\phi = 0$ on U. Setting $g(z) := e^{f(z)}$, one has that $|g(z)| = e^{u(x,y)}$, while setting $h(z) := e^{-if(z)}$ yields $|h(z)| = e^{v(x,y)}$. Note that both $g(z)$ and $h(z)$ are nonzero on U; hence, the Maximum Modulus Theorem and its corollaries yields the following important PDE result (also see [8, Theorem 7.2.1]):

Theorem 5.20. *Let $U \subset \mathbb{R}^2$ be an open and connected set, and let $u : U \mapsto \mathbb{R}$ be a harmonic function. One has that u achieves its maximum and minimum values only on ∂U.*

5.5. The Schwarz lemma

The following application of the Maximum Modulus Principle will play an important role in problems involving mappings of holomorphic functions.

Theorem 5.21 (Schwarz's lemma). *Let $f : D(0,R) \mapsto \mathbb{C}$ be holomorphic and satisfy $f(0) = 0$. Suppose that*

$$|f(z)| \le M < \infty$$

for all $z \in D(0,R)$. Then one has that

$$|f(z)| \le \frac{M}{R}|z|, \quad |f'(0)| \le \frac{M}{R}.$$

Furthermore, if equality is achieved for some $z \in D(0,R)$, then

$$f(z) = \frac{M}{R}e^{i\vartheta}z$$

for some $\vartheta \in [0, 2\pi)$.

Proof: Set $g(z) := f(z)/z$. Since $f(z)$ is holomorphic with $f(0) = 0$, one has that $g(z)$ is holomorphic on $D(0,R)$ with

$$\lim_{z\to 0} g(z) = f'(0).$$

Let $\epsilon > 0$ be given, and consider $g(z)$ on $\overline{D}(0, R-\epsilon)$. One has that

$$|g(z)| \le \frac{M}{R-\epsilon}, \quad z \in \partial D(0, R-\epsilon).$$

From the Maximum Modulus Theorem one then has that this estimate holds for all $z \in \overline{D}(0, R-\epsilon)$. Letting $\epsilon \to 0^+$ then yields the desired result. Note that if equality is achieved for $z \in D(0,R)$, then by the Maximum Modulus Principle $g(z)$ is constant on $D(0,R)$. This yields the second part of the lemma. □

Schwarz's lemma can be generalized by removing the hypothesis $f(0) = 0$. The result is the Schwarz-Pick Lemma [8, Theorem 5.5.2].

Theorem 5.22. *Let $f : D(0,1) \mapsto D(0,1)$ be holomorphic. Let $a \in D(0,1)$ be given. For $b = f(a)$ one has the estimate*

$$|f'(a)| \le \frac{1-|b|^2}{1-|a|^2}.$$

6. Holomorphic Functions as Geometric Mappings

Definition 6.1. Let $U, V \subset \mathbb{C}$ be open. $f : U \mapsto V$ is a conformal (biholomorphic) mapping if it is one-to-one.

Note that if f is conformal, then f^{-1} exists and is conformal. Thus, from the perspective of complex function theory the sets U and V are indistinguishable. Upon recalling the result of Lemma 4.33, one has that $f(z)$ is conformal at z_0 if $f'(z_0) \ne 0$. However, if $f'(z) \ne 0$ for all $z \in U$, this does not necessarily imply that f is conformal. For a counterexample, consider $e^z : \mathbb{C} \mapsto \mathbb{C}\backslash\{0\}$. Finally, if $f'(z_0) = 0$, then as an application of Theorem 5.10 one has that f is not conformal in any neighborhood of z_0.

6.1. Biholomorphic mappings of the complex plane to itself

Lemma 6.2. *Suppose that $f : \mathbb{C} \mapsto \mathbb{C}$ is a conformal mapping. Then*

$$\lim_{z\to\infty} |f(z)| = +\infty.$$

Remark 6.3. Since f is entire, this lemma implies that $f(z)$ must be a polynomial.

Proof: For each $\epsilon > 0$ set $R_\epsilon := \{z \in \mathbb{C} : |z| \le \epsilon^{-1}\}$. Since f^{-1} is holomorphic, and hence continuous, the set $S_\epsilon := f^{-1}(R_\epsilon)$ is compact. There is then a $C_\epsilon > 0$ such that $S_\epsilon \subset \overline{D}(0, C_\epsilon)$. If $w \notin \overline{D}(0, C_\epsilon)$, then $w \notin S_\epsilon$, so that $|f(w)| > 1/\epsilon$. This is the desired result. □

Now, since f is entire with a pole at ∞, by Lemma 4.29 one has that f must be a nonconstant polynomial. As a consequence of the above discussion, if f is conformal it must satisfy the minimal requirement that $f'(z) \neq 0$ for all $z \in \mathbb{C}$. This is possible if and only if $f(z) = az + b$, $a \in \mathbb{C}\backslash\{0\}$

Theorem 6.4. *$f : \mathbb{C} \mapsto \mathbb{C}$ is a conformal mapping if and only if*

$$f(z) = az + b, \quad a \neq 0.$$

6.2. Biholomorphic mappings of the unit disc to itself

In this section $D := D(0, 1)$.

Lemma 6.5. *Let $f : D \mapsto D$ be a conformal mapping such that $f(0) = 0$. There is an $\omega \in \mathbb{C}$ with $|\omega| = 1$ such that*

$$f(z) = \omega z.$$

Remark 6.6. Note that the statement is actually "if and only if".

Proof: Set $g(z) := f^{-1}(z)$. Since $f(0) = g(0) = 0$, by the Schwarz lemma

$$|f'(0)| \le 1, \quad |g'(0)| \le 1.$$

Since $f \circ g(z) \equiv z$, by the chain rule

$$1 = f'(0)g'(0);$$

hence, $|f'(0)| = |g'(0)| = 1$, i.e., $f'(0) = \omega$ with $|\omega| = 1$. By the Schwarz lemma one then has that

$$f(z) = \omega z.$$ □

Consider the Möbius transformation

$$\phi_a(z) := \frac{z - a}{1 - \bar{a}z}, \quad a \in D(0, 1).$$

It is an exercise to check that $\phi_a : D \mapsto D$, with $\phi_a^{-1} = \phi_{-a}$. Hence, ϕ_a is a conformal mapping of the unit disk to itself. Note that

$$f(z) = \omega\phi_a(z)$$

is a conformal self-map of $D(0, 1)$.

Theorem 6.7. *Let $f : D \mapsto D$ be holomorphic. f is conformal if and only if there exists $a \in D(0, 1)$ and $\omega \in \partial D(0, 1)$ such that*

$$f(z) = \omega\phi_a(z).$$

Proof: Let $f : D \mapsto D$ be a conformal mapping. Set $b := f(0)$, and consider the map $g := \phi_b \circ f$, which is clearly a conformal self-map of the disk. Noting that $g(0) = 0$, we have that there exists an $\omega \in \partial D(0,1)$ such that

$$g(z) = \omega z.$$

However, this implies that

$$\begin{aligned} f(z) &= \phi_{-b}(\omega z) \\ &= \omega\phi_{-b/\omega}(z). \end{aligned}$$

□

Remark 6.8. The conformal self-maps of the unit disk form a group under the operation of composition. This group is known as the automorphism group of D, and the maps themselves are referred to as the automorphisms of D.

6.3. Linear fractional transformations

Consider the linear fractional transformation

$$\ell(z) := \frac{az+b}{cz+d}, \quad ad - bc \neq 0.$$

The restriction on the coefficients follows from the fact that

$$\ell'(z) = \frac{ad-bc}{(cz+d)^2},$$

and that a minimal condition for ℓ to be a conformal map is that $\ell'(z) \neq 0$. If $c \neq 0$, then $\ell(z)$ has a simple pole at $z = -d/c$; otherwise, it has a simple pole at ∞. Upon defining a neighborhood of ∞ as $\mathbb{C}\backslash\overline{D}(0,R)$ for each $R > 0$, one has that $\ell(z)$ is continuous on $\mathbb{C} \cup \{\infty\}$. It is easy to check that

$$\ell^{-1}(z) = -\frac{dz-b}{cz-a}.$$

Hence, $\ell(z)$ is a conformal mapping of $\mathbb{C} \cup \{\infty\}$ to itself. Finally, it is easy to check that if $\ell_1, \ell_2 : \mathbb{C} \cup \{\infty\} \mapsto \mathbb{C} \cup \{\infty\}$, then $\ell_1 \circ \ell_2$ is also a linear fractional transformation.

Now suppose that $f : \mathbb{C}\cup\{\infty\} \mapsto \mathbb{C}\cup\{\infty\}$ is a conformal mapping. Let ℓ be a linear fractional transformation such that $\ell \circ f(\infty) = \infty$; hence, $\ell \circ f : \mathbb{C} \cup \{\infty\} \mapsto \mathbb{C} \cup \{\infty\}$ is a conformal mapping which maps ∞ to ∞. Thus, by Theorem 6.4 there exist constants $a_1, a_2 \in \mathbb{C}$ such that

$$\ell \circ f(z) = a_1 z + a_2.$$

Inverting ℓ yields that $f(z)$ is also a linear fractional transformation.

Theorem 6.9. *$f : \mathbb{C} \cup \{\infty\} \mapsto \mathbb{C} \cup \{\infty\}$ is a conformal mapping if and only if f is a linear fractional transformation.*

As seen in the next result, linear fractional transformations have very nice mapping properties.

Theorem 6.10. *Let C be the set of subsets of $\mathbb{C} \cup \{\infty\}$ consisting of:*

(a) circles

(b) $\mathcal{L} \cup \{\infty\}$, where $\mathcal{L}$ is a line.

If ℓ is a linear fractional transformation, and if $\phi \in C$, then $\ell(\phi) \in C$.

Proof: Set

$$\ell_{\mathrm{d}}(z) := az, \quad \ell_{\mathrm{t}}(z) = z + b, \quad \ell_{\mathrm{i}} z = \frac{1}{z}.$$

It can be shown that every linear fractional transformation is a composition of $\ell_{\mathrm{d}}, \ell_{\mathrm{t}}, \ell_{\mathrm{i}}$, so that it is enough to verify the statement for each of these transformations. Note that the result is clearly true when considering ℓ_{d} and ℓ_{t}. Now consider the mapping ℓ_{i} applied to the arbitrary circle

$$x^2 + y^2 + ax + by + c = 0.$$

Setting $w := 1/z$ with $z \neq 0, \infty$. With $w = u + \mathrm{i}v$ one finds that

$$x = \operatorname{Re} z = \frac{u}{u^2 + v^2}, \quad y = \operatorname{Im} z = -\frac{v}{u^2 + v^2}.$$

Substituting into the equation for the circle and simplifying yields that

$$c(u^2 + v^2) + au - bv + 1 = 0,$$

which is an element of C. □

Example. The Cayley transform is given by

$$\ell(z) = \frac{z - \mathrm{i}}{z + \mathrm{i}}.$$

For the Cayley transform one has that

$$\ell(\infty) = 1, \quad \ell(\pm 1) = \mp\mathrm{i},$$

so that three points on $\mathbb{R} \cup \{\infty\}$ are sent to $\partial D(0, 1)$. By Theorem 6.10 one then has that $\ell(\mathbb{R} \cup \{\infty\}) = \partial D(0, 1)$. Since $\ell(\mathrm{i}) = 0$, by continuity one necessarily has that the Cayley transform sends the upper half plane to the interior of the unit disk.

6.5. Normal families

Definition 6.11. Let $f_j : U \mapsto \mathbb{C}$, where $U \subset \mathbb{C}$ is open. $f_j \to f_0$ normally on U if $f_j \to f_0$ uniformly on each compact $K \subset U$.

Remark 6.12. If each f_j is holomorphic, then one has that f_0 is necessarily holomorphic.

Example. If $f_j(z) = z^j + 1/j$, then $f_j \to f_0 \equiv 0$ normally on $D(0, 1)$. However, f_j does not converge to f_0 uniformly on all of $D(0, 1)$.

Definition 6.13. Let $\mathcal{F} = \{f_a\}_{a \in A}$ be a family of holomorphic functions on an open set $U \subset \mathbb{C}$. One has that $\mathcal{F}$ is bounded on compact sets if for each compact $K \subset U$ there is a constant M_K such that for all $a \in A$ and $z \in K$,

$$|f_a(z)| \leq M_K.$$

Remark 6.14. The family $\{z^j\}$ is bounded on compact sets of $D(0, 1)$, whereas the family $\{\sin jz\}$ is not, as

$$\sin jz = \sin jx \cosh jy + \mathrm{i} \cos jx \sinh jy.$$

Theorem 6.15 (Montel's theorem)**.** *Let $U \subset \mathbb{C}$ be open and let $\mathcal{F}$ be bounded on compact sets. For every sequence $\{f_j\} \subset \mathcal{F}$ there is a subsequence $\{f_{j_k}\}$ which converges normally on U to a limit function f_0.*

Remark 6.16. It is important to realize that the subsequence is independent of the compact set $K \subset U$. It is also important to note that the subsequence is not necessarily unique.

Proof: Let a compact $K \subset U$ be given, and choose a compact $L \subset U$ such that $K \subset \operatorname{int}(L)$. There is an $\eta > 0$ such that for any two points $z, w \in K$ with $|z - w| < \eta$ one has that $\gamma(t) := w + t(z - w) \in L$ for $t \in [0, 1]$. Since L is compact, there is an $r > 0$ such that for each $\ell \in L$, $\overline{D}(\ell, r) \subset U$. The Cauchy estimates then yield that for each $f \in \mathcal{F}$,

$$|f'(\ell)| \leq \frac{M_L}{r} := C.$$

Let $z, w \in K$ be given with $|z - w| < \eta$, and fix an $f \in \mathcal{F}$. One then has that

$$\begin{aligned}|f(z) - f(w)| &= \left|\oint_\gamma f'(\eta)\,\mathrm{d}\eta\right| \\ &\le C\int_0^1 |\gamma'(t)|\,\mathrm{d}t \\ &= C|z-w|.\end{aligned}$$

This estimate is independent of f; hence, for all $f \in \mathcal{F}$ one has that if $|z - w| < \eta$, then

$$|f(z) - f(w)| \le C|z - w|.$$

The family $\mathcal{F}$ is then equicontinuous, so by the Ascoli-Arzela theorem one has that for any $\{f_j\} \subset \mathcal{F}$ there is a uniformly convergent subsequence.

It must now be shown that there is a subsequence which converges on every compact $K \subset U$. Let $K_1 \subset K_2 \subset \cdots$ be such that $K_j \subset \mathrm{int}(K_{j+1})$ and $\cup_j K_j = U$. By the above argument there is a sequence $\{f_{j_1}\} \subset \mathcal{F}$ which converges uniformly on K_1. There exists a subsequence $\{f_{j_2}\} \subset \{f_{j_1}\}$ which converges uniformly on K_2. Continuing in this fashion, there is a subsequence $\{f_{j_m}\} \subset \{f_{j_{m-1}}\}$ which converges uniformly on K_m for each $m \ge 2$. Define the sequence

$$g_k := f_{k_k}.$$

One has that $\{g_k\} \subset \mathcal{F}$, and that $\{g_k\} \subset \{f_{j_m}\}$ for each $m \ge 1$. Hence, $\{g_k\}$ converges uniformly on each K_m.

Finally, let $L \subset U$ be compact. Since $L \subset \cup_m \mathrm{int}(K_m)$ and L is compact, there is a finite number of the $\mathrm{int}(K_m)$'s which cover L. Hence, by definition there is an M such that $L \subset K_M$. The sequence $\{g_k\}$ converges uniformly on K_M, and hence L. □

Remark 6.17. Suppose that $f_{j_1} \to f_0$ uniformly on K_1, and $f_{j_2} \to \tilde{f}_0$ uniformly on K_2. Since $\{f_{j_2}\} \subset \{f_{j_1}\}$ and $K_1 \subset K_2$, one must have that $\tilde{f}_0(z) = f_0(z)$ for all $z \in K_1$. Assuming that $K_1 \subset K_2$ has an accumulation point, one then has that $\tilde{f}_0(z) = f_0(z)$ for all $z \in K_2$, and hence one can choose $\{f_{j_2}\} = \{f_{j_1}\}$. This idea can be repeated ad nauseam.

Example. Let $\mathcal{F} = \{z/j\}_{n\in\mathbb{N}}$. For each compact $K \subset \mathbb{C}$ there is an M_K such that $|z/j| < M_k$ for all $j \in \mathbb{N}$. One clearly has that $z/j \to 0$ normally on $\mathbb{C}$.

Corollary 6.18. *Let $U \subset \mathbb{C}$ be open, and let $P \in U$ be fixed. Let $\mathcal{F}$ be a holomorphic family from U to $D(0, 1)$ such that $f(P) = 0$ for all $f \in \mathcal{F}$. There is a sequence $\{f_j\} \subset \mathcal{F}$ which converges normally to $f_0 : U \mapsto D(0, 1)$ such that*

$$|f_0'(P)| \ge |f'(P)|$$

for all $f \in \mathcal{F}$.

Proof: Choose $r > 0$ such that $\overline{D}(P, r) \subset U$. Since $|f(z)| \le 1$ for all $z \in U$ and all $f \in \mathcal{F}$, by the Cauchy estimates one has that $|f'(z)| \le 1/r$ for all $z \in U$ and all $f \in \mathcal{F}$. Set

$$\mathfrak{n} := \sup\{|f'(P)| : f \in \mathcal{F}\}.$$

There exists a sequence $\{f_j\} \subset \mathcal{F}$ such that $|f_j'(P)| \to \mathfrak{n}$. Since $|f_j(z)| \le 1$, by Montel's theorem there exists a subsequence $\{f_{j_k}\}$ which converges normally on U to f_0. By the Cauchy estimates, $f_{j_k}'(P) \to f_0'(P)$; hence, $|f_0'(P)| = \mathfrak{n}$. The estimate has now been proven. Now suppose that there exists a $z \in U$ such that $f_0(z) \in \partial D(0, 1)$. By the Maximum Modulus Theorem this then implies that $f_0(z)$ is a constant with $|f_0(z)| \equiv 1$. Since $f_0(P) = 0$, one then has that $f_0 : U \mapsto D(0, 1)$. □

Corollary 6.19. *Let*

$$\mathcal{F} := \{f : U \mapsto D(0, 1) : f(P) = 0 \text{ and } f \text{ is one-to-one}\}.$$

If $\mathcal{F}$ is nonempty, the function f_0 is then one-to-one.

Proof: Let $b \in U$ be given, and consider $g_j(z) := f_j(z) - f_j(b)$ on $U\backslash\{b\}$. Since f_j is one-to-one, $g_j(z) \neq 0$ for $z \in U\backslash\{b\}$. By Hurwitz's theorem the limit function $f_0(z) - f_0(b)$ is either nowhere vanishing on $U\backslash\{b\}$, or is identically zero. Now, one has that $f'(P) \neq 0$ for each $f \in \mathcal{F}$, for otherwise f would not be one-to-one. Since $\mathcal{F}$ is nonempty, the function f_0 satisfies $f_0'(P) \neq 0$. Hence, $f_0(z) - f_0(b)$ is nowhere zero on $U\backslash\{b\}$. Since b is arbitrary, f_0 is one-to-one. □

Remark 6.20. If f_0 is onto, then one has that f_0 is a conformal map from U to D. Some restrictions on U may be necessary to achieve this result. For example, if $U = \mathbb{C}$, then each $f \in \mathcal{F}$ is entire with $|f(z)| \leq 1$. This implies that each f is constant, and hence not one-to-one.

6.6. Holomorphically simply connected domains

After our discussion on multi-valued functions in Section 4.8, the following two results should come as no surprise.

Lemma 6.21 (Holomorphic logarithm lemma)**.** *Let $U \subset \mathbb{C}$ be open and simply connected. If $f : U \mapsto \mathbb{C}\backslash\{0\}$ is holomorphic, then there exists a holomorphic $h : U \mapsto \mathbb{C}$ such that*

$$f(z) = \mathrm{e}^{h(z)}, \quad z \in U.$$

Proof: Set

$$g(z) := \frac{f'(z)}{f(z)} = \frac{\mathrm{d}}{\mathrm{d}z} \ln(f(z)).$$

Since f is nowhere zero on U, g is holomorphic. Since U is simply connected, there exists a holomorphic $h : U \mapsto \mathbb{C}$ such that $h'(z) = g(z)$. Let $z_0 \in U$ be given, and suppose that $f(z_0) = \mathrm{e}^{h(z_0)}$. Set $G(z) := f(z)\mathrm{e}^{-h(z)}$. By construction, $G(z_0) = 1$. Furthermore, by the product rule and the fact that $h' = g$ one gets that $G'(z) \equiv 0$. Hence, $G(z) \equiv 1$, which proves the lemma. □

Remark 6.22. The fact that U is simply connected is crucial. If, for example, one considers the annulus $D(0,2)\backslash\overline{D}(0,1)$, then a holomorphic h does not exist even for $f(z) = z$ (see [**8**, Problem 6.13]).

Corollary 6.23. *For each $n \in \mathbb{N}$ there is a holomorphic $g : U \mapsto \mathbb{C}\backslash\{0\}$ such that*

$$f(z) = [g(z)]^n.$$

Proof: Set $g(z) := \mathrm{e}^{h(z)/n}$, where h is given in Lemma 6.21. □

6.7. The proof of the analytic form of the Riemann mapping theorem

Let $U \subset \mathbb{C}$ be open and simply connected; however, $U \neq \mathbb{C}$. Let $P \in U$ be fixed, and let

$$\mathcal{F} := \{f : U \mapsto D(0,1) : f(P) = 0 \text{ and } f \text{ is one-to-one}\}.$$

Lemma 6.24. *$\mathcal{F}$ is nonempty.*

Proof: Let $Q \in \mathbb{C}\backslash U$, and set $\phi(z) := z - Q$. By Corollary 6.23 one has that there is a holomorphic h such that $h^2 = \phi$. Since ϕ is one-to-one, h is one-to-one; furthermore, there does not exist distinct $z_1, z_2 \in U$ such that $h(z_1) = -h(z_2)$. Since h is nonconstant, it is an open mapping. Set $b := h(P)$. One can then choose an $r > 0$ such that the image of h contains $D(b,r)$, and yet is disjoint from $D(-b,r)$. Therefore,

$$f(z) := \frac{r}{2(h(z)+b)}$$

is holomorphic and one-to-one. Furthermore, since $|h(z)+b| > r$ for $z \in U$, one has that $f : U \mapsto D(0,1)$. Set $c := f(P)$ (note that $|c| < 1$), and set

$$\phi_c(z) := \frac{z-c}{1-\bar{c}z}.$$

The function $f_c := \phi_c \circ f : U \mapsto D(0,1)$ is one-to-one with $f_c(P) = 0$. □

Lemma 6.25. *If $g \in \mathcal{F}$ is such that*
$$|g'(P)| = \sup_{h\in\mathcal{F}} |h'(P)|,$$
then $g : U \mapsto D(0,1)$ is onto.

Proof: Suppose that there is a point $R \in D(0,1)$ such that the image of g does not contain R. Set
$$\phi_R(z) := \frac{z-R}{1-\bar{R}z}, \quad \phi(z) := \phi_R \circ g(z).$$
By construction $\phi : U \mapsto D(0,1)\backslash\{0\}$. By Corollary 6.23 one has that there is a holomorphic ψ such that $\psi^2 = \phi$. One has that ψ is one-to-one and has range contained in $D(0,1)$. However, $\psi \notin \mathcal{F}$, since it is nonvanishing. Set $Q := \psi(P)$, and
$$\phi_Q(z) := \frac{z-Q}{1-\bar{Q}z}, \quad \rho(z) := \phi_Q \circ \psi(z).$$
Then $\rho : U \mapsto D(0,1)$ is such that $\rho(P) = 0$; furthermore, ρ is one-to-one. Now,
$$\rho'(P) = \frac{1}{1-|\psi(P)|^2}\psi'(P);$$
furthermore, upon using the fact that $g(P) = 0$ one has that
$$2\psi(P)\psi'(P) = \phi'(P) = (1-|R|^2)g'(P).$$
After substituting, some lengthy algebra, and using $|\phi(P)| = |R|$, one gets that
$$\rho'(P) = \frac{1+|R|}{2\psi(P)}g'(P).$$
Since $R \neq 0$ (as g is one-to-one with $g(P) = 0$) and $|\psi(P)| = \sqrt{|R|}$, one then gets that
$$|\rho'(P)| > |g'(P)|.$$
Thus, if g is not onto, then the assumption on g' is violated. Hence, g must be onto. □

We now have the following result.

Theorem 6.26 (Riemann Mapping Theorem)**.** *Let $U \subset \mathbb{C}$ be simply connected with $U \neq \mathbb{C}$. Then U is conformally equivalent to $D(0,1)$.*

Now let $f_U : U \mapsto D(0,1)$ be a conformal map, and $f_V : V \mapsto D(0,1)$ be a conformal map. If one sets $f := f_V^{-1} \circ f_U$, then one has a conformal map from U to V.

Corollary 6.27. *If $U, V \subset \mathbb{C}$ are both open, simply connected, and not equal to $\mathbb{C}$, then U is conformally equivalent to V.*

7. Infinite Series and Products

We have seen that an entire function is defined by its derivatives at a single point. In particular, knowing the values of the derivatives at a single point allows us to reconstruct the function for all $z \in \mathbb{C}$. Suppose that one wishes to reconstruct a function knowing only the location of its zeros. For example, if $z_1, \dots, z_n$ represent the zeros of a polynomial $p(z)$, then one has that
$$p(z) = a\prod_{j=1}^{n}(z-z_j), \quad a \in \mathbb{C}\backslash\{0\}.$$
Can one write $\sin z$ in a similar manner?

7.1. Basic concepts concerning infinite sums and products

For each $N \in \mathbb{N}$ define the partial products by

$$P_N := \prod_{j=1}^{N}(1+a_j), \quad a_j \in \mathbb{C}.$$

Definition 7.1. The infinite product

$$\prod_{j=1}^{\infty}(1+a_j)$$

is said to converge if

(a) only a finite number of $\{a_1, a_2, \dots\}$ are equal to -1

(b) if N_0 is such that $a_j \neq -1$ for $j > N_0$, then

$$\lim_{N\to+\infty}\prod_{j=N_0+1}^{N}(1+a_j)$$

exists and is nonzero.

If the product converges, its value is given by

$$P_{N_0}\cdot\lim_{N\to+\infty}\prod_{j=N_0+1}^{N}(1+a_j).$$

Remark 7.2. One has that if the product converges:

(a) then

$$\lim_{M,N\to+\infty}\prod_{j=N}^{M}(1+a_j)=1$$

[8, Problem 8.1]

(b) then

$$\prod_{j=1}^{\infty}(1+a_j)=\lim_{N\to+\infty}P_N;$$

however, the converse is not true (suppose that $a_j = -1/2$ for all j).

The question to be answered is: what conditions are necessary on the sequence $\{a_j\}$ to ensure that the infinite product converges?

Proposition 7.3. *For* $0 \le x \le 1$,

$$1+x \le \mathrm{e}^x \le 1+2x.$$

Proof: Use the Taylor expansion for e^x and the estimate

$$\sum_{j=2}^{\infty}\frac{1}{j!} \le \sum_{j=2}^{\infty}\frac{1}{2^{j-1}} = 1. \qquad \square$$

Proposition 7.4. *Let* $\{a_j\} \subset \mathbb{C}$ *be such that* $a_j \in D(0,1)$ *for all* j*. The partial product*

$$P_N^{\mathrm{a}} := \prod_{j=1}^{N}(1+|a_j|)$$

satisfies the estimate

$$\exp\left(\frac{1}{2}\sum_{j=1}^{N}|a_j|\right) \le P_N^{\mathrm{a}} \le \exp\left(\sum_{j=1}^{N}|a_j|\right).$$

Proof: Using the previous proposition yields that $1 + |a_j| \le e^{|a_j|}$, and hence the right-hand inequality. Similarly,

$$1 + |a_j| = 1 + 2(\frac{1}{2}|a_j|) \ge e^{|a_j|/2}$$

yields the left-hand inequality. □

Corollary 7.5. *The infinite product*

$$\prod_{j=1}^{\infty}(1 + |a_j|)$$

converges if and only if

$$\sum_{j=1}^{\infty}|a_j|$$

converges.

Proof: Suppose that the infinite sum converges, so that $P_N^{\mathrm{a}} \le C$. Since the sequence $\{P_N^{\mathrm{a}}\}$ is monotone increasing and $P_1^{\mathrm{a}} > 0$, the sequence of partial products converges to a nonzero limit. Now suppose that the infinite product converges. The monotone sequence

$$\{\exp\left(\frac{1}{2}\sum_{j=1}^{N}|a_j|\right)\}$$

is then bounded above, and hence converges. Since e^x is one-to-one and continuous, this them implies that the monotone sequence of partial sums converges. □

Now the question is: does the "absolute" convergence imply convergence? It is certainly true when considering infinite sums.

Proposition 7.6. *Let $\{a_j\} \subset \mathbb{C}$, and set*

$$P_N := \prod_{j=M+1}^{N}(1 + a_j), \quad P_N^{\mathrm{a}} := \prod_{j=M+1}^{N}(1 + |a_j|).$$

Then $|P_N - 1| \le P_N^{\mathrm{a}} - 1$.

Proof: One has that $P_N = 1+$ monomial terms consisting of products of the a_j's, whereas $P_N^{\mathrm{a}} = 1+$ the absolute value of the same monomials. The result now follows from the triangle inequality. □

Lemma 7.7. *If the infinite product*

$$\prod_{j=1}^{\infty}(1 + |a_j|)$$

converges, then so does

$$\prod_{j=1}^{\infty}(1 + a_j).$$

Proof: One has that $\prod(1 + |a_j|)$ converges if and only if $\sum |a_j|$ converges; hence, $a_j \to 0$ as $j \to +\infty$, so that there is an N_0 such that $a_j \ne -1$ for $j > N_0$. For $J > N_0$ write

$$Q_J := \prod_{j=N_0+1}^{J}(1 + a_j), \quad Q_J^{\mathrm{a}} := \prod_{j=N_0+1}^{J}(1 + |a_j|).$$

For $M > N > N_0$ one then has that

$$\begin{aligned}|Q_M - Q_N| &= |Q_N| \left| \prod_{j=N+1}^{M} (1 + a_j) - 1 \right| \\ &\le Q_N^{\mathrm{a}} \left(\prod_{j=N+1}^{M} (1 + |a_j|) - 1 \right) \\ &= |Q_M^{\mathrm{a}} - Q_N^{\mathrm{a}}|.\end{aligned}$$

The second line follows from Proposition 7.6. Thus, the convergence of $\{Q_J^{\mathrm{a}}\}$ implies the convergence of $\{Q_J\}$. Finally, it must be shown that $|Q_J| \ge \delta > 0$ for all $J > N_0$. Choose $M > N_0 + 1$ sufficiently large so that

$$\prod_{j=M}^{N} (1 + |a_j|) - 1 < \frac{1}{2}.$$

By Proposition 7.6 one then has that for $N > M$,

$$\left| \prod_{j=M}^{N} (1 + a_j) - 1 \right| < \frac{1}{2},$$

so that

$$\left| \prod_{j=M}^{N} (1 + a_j) \right| > \frac{1}{2}.$$

Hence,

$$\begin{aligned}|Q_N| &= \left| \prod_{j=N_0+1}^{M-1} (1 + a_j) \right| \cdot \left| \prod_{j=M}^{N} (1 + a_j) \right| \\ &\ge \frac{1}{2} \left| \prod_{j=N_0+1}^{M-1} (1 + a_j) \right|,\end{aligned}$$

which is the desired result. □

Corollary 7.8. *If*

$$\sum_{j=1}^{\infty} |a_j|$$

converges, then

$$\prod_{j=1}^{\infty} (1 + a_j)$$

converges.

We know quite a bit about the limit of sums of holomorphic functions. The above corollary allows us to then talk about the infinite products of holomorphic functions in a concrete way.

Definition 7.9. The infinite product

$$\prod_{j=1}^{\infty} (1 + f_j(z))$$

converges uniformly on a set $E \subset \mathbb{C}$ if

(a) if converges for each $z \in E$

(b) the sequence

$$\{\prod_{j=1}^{N} (1 + f_j(z))\}$$

converges uniformly on E to the infinite product.

Theorem 7.10. *Let $U \subset \mathbb{C}$ be open. Let $f_j : U \mapsto \mathbb{C}$ be holomorphic, and suppose that $\sum |f_j|$ converges uniformly on compact sets. Then the infinite product $\prod(1 + f_j(z))$ converges uniformly on compact sets. In particular, the limit $f(z)$ is a holomorphic function. Finally, $f(z_0) = 0$ for some $z_0 \in U$ if and only if $f_j(z_0) = -1$ for some j. The multiplicity of the zero at z_0 is the sum of the multiplicities of the zeros of $1 + f_j$ at z_0.*

Proof: Let a compact $K \subset U$ be given. Since $\sum |f_j|$ converges uniformly on K, the partial sums are uniformly bounded on K by some constant C. Therefore, by Proposition 7.4 the partial products

$$P_N(z) := \prod_{j=1}^{N}(1 + |f_j(z)|)$$

are uniformly bounded on K by e^C. Let $0 < \epsilon < 1$ be given, and choose L sufficiently large so that for $M \geq N \geq L$ one has

$$\sum_{j=N}^{M} |f_j(z)| < \epsilon, \quad z \in K.$$

One then has that

$$\begin{aligned} |P_M(z) - P_N(z)| &\leq |P_N(z)| \left| \prod_{j=N+1}^{M} (1 + |f_j(z)|) - 1 \right| \\ &\leq |P_N(z)| \left| \exp\left(\sum_{j=N+1}^{M} |f_j(z)| \right) - 1 \right| \\ &\leq e^C(e^\epsilon - 1). \end{aligned}$$

Letting $\epsilon \to 0^+$ yields that the sequence $\{P_N(z)\}$ is uniformly Cauchy on K. Hence, by Lemma 7.7 the sequence

$$\{\prod_{j=1}^{N}(1 + f_j(z))\}$$

is uniformly Cauchy on K, and converges to a holomorphic function $f(z)$.

Suppose that $f(z_0) = 0$ for some $z_0 \in K$. By definition, there exists an N_0 such that

$$F_{N_0}(z) := \lim_{N\to+\infty} \prod_{j=N_0+1}^{N} (1 + f_j(z)) \neq 0, \quad z = z_0.$$

Furthermore, this $F_{N_0}(z)$ is holomorphic, and hence nonvanishing in $D(z_0, r)$ for some $r > 0$. Since

$$f(z) = \prod_{j=1}^{N_0}(1 + f_j(z)) \cdot F_{N_0}(z),$$

and the second factor is nonzero on $D(z_0, r)$, the statement about the zeros of $f(z)$ and their multiplicities follows by inspection of the first factor. □

Remark 7.11. The statement that $f(z_0) = 0$ if and only if $f_j(z_0) = -1$ is a primary reason that one restricts the definition of the convergence of infinite products as we do.

7.2. The Weierstrass factorization theorem

Let $\{a_j\} \subset \mathbb{C}\backslash\{0\}$ have no finite accumulation point, and suppose that

$$\sum_{j=1}^{\infty} \frac{1}{|a_j|} < \infty.$$

As a consequence of Theorem 7.10 one has that the function

$$f(z) := \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right)$$

is entire and satisfies $f(z) = 0$ if and only if $z = a_j$ for some $j \in \mathbb{N}$. However, if the summability constraint is not satisfied, then it is not clear that $f(z)$ as defined above is entire. In general, this constraint is too strict.

Definition 7.12. For $p \in \mathbb{N}_0$ the Weierstrass elementary factors are given by

$$E_0(z) := 1 - z, \quad E_p(z) := E_0(z) \exp\left(\sum_{j=1}^{p} \frac{z^j}{j}\right).$$

It is clear that $E_p(z)$ is entire for each $p \in \mathbb{N}_0$, and that $E_p(z/a) = 0$ if and only if $z = a$. Furthermore, if $z \in D(0,1)$ one has that

$$\lim_{p\to\infty} E_p(z) = 1.$$

The next result specifies the rate of convergence.

Proposition 7.13. *If $z \in D(0,1)$, then*

$$|1 - E_p(z)| \le |z|^{p+1}.$$

Proof: See [8, Lemma 8.2.1]. □

Lemma 7.14. *Let $\{a_j\} \subset \mathbb{C}\backslash\{0\}$ have no finite accumulation point. Then*

$$F(z) := \prod_{n=1}^{\infty} E_{n-1}\left(\frac{z}{a_n}\right)$$

is entire; furthermore, the zeros of F are precisely the points $\{a_j\}$, counted with multiplicity.

Proof: Let $r > 0$ be fixed. There is a $N \in \mathbb{N}$ such that for $n \ge N$ one has $|a_n| > 2r$. Thus, for $n \ge N$ and $z \in \overline{D}(0,r)$ one has from Proposition 7.13 that

$$\left|E_{n-1}\left(\frac{z}{a_n}\right) - 1\right| \le \left|\frac{z}{a_n}\right|^n \le \left|\frac{r}{a_n}\right|^n \le 2^{-n},$$

which implies that

$$\sum_{n=N}^{\infty} \left|E_{n-1}\left(\frac{z}{a_n}\right) - 1\right| < \infty.$$

The series then converges uniformly on $\overline{D}(0,r)$ by the Weierstrass M-test, so by Theorem 7.10 the infinite product

$$F(z) = \prod_{n=1}^{\infty} \left(1 + \left(E_{n-1}\left(\frac{z}{a_n}\right) - 1\right)\right)$$

converges uniformly on $\overline{D}(0,r)$. Since $r > 0$ is arbitrary, $F(z)$ is entire. The statement about the zeros follows immediately from Theorem 7.10. □

Remark 7.15. If one assumes that there exists $\{p_j\} \subset \mathbb{N}$ such that for each $r \in \mathbb{R}^+$,

$$\sum_{j=1}^{\infty} \left(\frac{r}{|a_j|}\right)^{p_j+1} < \infty,$$

then the result of Lemma 7.14 holds for

$$F(z) := \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

[8, Theorem 8.2.2].

Corollary 7.16. *Let $\{a_j\} \subset \mathbb{C}\backslash\{0\}$ have no finite accumulation point. There exists an entire function f with a zero of order k at $z = 0$ and all other zeros precisely equal to $\{a_j\}$.*

Proof: By Lemma 7.14 the entire function

$$z^k \prod_{n=1}^{\infty} E_{n-1}\left(\frac{z}{a_n}\right)$$

is the desired function. □

Remark 7.17. Note that by the discussion at the beginning of this section it may not be necessary to use the Weierstrass elementary factors in order to construct the desired entire function.

Theorem 7.18 (Weierstrass factorization theorem)**.** *Let $f : \mathbb{C} \mapsto \mathbb{C}$ be entire. Suppose that f has a zero of order k at $z = 0$, and let $\{a_j\} \subset \mathbb{C}\backslash\{0\}$ be the other zeros of f. There exists an entire function g such that*

$$f(z) = e^{g(z)} z^k \prod_{n=1}^{\infty} E_{n-1}\left(\frac{z}{a_n}\right).$$

Proof: Set

$$h(z) := z^k \prod_{n=1}^{\infty} E_{n-1}\left(\frac{z}{a_n}\right).$$

From Corollary 7.16 one has that h has the same zeros as f, counting multiplicities, so that the function f/h is entire and nonvanishing. Thus, by Lemma 6.21 there exists an entire g such that

$$\frac{f(z)}{h(z)} = e^{g(z)},$$

which proves the theorem. □

At this point, there is little understanding as to how one can derive a product expansion for a given function. For example, by following the argument in the proof of Theorem 7.18 one has that

$$\sin \pi z = e^{g(z)} z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right).$$

Upon using the fact that

$$\lim_{z \to 0} \frac{\sin \pi z}{z} = \pi,$$

one can deduce that $g(0) = \ln \pi$. What is $g(z)$ for $z \in \mathbb{C}\backslash\{0\}$?

Suppose that

$$F(z) := \prod_{n=1}^{\infty}(1 + a_n(z)).$$

By considering the partial product expansion and using the continuity of $\ln(z)$, it can be shown that at all points for which $F(z) \neq 0$,

$$\ln F(z) = \sum_{n=1}^{\infty} \ln(1 + a_n(z)).$$

A formal term-by-term differentiation then yields

$$\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{a_n'(z)}{1 + a_n(z)}.$$

This formal argument can be made rigorous [10, Theorem 8.4].

Now let us revisit the previous example. By considering the function

$$f(\xi) := \frac{\pi \cot \pi\xi}{z^2 - \xi^2},$$

and the contour comprising the boundary of the rectangular domain

$$D_N := \{z \in \mathbb{C} : |\mathrm{Re}\, z| < N + \frac{1}{2},\ |\mathrm{Im}\, z| < N\},$$

one can show that

$$\begin{aligned} \pi \cot \pi z &= \lim_{N\to+\infty} \frac{1}{2\pi \mathrm{i}} \oint_{\partial D_N} f(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \\ &= \frac{1}{1 + (z-1)} + \sum_{n=1}^{\infty} \frac{-2z/n^2}{1 - z^2/n^2}. \end{aligned}$$

(see [10, Example 6.2.2] for the details). Since

$$\pi \cot \pi z = \frac{\mathrm{d}}{\mathrm{d}z} \ln(a \sin \pi z)$$

for any $a \in \mathbb{C}\backslash\{0\}$, from the above argument one has that

$$a \sin \pi z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Dividing both sides by z and taking the limit at $z = 0$ yields that $a = 1/\pi$; hence, we have the result

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Remark 7.19. Following [10, Problem 6.8.8], one gets that

$$\cos \pi z = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{(n-1/2)^2}\right).$$

7.3. *The theorems of Weierstrass and Mittag-Leffler: interpolation problems*

We now consider the question as to what happens if there is a finite accumulation point for the zeros. If the function is to be nontrivial, then these accumulation points must lie on the boundary of the domain of f.

Theorem 7.20. *[Weierstrass] Let $U \subset \mathbb{C}$ be open, and let $\{a_j\} \subset U$ have no accumulation point in U. There exists a holomorphic $f : U \mapsto \mathbb{C}$ whose zero set is precisely $\{a_j\}$.*

Proof: If the sequence is finite, then the desired function is a polynomial. Hence, it may be assumed that the sequence is infinite.

Set $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, and for a given $p \in U\backslash\{a_j\}$ apply the linear fractional transformation $z \mapsto 1/(z-p)$. This yields $\infty \in U$ with $\partial U \subset \widehat{\mathbb{C}}\backslash\{\infty\}$, so that after the transformation:

(a) $U \subset \widehat{\mathbb{C}}$ with $U \neq \widehat{\mathbb{C}}$

(b) $\widehat{\mathbb{C}}\backslash U$ is compact

(c) $\{a_j\} \cup \{\infty\} \subset U$

(d) $\{a_j\} \cap \{\infty\} = \emptyset$.

Thus, by hypothesis the accumulation points are finite and contained solely in ∂U, so that any compact subset of U contains only finitely many a_j's. As a consequence of (b), for each a_j there exists $\hat{a}_j \in \partial U$ which is closest to a_j; thus, for $d_j := |a_j - \hat{a}_j|$ one has $d_j \to 0$ as $j \to +\infty$. Let $K \subset U$ be compact. There exists a $\delta > 0$ such that

$$|z - w| \geq \delta, \quad z \in K, \ w \in \widehat{\mathbb{C}} \backslash U,$$

which implies that $|z - \hat{a}_j| \geq \delta$ for all $z \in K$ and all j. There is then a j_0 such that for $j > j_0$, $d_j < |z - \hat{a}_j|/2$, i.e.,

$$\left| \frac{a_j - \hat{a}_j}{z - \hat{a}_j} \right| < \frac{1}{2}.$$

As a consequence of Proposition 7.13 (with $p = j$) and Theorem 7.10 one has that

$$f(z) := \prod_{j=1}^{\infty} E_j \left(\frac{a_j - \hat{a}_j}{z - \hat{a}_j} \right) \tag{7.1}$$

converges uniformly on K. Since $K \subset U$ is arbitrary, f is holomorphic on U and has the desired properties.
It is an exercise to check what happens at the point ∞. □

As a consequence of Theorem 7.20, one can now completely classify meromorphic functions $m : U \mapsto \mathbb{C}$ (recall Definition 4.26). Recall Lemma 4.32 in the case that $U = \mathbb{C}$.

Corollary 7.21. *Let $m : U \mapsto \mathbb{C}$ be meromorphic. There exists holomorphic $f, g : U \mapsto \mathbb{C}$ such that*

$$m(z) = \frac{f(z)}{g(z)}.$$

Proof: Let $\{a_j\} \subset U$ represent the poles of m, counting multiplicity. By Theorem 7.20 there exists a holomorphic $g : U \mapsto \mathbb{C}$ whose zero set is precisely $\{a_j\}$. Since $f(z) := m(z)g(z)$ is holomorphic on U, one has the desired result. □

The question to now be considered is: what happens to f as $z \to \partial U$? Can it be holomorphically extended to a $\widehat{U}$ with $U \subset \widehat{U}$?

Definition 7.22. Let $U \subset \mathbb{C}$ be open, and let $f : U \mapsto \mathbb{C}$ be holomorphic. For $P \in \partial U$, f is said to be regular at P if there is an $r > 0$ and a holomorphic $\tilde{f} : D(P, r) \mapsto \mathbb{C}$ such that

$$f|_{D(P,r)\cap U} = \tilde{f}|_{D(P,r)\cap U}.$$

When considering equation (7.1), first note $f(z)$ has a singularity at each $\hat{a}_j$. Now, it can be shown that if $z \in \partial U$ is such that $|z - \hat{a}_j| > \delta > 0$ for all $j \in \mathbb{N}$, then $f(z)$ is holomorphic at z. However, such a δ may not exist. For example, set $U = D(0, 1)$, and for each $n \in \mathbb{N}$ place n zeros on $\partial D(0, 1 - 1/n)$ via the prescription

$$a_j^n := \left(1 - \frac{1}{n}\right) e^{i2\pi j/n}, \quad j = 0, \ldots, n-1.$$

One then has that

$$\hat{a}_j^n = e^{i2\pi j/n}, \quad j = 0, \ldots, n-1,$$

so that $\{\hat{a}_j^n\} \subset \partial D(0, 1)$ is a countable dense set. While it will not be done so herein, this example can be generalized to arbitrary open sets $U \subset \mathbb{C}$.

Corollary 7.23. *Let $U \subset \mathbb{C}$ be open and connected with $U \neq \mathbb{C}$. There is a holomorphic $f : U \mapsto \mathbb{C}$ such that no $P \in \partial U$ is regular for f.*

Proof: By [**8**, Lemma 8.3.2] there exists a countably infinite $A := \{a_j\} \subset U$ such that A has no accumulation point in U, and every $P \in \partial U$ is an accumulation point of A. Applying Weierstrass theorem with this set A yields a holomorphic $f : U \mapsto \mathbb{C}$ whose zero set is precisely A. Suppose that there is a point $P \in \partial U$ which is regular. Since $\tilde{f} : D(P, r) \mapsto \mathbb{C}$ is holomorphic with zeros accumulating at P, one has that $\tilde{f} \equiv 0$. This then implies that $f \equiv 0$ on $D(P, r)$, which is a contradiction. □

Recall that a meromorphic function has an associated Laurent series about each given point. The question to be answered next is: given the principal part of a Laurent series at a collection of points, is there a meromorphic function with the given principal parts?

Theorem 7.24. *[Mittag-Leffler theorem] Let $\{a_j\} \subset \mathbb{C}\backslash\{0\}$ be such that $|a_j| < |a_{j+1}|$ for $j \in \mathbb{N}$ with $|a_j| \to +\infty$. Set*

$$s_j(z) := \sum_{\ell=-p(j)}^{-1} a_\ell^j (z-a_j)^\ell.$$

There is a meromorphic $m : \mathbb{C} \mapsto \mathbb{C}$ whose principal part at a_j is $s_j(z)$, and which has no other poles.

Proof: If the sequence is finite, the desired function is found by adding the s_j's; thus, assume that the sequence is infinite. For each $j \in \mathbb{N}$ one has the Taylor expansion

$$s_j(z) = \sum_{k=0}^{\infty} a_k^j z^k, \quad z \in D(0, |a_j|),$$

which converges uniformly on

$$D_j := D(0, |a_j|/2).$$

Let $\{\epsilon_k\} \subset \mathbb{R}^+$ be such that

$$\sum_{k=1}^{\infty} \epsilon_k < \infty,$$

and choose integers ℓ_j such that for $j \in \mathbb{N}$,

$$|s_j(z) - \sum_{k=0}^{\ell_j} a_k^j z^k| < \epsilon_j, \quad z \in D_j. \tag{7.2}$$

Set

$$P_j(z) := -\sum_{k=0}^{\ell_j} a_k^j z^k, \quad j \in \mathbb{N}.$$

For a given $R \in \mathbb{R}^+$, set

$$D_R := D(0, R),$$

and let $N \in \mathbb{N}$ be the least integer such that $|a_j| > 2R$ for $j > N$. Set

$$f_N(z) := \sum_{j=N+1}^{\infty} [s_j(z) + P_j(z)]. \tag{7.3}$$

Since $D_R \subset D_j$ for all $j > N$, and since D_R contains none of the points $a_{N+1}, a_{N+2}, \dots,$ as a consequence of equation (7.2) one has that

$$|s_j(z) + P_j(z)| < \epsilon_j, \quad z \in D_R.$$

Thus, by the Weierstrass M-test the series in equation (7.3) is uniformly convergent, so that $f_N(z)$ is holomorphic on D_R.

Now set

$$\begin{aligned} f(z) &:= \sum_{j=1}^{N} [s_j(z) + P_j(z)] + f_N(z) \\ &= g_R(z) + f_N(z). \end{aligned} \tag{7.4}$$

One has that $f_N(z)$ is holomorphic on D_R, whereas $g_R(z)$ is a rational function whose poles in D_R are precisely $a_1, \dots, a_N$. Furthermore, the principal part of $g_R(z)$ at each $a_j \in D_R$ is precisely $s_j(z)$. The result now follows upon noting that $R > 0$ is arbitrary. □

Remark 7.25. One has that:

(a) this statement and proof of the Mittag-Leffler theorem can be found in [**11**, II.10.51]

(b) if $a_0 = 0$, then the polynomial $P_0(z)$ can be arbitrarily defined

(c) the assumption that $|a_j| < |a_{j+1}|$ is made for convenience only; otherwise, one must just be more careful when discussing the set D_R, and the location and number of poles therein.

Corollary 7.26. *Let $f : \mathbb{C} \mapsto \mathbb{C}\backslash\{0\}$ be a meromorphic function whose poles are given by an increasing sequence of distinct $\{a_j\} \subset \mathbb{C}$, and with principal part $s_j(z)$ at a_j. Then*

$$f(z) = g(z) + \sum_{j=1}^{\infty}[s_j(z) + P_j(z)],$$

where $g : \mathbb{C} \mapsto \mathbb{C}$ is entire, and each $P_j(z)$ is a polynomial.

Proof: As a consequence of equation (7.4) in the proof of Theorem 7.24, one has that there exists a meromorphic function $\varphi(z)$ of the form

$$\varphi(z) := \sum_{j=1}^{\infty}[s_j(z) + P_j(z)]$$

which has the same poles and principal parts as $f(z)$. Thus, $g(z) := f(z) - \varphi(z)$ is entire. □

Example. Let $\{a_j\} \subset \mathbb{C}\backslash\{0\}$ be such that $|a_j| < |a_{j+1}|$ for $j \in \mathbb{N}$ with $|a_j| \to +\infty$, and let the principal part at a_j be simply

$$\frac{1}{z - a_j}, \quad j \in \mathbb{N}.$$

Following the proof of Theorem 7.24, set $\epsilon_j := 2^{-(j-1)}$. One then has that the corresponding $P_j(z)$ which allows equation (7.2) to be satisfied is given by

$$P_j(z) := \frac{1}{a_j}\sum_{k=0}^{j-1}\left(\frac{z}{a_j}\right)^k = \frac{1}{a_j} + \frac{z}{a_j^2} + \cdots + \frac{z^{j-1}}{a_j^j}.$$

As a consequence of Corollary 7.26 the meromorphic function with the given principal part is given by

$$f(z) = g(z) + \sum_{j=1}^{\infty}\left(\frac{1}{z - a_j} + \frac{1}{a_j} + \frac{z}{a_j^2} + \cdots + \frac{z^{j-1}}{a_j^j}\right),$$

where $g(z)$ is entire.

Corollary 7.27. *Let $\{a_j\} \subset \mathbb{C}\backslash\{0\}$ be such that $|a_j| < |a_{j+1}|$ for $j \in \mathbb{N}$ with $|a_j| \to +\infty$. Let $\{\beta_j\} \subset \mathbb{C}$ be arbitrary. There exists an entire function $f(z)$ such that*

$$f(a_j) = \beta_j, \quad j \in \mathbb{N}.$$

Proof: As a consequence of Theorem 7.18 one has that

$$g(z) := \prod_{n=1}^{\infty} E_{n-1}\left(\frac{z}{a_n}\right)$$

is entire with simple zeros at a_j. As a consequence of Theorem 7.24 there exists a meromorphic function $\varphi(z)$ with simple poles at a_j and corresponding principal parts

$$\frac{\beta_j/g'(a_j)}{z - a_j}, \quad j \in \mathbb{N}.$$

Note that this implies that

$$\lim_{z \to a_j} \varphi(z)(z - a_j) = \frac{\beta_j}{g'(a_j)}.$$

The function

$$f(z) := g(z)\varphi(z)$$

is clearly entire, and moreover satisfies

$$\begin{aligned} f(a_j) &= \lim_{z\to a_j} g(z)\varphi(z) \\ &= \lim_{z\to a_j} \frac{g(z)-g(a_j)}{z-a_j}\varphi(z)(z-a_j) \\ &= g'(a_j)\frac{\beta_j}{g'(a_j)}. \end{aligned}$$

□

It is natural to wonder if Corollary 7.27 can be generalized to cover the case of a finite Taylor expansion at each point a_j. In other words, is Theorem 7.24 still valid when "singular part" is replace by "finite Laurent expansion"? The answer is yes, and all that is required to prove it is the proper generalization of the proof of Corollary 7.27 [8, Lemma 8.3.7].

Theorem 7.28. *Let $\{a_j\} \subset \mathbb{C}\backslash\{0\}$ be such that $|a_j| < |a_{j+1}|$ for $j \in \mathbb{N}$ with $|a_j| \to +\infty$. Set*

$$s_j(z) := \sum_{\ell=-M(j)}^{N(j)} a_\ell^j (z-a_j)^\ell,$$

with $M(j), N(j) \geq 0$. There is a meromorphic $m : C \mapsto \mathbb{C}$, holomorphic on $\mathbb{C}\backslash\{a_j\}$, such that the ℓ-th coefficient of $m(z)$ at a_j is a_ℓ^j for $-M(j) \leq \ell \leq N(j)$.

Proof: By Theorem 7.18 there is a holomorphic $h : \mathbb{C} \mapsto \mathbb{C}$ with a zero of order $M(j)$ at a_j for each j and no others. Set $\tilde{s}_j(z) := h(z)s_j(z)$, and note that $\tilde{s}_j$ is holomorphic at a_j. Let $\tilde{\tilde{s}}_j$ denote the Taylor polynomial of degree $N(j)+M(j)$ associated with $\tilde{s}_j$. Again by Theorem 7.18 there exists a holomorphic $g : \mathbb{C} \mapsto \mathbb{C}$ with zeros of order $N(j)+M(j)+1$ at each a_j, and no others. By [8, Lemma 8.3.7] there is a Laurent polynomial $v_j(z)$ with a pole of order $N(j)+M(j)+1$ at a_j such that for each j,

$$g(z)v_j(z) = \tilde{\tilde{s}}_j(z) + \text{higher order terms}.$$

Now, by Theorem 7.24 there exists a meromorphic function $k : \mathbb{C} \mapsto \mathbb{C}$ with principal part $v_j(z)$ at a_j for each j. But $g(z)k(z)$ will have no poles, and furthermore will have an $N(j)+M(j)$ degree Taylor polynomial at a_j which is equal to $\tilde{s}_j(z)$. Hence,

$$m(z) := \frac{g(z)k(z)}{h(z)}$$

satisfies the conclusion of the theorem. □

In the event that the poles have an accumulation point in $\mathbb{C}$, one can no longer apply the proof of Theorem 7.24; furthermore, the subsequent results are no longer valid. However, there is still a definitive result.

Theorem 7.29. *[Mittag-Leffler theorem] Let $U \subset \mathbb{C}$ be open, and let $\{a_j\} \subset U$ have no accumulation points in U. Set*

$$s_j(z) := \sum_{\ell=-p(j)}^{-1} a_\ell^j (z-a_j)^\ell.$$

There is a meromorphic $m : U \mapsto \mathbb{C}$ whose principal part at a_j is $s_j(z)$, and which has no other poles.

Proof: Assume that U has the same properties as in the proof of Theorem 7.20. For each j let $\hat{a}_j \in \widehat{\mathbb{C}}\backslash U$ be the nearest point to a_j, and set $d_j := |\hat{a}_j - a_j|$.

As a consequence of the pole-pushing lemma [8, Lemma 8.3.5], for each j there exists a finite Laurent expansion $t_j(z)$ in powers of $(z-\hat{a}_j)$ such that

$$|s_j(z) - t_j(z)| < \frac{1}{2^j}, \quad z \in \widehat{\mathbb{C}}\backslash\overline{D}(\hat{a}_j, 2d_j).$$

The claim is that

$$m(z) := \sum_{j=1}^{+\infty} (s_j(z) - t_j(z))$$

is the desired meromorphic function. Note that only the terms $s_j(z)$ contribute the poles.

Fix $\overline{D}(a, r) \subset U \backslash \{a_j\}$. Since $d_j \to 0$, choose J sufficiently large so that $j \geq J$ implies that

$$2d_j < \operatorname{dist}\left(\overline{D}(a, r), \widehat{\mathbb{C}} \backslash U\right).$$

Thus, for $j \geq J$ one has that $\overline{D}(a, r) \subset U \backslash \overline{D}(\hat{a}_j, 2d_j)$, so that

$$|s_j(z) - t_j(z)| < \frac{1}{2^j}, \quad z \in \overline{D}(a, r).$$

The Weierstrass M-test then yields uniform convergence on $\overline{D}(a, r)$. Since $\overline{D}(a, r)$ is arbitrary, the series converges uniformly on compact subsets of $U \backslash \{a_j\}$, which yields that $m(z)$ is the desired meromorphic function. □

Remark 7.30. The pole-pushing lemma essentially says that the principal part of a Laurent expansion expanded about $z = a$ can be uniformly approximated by a truncation of a Laurent expansion expanded about $z = \beta$ on $\widehat{\mathbb{C}} \backslash D(\beta, r)$, where $r > |a - \beta|$.

Combining Weierstrass theorem with the Mittag-Leffler theorem yields the following approximation theorem.

Theorem 7.31. *Let $U \subset \mathbb{C}$ be open, and let $\{a_j\} \subset U$ have no accumulation points in U. Set*

$$s_j(z) := \sum_{\ell=-M(j)}^{N(j)} a_\ell^j (z - a_j)^\ell,$$

with $M(j), N(j) \geq 0$. There is a meromorphic $m : U \mapsto \mathbb{C}$, holomorphic on $U \backslash \{a_j\}$, such that the ℓ-th coefficient of $m(z)$ at a_j is a_ℓ^j for $-M(j) \leq \ell \leq N(j)$.

Proof: The proof is the same as that for Theorem 7.28. All that must be done is follow the same line of reasoning, and instead appeal to the results of Theorem 7.20 and Theorem 7.29. □

8. Applications of Infinite Sums and Products

It has been seen that the zeros of a holomorphic function can be arbitrarily placed. The converse question will now be considered. Does the behavior of the holomorphic function near the boundary of its domain of existence control the number and placement of the zeros? For example, the higher the order of a polynomial, the greater the number of zeros, and the faster the growth rate as $|z| \to \infty$. What if the growth rate is exponential?

8.1. Jensen's formula and an introduction to Blaschke products

Definition 8.1. If $a \in D(0, 1)$, the Blaschke factor is given by

$$B_a(z) := \frac{z - a}{1 - \bar{a}z}.$$

Note that a Blaschke factor is actually a Mőbius transformation. It is known that $B_a(z)$ is a conformal self-map of $D(0, 1)$, with $|B_a(z)| = 1$ for $z \in \partial D(0, 1)$. Furthermore, since the pole is located at $z = 1/\bar{a}$, one has that $B_a(z)$ is holomorphic on a neighborhood of $\overline{D}(0, 1)$.

Suppose that $g := u + \mathrm{i}v$ is nonzero and holomorphic on $\overline{D}(0,r)$. It can be easily checked that

$$\ln|g| = \frac{1}{2}\ln(u^2+v^2)$$

is harmonic on $\overline{D}(0,r)$, i.e., $\Delta \ln|g| = 0$. As a consequence one has the mean value property [8, Theorem 7.2.5], i.e.,

$$\ln|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|g(r\mathrm{e}^{\mathrm{i}\vartheta})|\,\mathrm{d}\vartheta. \tag{8.1}$$

This result follows immediately from the Cauchy integral formula. Equation (8.1) will be useful in the following theorem, which can be considered to be a generalization.

Theorem 8.2. *[Jensen's formula] Let f be holomorphic on a neighborhood of $\overline{D}(0,r)$, and suppose that $f(0) \neq 0$. Let $a_1,\dots,a_k \in D(0,r)$ be the zeros of f. Then*

$$\ln|f(0)| + \sum_{j=1}^{k} \ln\left|\frac{r}{a_j}\right| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(r\mathrm{e}^{\mathrm{i}\vartheta})|\,\mathrm{d}\vartheta.$$

Remark 8.3. Note that this implies

$$\ln|f(0)| \le \frac{1}{2\pi}\int_0^{2\pi} \ln|f(r\mathrm{e}^{\mathrm{i}\vartheta})|\,\mathrm{d}\vartheta,$$

(*Jensen's inequality*), with equality if and only if f has no zeros in $D(0,r)$.

Proof: For $j = 1,\dots,k$ consider

$$f_j(z) := B_{a_j/r}\left(\frac{z}{r}\right).$$

One has that each f_j is holomorphic on $\overline{D}(0,r)$ with a simple zero only at $z = a_j$; furthermore, $|f_j(z)| = 1$ for $z \in \partial D(0,r)$. Thus, the function

$$g(z) := \frac{f(z)}{\prod_{j=1}^{k} f_j(z)}$$

is holomorphic and nonzero on $\overline{D}(0,r)$. It is clear that

$$\ln|f_j(0)| = \ln\left|\frac{a_j}{r}\right|,$$

and that

$$\ln|f_j(\mathrm{e}^{\mathrm{i}\vartheta})| = 0;$$

hence, upon applying equation (8.1) to $g(z)$ and simplifying one gets the desired result. □

The following theorem details the placement of the zeros of a bounded holomorphic function. In particular, the zeros must concentrate on the boundary sufficiently quickly.

Theorem 8.4. *Let $f : D(0,1) \mapsto \mathbb{C}$ be bounded and holomorphic, and let $\{a_j\} \subset D(0,1)$ be the zeros of f. One then has*

$$\sum_{j=1}^{\infty} (1 - |a_j|) < \infty. \tag{8.2}$$

Proof: First suppose that $f(0) \neq 0$. If $f(0) = 0$, apply the below argument to $g(z) := f(z)/z^m$, where m is the order of the zero. There exists numbers $0 < r \ll 1$ such that $|a_j| \neq r$ for all j. Given such an r, let $n(r)$ represent the number of zeros in $D(0,r)$. By Theorem 8.2 one has that

$$\ln|f(0)| + \sum_{j=1}^{n(r)} \ln\left|\frac{r}{a_j}\right| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(r\mathrm{e}^{\mathrm{i}\vartheta})|\,\mathrm{d}\vartheta.$$

Since $|f(z)| \le M$, upon letting $r \to 1^-$ one sees that

$$-\sum_{j=1}^{\infty} \ln|a_j| = \sum_{j=1}^{\infty} \ln\frac{1}{|a_j|} \le \ln M - \ln|f(0)|.$$

Since

$$-\ln a = -\ln(1-(1-a)) = \sum_{n=1}^{\infty} \frac{(1-a)^n}{n}$$

for any $a \in (0,1)$, one gets that $-\ln a > 1-a$. Hence,

$$\sum_{j=1}^{\infty}(1-|a_j|) \le \ln M - \ln|f(0)|,$$

which proves the result. □

Remark 8.5. There is an obvious modification of the theorem in the event that $f : D(0,r) \mapsto \mathbb{C}$ is bounded and holomorphic.

Definition 8.6. A Blaschke product if of the form

$$B(z) := z^m \left\{ \prod_{j=1}^{\infty} -\frac{\bar{a}_j}{|a_j|} B_{a_j}(z) \right\},$$

where $\{a_j\} \subset D(0,1)$.

Note that $|B_{a_j}(z)| \le 1$ on $\overline{D}(0,1)$, so that $|B(z)| \le 1$ on $\overline{D}(0,1)$. In order to determine if the Blaschke product is convergent, one needs to study the convergence of the sum

$$\sum_{j=1}^{\infty} \left| 1 + \frac{\bar{a}_j}{|a_j|} B_{a_j}(z) \right|$$

on $\overline{D}(0,r)$ for any $0 < r < 1$. Assume that equation (8.2) holds. A routine calculation shows that

$$\begin{aligned} \left| 1 + \frac{\bar{a}_j}{|a_j|} B_{a_j}(z) \right| &\le \frac{(1+r)(1-|a_j|)}{|a_j|(1-r)} \\ &\le 2\frac{1+r}{1-r}(1-|a_j|). \end{aligned}$$

The second inequality holds for $j > J_0$, where $|a_j| \ge 1/2$ for $j > J_0$. Thus, by the Weierstrass M-test the series converges uniformly on $\overline{D}(0,r)$. Under the assumption of equation (8.2) one then has that the Blaschke product is convergent on $D(0,1)$ to a bounded holomorphic function $B(z)$ whose zeros are precisely at $z = a_j$.

Lemma 8.7. *Suppose that $f : D(0,1) \mapsto \mathbb{C}$ is bounded with zeros at $\{a_j\} \subset D(0,1)$. Then*

$$f(z) = F(z)B(z),$$

where $F : D(0,1) \mapsto \mathbb{C}\backslash\{0\}$ is bounded. Furthermore,

$$\sup_{z \in \partial D(0,1)} |f(z)| = \sup_{z \in \partial D(0,1)} |F(z)|.$$

Proof: If one defines $F(z) := f(z)/B(z)$, then it is clear that F is holomorphic and nonzero on $D(0,1)$. Upon applying the maximum modulus theorem (Corollary 5.18), and using the facts that

$$|z_1 z_2| = |z_1|\,|z_2|,$$

and $|B(z)| \equiv 1$ for $z \in \partial D(0,1)$, one immediately gets the second result. □

In conclusion, one can then say that $f : D(0,1) \mapsto \mathbb{C}$ is bounded with zeros at $\{a_j\} \subset D(0,1)$ if and only if equation (8.2) holds. Furthermore, up to a nonzero multiplicative factor the function is a Blaschke product.

8.2. The Hadamard Gap Theorem

The Blaschke product $B(z)$ given in Definition 8.6 is bounded and holomorphic on $D(0,1)$. It is natural to then wonder if it can be holomorphically extended onto $\overline{D}(0,1)$. Of course, it depends upon the distribution of the zeros near $\partial D(0,1)$; in particular, is the limit of the zeros dense on $\partial D(0,1)$ (recall the discussion before Corollary 7.23)? Now let us construct a holomorphic function on $D(0,1)$ for which no $z \in \partial D(0,1)$ is regular. This will be accomplished without considering the placement of the zeros. Consider

$$f(z) := \sum_{n=1}^{\infty} z^{2^n}, \quad z \in D(0,1).$$

The sum is clearly uniformly convergent, and hence $f(z)$ is holomorphic. For a given $N \in \mathbb{N}$ choose $w \in \partial D(0,1)$ such that $w^{2^N} = 1$, and let $r \in (0,1)$. One has that

$$\begin{aligned} f(rw) &= \sum_{n=1}^{N-1} (rw)^{2^n} + \sum_{n=N}^{\infty} (rw)^{2^n} \\ &= \sum_{n=1}^{N-1} (rw)^{2^n} + \sum_{n=N}^{\infty} r^{2^n}, \end{aligned}$$

so that

$$\lim_{r \to 1^-} |f(rw)| = +\infty.$$

Since the set of points $w \in \partial D(0,1)$ is dense, one then immediately gets that no $z \in \partial D(0,1)$ is regular. This example is a particular case of the following theorem:

Theorem 8.8. *[Ostrowski-Hadamard] Let $0 < p_1 < p_2 < \cdots \in \mathbb{N}$ satisfy*

$$\frac{p_{j+1}}{p_j} > \lambda > 1.$$

If

$$f(z) := \sum_{j=1}^{\infty} a_j z^{p_j}$$

is holomorphic on $D(0,1)$, then no point of $\partial D(0,1)$ is regular.

Proof: See [8, Theorem 9.2.1]. □

8.3. Entire functions of finite order

Throughout this subsection it will be assumed that $f : \mathbb{C} \mapsto \mathbb{C}$ is entire. It will also be assumed that the zeros $\{a_j\}$ are ordered so that $|a_1| \le |a_2| \le \cdots$. A primary purposes of this subsection is to present a refinement of the Weierstrass factorization theorem (Theorem 7.18).

Definition 8.9. f is of finite order if there exist $a, r \in \mathbb{R}^+$ such that

$$|f(z)| \le \exp(|z|^a), \quad |z| > r.$$

The order of f is given by

$$\lambda := \inf\{a \in \mathbb{R}^+ : |f(z)| \le \exp(|z|^a), |z| > r\}.$$

Note that a polynomial has order zero, whereas $\sin z$ has order one. The aim is to determine the manner in which the order of f dictates the rate at which the zeros tend towards infinity. In all that follows it will typically be assumed that $f(0) = 1$. This can be done without loss of generality, for the simple scaling

$$f(z) \mapsto c\frac{f(z)}{z^m}, \quad \lim_{z \to 0} c\frac{f(z)}{z^m} = 1,$$

allows f to satisfy the property without changing the location of any of the other zeros.

Definition 8.10. Set $n(r)$ to be the number of zeros of f in $D(0,r)$, and set

$$M(r) := \max_{z\in\partial D(0,r)} |f(z)|.$$

Lemma 8.11. *If $f(0)=1$, then*

$$n(r) \le \frac{\ln M(2r)}{\ln 2}.$$

Proof: Let $a_1,\dots,a_{n(2r)} \in D(0,2r)$ denote the zeros. Since they are ordered, they satisfy $|a_1| \le |a_2| \le \cdots$, so that $a_1,\dots,a_{n(r)} \in D(0,r)$. Further suppose that no zeros lie on $\partial D(0,2r)$. Using the fact that $f(0)=1$ and applying Theorem 8.2 yields

$$\begin{aligned}\sum_{k=1}^{n(2r)} \ln\left(\frac{2r}{|a_k|}\right) &= \frac{1}{2\pi}\int_0^{2\pi} \ln|f(2re^{i\vartheta})|\,d\vartheta\\ &\le \ln M(2r).\end{aligned}$$

Since

$$\sum_{k=1}^{n(r)} \ln 2 \le \sum_{k=1}^{n(r)} \ln\left(\frac{2r}{|a_k|}\right) \le \sum_{k=1}^{n(2r)} \ln\left(\frac{2r}{|a_k|}\right),$$

the result is now proven. □

Now suppose that f has finite order, so that

$$M(r) \le \exp(r^{\beta+\epsilon/2})$$

for any $\epsilon > 0$ and r sufficiently large. By Lemma 8.11 this implies that

$$n(r) \le \frac{(2r)^{\beta+\epsilon/2}}{\ln 2},$$

which, upon multiplying both sides by $r^{-(\beta+\epsilon)}$ and taking the limit $r \to +\infty$, in turn implies that

$$n(r) \le r^{\beta+\epsilon}$$

for r sufficiently large. Since $\{a_1,\dots,a_j\} \subset \overline{D}(0,|a_j|)$, for j sufficiently large and $\delta > 0$ arbitrarily small one has that

$$j \le n(|a_j|+\delta) \le |a_j|^{\beta+\epsilon}.$$

Letting $\delta \to 0^+$ then yields that for any $\mu > \epsilon$,

$$|a_j|^{-(\beta+\mu)} \le j^{-(\beta+\mu)/(\beta+\epsilon)}.$$

Thus,

$$\sum_{j=1}^{\infty} |a_j|^{-(\beta+\mu)} < \infty$$

for any $\mu > 0$ (since $\epsilon > 0$ is arbitrarily small). The following theorem has now been proved.

Theorem 8.12. *If f is of finite order β and satisfies $f(0)=1$, then the zeros $a_1, a_2, \dots$ satisfy*

$$\sum_{j=1}^{\infty} |a_j|^{-(\beta+\mu)} < \infty$$

for any $\mu > 0$.

Let $[\lambda]$ represent the largest integer less than or equal to λ. As a consequence of Theorem 8.12 and Remark 7.15 one has that

$$P(z) := \prod_{j=1}^{\infty} E_{[\lambda]}\left(\frac{z}{a_j}\right) \tag{8.3}$$

converges uniformly on compact subsets of $\mathbb{C}$. Furthermore, the Weierstrass factorization theorem (Theorem 7.18) states that for a function f of finite order λ one has

$$f(z) = \mathrm{e}^{g(z)} z^m P(z) \tag{8.4}$$

for some entire function $g(z)$. The next question to answer: what are the properties of $g(z)$? The fact that it is entire implies that it has a convergent Taylor series. Does the order of f imply that the series is finite? Assuming that $f(0) = 1$ and applying logarithmic differentiation to equation (8.4) yields

$$\frac{f'(z)}{f(z)} = g'(z) + \frac{P'(z)}{P(z)}; \tag{8.5}$$

hence, if one better understands the logarithmic differentiation of f and P, one will better understand the behavior of g. The following technical lemma, which is proven in [**8**, Lemmas 9.3.3,9.3.4], is first needed.

Lemma 8.13. *Let f be of finite order λ with $f(0) = 1$, and let $p \in \mathbb{N}$ satisfy $p > \lambda - 1$. Then for any fixed $z \in \mathbb{C}$,*

$$\lim_{r\to+\infty} \sum_{k=1}^{n(r)} \bar{a}_k^{p+1}(r^2 - \bar{a}_k z)^{-(p+1)} = 0$$

$$\lim_{r\to+\infty} \int_0^{2\pi} 2r\mathrm{e}^{\mathrm{i}\vartheta}(r\mathrm{e}^{\mathrm{i}\vartheta} - z)^{-(p+2)} \ln|f(r\mathrm{e}^{\mathrm{i}\vartheta})| \,\mathrm{d}\vartheta = 0.$$

Let $z \in D(0, 2r)$. As a consequence of the Poisson-Jensen formula [**8**, Problem 9.1] one has that

$$\ln|f(z)| = -\sum_{j=1}^{n(r)} \ln\left|\frac{r^2 - \bar{a}_j z}{r(z - a_j)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{r\mathrm{e}^{\mathrm{i}\vartheta} + z}{r\mathrm{e}^{\mathrm{i}\vartheta} - z}\right) \ln|f(r\mathrm{e}^{\mathrm{i}\vartheta})| \,\mathrm{d}\vartheta.$$

Logarithmic differentiation in z then yields

$$\frac{f'(z)}{f(z)} = -\sum_{j=1}^{n(r)} (a_j - z)^{-1} + \sum_{j=1}^{n(r)} \bar{a}_j(r^2 - \bar{a}_j z)^{-1} + \frac{1}{2\pi}\int_0^{2\pi} 2r\mathrm{e}^{\mathrm{i}\vartheta}(r\mathrm{e}^{\mathrm{i}\vartheta} - z)^{-2} \ln|f(r\mathrm{e}^{\mathrm{i}\vartheta})| \,\mathrm{d}\vartheta.$$

Let $p \in \mathbb{N}$ be such that $p > \lambda - 1$. Differentiate both sides of the above p times. Upon doing so, take the limit $r \to +\infty$ and use the result of Lemma 8.13 to see that

$$\frac{\mathrm{d}^p}{\mathrm{d}z^p}\frac{f'(z)}{f(z)} = -p! \sum_{j=1}^{\infty} (a_j - z)^{-(p+1)}. \tag{8.6}$$

A similar argument yields that for $P(z)$ defined in equation (8.3),

$$\frac{\mathrm{d}^p}{\mathrm{d}z^p}\frac{P'(z)}{P(z)} = -p! \sum_{j=1}^{\infty} (a_j - z)^{-(p+1)}. \tag{8.7}$$

Now consider equation (8.5). Differentiating p times, and using the results of equation (8.6) and equation (8.7) yields

$$\frac{\mathrm{d}^{p+1}}{\mathrm{d}z^{p+1}} g(z) \equiv 0;$$

hence, $g(z)$ is a polynomial of degree at most p. Note that one can have $\lambda - 1 < p \le \lambda$ in the above argument. The following theorem has now been proven.

Theorem 8.14. *Suppose that f is of finite order ρ with $f(0) = 1$. The Weierstrass canonical product*

$$f(z) = e^{g(z)}P(z)$$

has the property that g is a polynomial of degree at most ρ.

Remark 8.15. One has that:

(a) As noted above, the above theorem will apply if $f(0) = 0$. Simply set $\tilde{f}(z) = cf(z)/z^m$, where $c \in \mathbb{C}$ is chosen so that $\tilde{f}(0) = 1$, and apply the theorem to $\tilde{f}$ to eventually get

$$f(z) = \frac{1}{c}z^m e^{g(z)}P(z),$$

where again g is a polynomial of degree at most ρ.

(b) The synthesis of Theorem 8.12 and Theorem 8.14 is known as that *Hadamard factorization theorem*.

Corollary 8.16. *Suppose that f is of finite order ρ with $\rho \notin \mathbb{N}$. If $a \in \mathbb{C}$ lies in the image of f, then there exist infinitely many values $z_j \in \mathbb{C}$ such that $f(z_j) = a$.*

Proof: Without loss of generality assume that $a = 0$, and suppose that the assertion is false. One then has that

$$f^{-1}(0) = \{a_1, \dots, a_N\}, \quad N \in \mathbb{N},$$

so that by Theorem 8.14

$$f(z) = e^{g(z)}P(z), \quad P(z) := \prod_{j=1}^{N}(z - a_j),$$

where $g(z)$ is a polynomial of degree not exceeding ρ. It is clear that the order of $e^{g(z)}$ is $q \in \mathbb{N}_0$, where q is the degree of g. Let $\epsilon > 0$ be given. It is clear that for $|z|$ sufficiently large,

$$|e^{g(z)}| = \frac{|f(z)|}{|P(z)|} \le e^{|z|^{\rho+\epsilon}};$$

similarly, $|e^{g(z)}| \ge e^{|z|^{\rho-\epsilon}}$. Hence, since $\epsilon > 0$ is arbitrary, the order of $e^{g(z)}$ is ρ. Thus, $\rho = q \in \mathbb{N}_0$, which is a contradiction. □

Corollary 8.17. *Suppose that f is nonconstant and is of finite order ρ. The image of f contains all complex numbers except possibly one.*

Proof: Suppose that the image of f omits a_1 and a_2. Set $h(z) := f(z) - a_1$. Since $h(z) \neq 0$, one has that $h(z) = e^{g(z)}$ for some polynomial $g(z)$. Since there exists no z such that $h(z) = a_2 - a_1$, one has that $g(z)$ omits the value $\ln(a_2 - a_1)$. Thus, the polynomial $g(z) - \ln(a_2 - a_1)$ never vanishes, which violates the fundamental theorem of algebra unless g is constant. This is possible only if f is constant, which contradicts the hypothesis. □

To paraphrase the above results, one can state that a nonconstant entire function of nonintegral finite order will achieve every value except possibly one an infinite number of times. As we will see in the next section, this is a consequence of *Picard's little theorem*.

8.4. Picard's theorems

The primary purposes of this subsection is to remove the restriction that $\rho \notin \mathbb{N}$ in Corollary 8.16, and to strengthen the result of Corollary 8.17. We start with a preliminary result, which follows immediately from Lemma 6.21.

Definition 8.18. If $f : U \mapsto \mathbb{C}$ is holomorphic, then we say that $A_0 \in \mathbb{C}$ is an A-point if there exists a $z_0 \in U$ such that $f(z_0) = A_0$.

Lemma 8.19. *Suppose that $f : \mathbb{C} \mapsto \mathbb{C}$ is entire, and further suppose that f has no A_0-points. Then*

$$f(z) = A_0 + e^{g(z)},$$

where $g : \mathbb{C} \mapsto \mathbb{C}$ is entire.

Now suppose that in addition to having no A_0-points, the function f has finite order ρ. As a consequence of Theorem 8.14 one has that

$$f(z) - A_0 = e^{P(z)},$$

where $P(z)$ is a polynomial of degree $n \le \rho$. Since the order of $e^{P(z)}$ is n (see the proof of Corollary 8.16), and since $f(z)$ and $f(z) - A_0$ have the same order, the following result has now been proven.

Lemma 8.20. *Suppose that $f : \mathbb{C} \mapsto \mathbb{C}$ is entire, and further suppose that f has no A_0-points. If f has finite order ρ, then $\rho = n \in \mathbb{N}_0$, and*

$$f(z) = A_0 + e^{P(z)},$$

where $P(z)$ is a polynomial of degree n.

Remark 8.21. Note that as a consequence of Corollary 8.17 one has that the image of $f(z)$ is $\mathbb{C}\backslash\{A_0\}$.

Now suppose that $z_1, \dots, z_m \in \mathbb{C}$ are distinct A_0-points, where z_j is of order $k_j \in \mathbb{N}$, i.e., $f(z) - A_0$ has a zero of order k_j at z_j. Upon applying Lemma 8.19 to the function

$$F(z) := \frac{f(z) - A_0}{(z - z_1)^{k_1} \dots (z - z_m)^{k_m}},$$

one has that

$$f(z) = A_0 + e^{g(z)} \prod_{j=1}^{m} (z - z_j)^{k_j}. \tag{8.8}$$

If one further assumes that $f(z)$ has finite order ρ, then upon applying the proof of Theorem 8.14 to equation (8.8) one sees that $g(z) = P(z)$, where $P(z)$ is a polynomial of degree n. Finally, as in the discussion preceding Lemma 8.20, one can conclude that $\rho = n$.

Lemma 8.22. *Suppose that $f : \mathbb{C} \mapsto \mathbb{C}$ is entire, and further suppose that $z_1, \dots, z_m \in \mathbb{C}$ are distinct A_0-points of order $k_j \in \mathbb{N}$. If f has finite order ρ, then $\rho = n \in \mathbb{N}_0$, and*

$$f(z) = A_0 + e^{P(z)} \prod_{j=1}^{m} (z - z_j)^{k_j},$$

where $P(z)$ is a polynomial of degree n.

Remark 8.23. As a consequence of Corollary 8.17 and Lemma 8.22, one has that if f has order $\rho \notin \mathbb{N}$, then its image is all of $\mathbb{C}$.

8.4.1. Picard's little theorem

We are now ready to prove *Picard's little theorem*. Let $f(z)$ be an entire function of finite order ρ. Note that if $\rho = 0$, then $f(z)$ is a polynomial, and hence $f(z)$ has finitely many A-points for any $A \in \mathbb{C}$. As a consequence, assume that $\rho > 0$. Recall the results of Corollary 8.16 and Corollary 8.17. If $\rho \notin \mathbb{N}$, then there exists at most one point $A_0 \in \mathbb{C}$ that is not an A-point for f. Furthermore, for each $A \neq A_0$ there exists an infinite number of points $z \in \mathbb{C}$ such that $f(z) = A$.

Now assume that $A_0 \in \mathbb{C}$ is an A-point of finite order. As a consequence of Lemma 8.22, one then has that $\rho \in \mathbb{N}$; furthermore,

$$f(z) = A_0 + e^{P(z)} \prod_{j=1}^{m} (z - z_j)^{k_j},$$

where $P(z)$ is a polynomial of degree ρ. Suppose that there is a $B_0 \neq A_0$ such that $f(z)$ has finitely many B_0-points. As a consequence of Lemma 8.22 one has that

$$f(z) = B_0 + e^{Q(z)} \prod_{j=1}^{n} (z - \zeta_j)^{\ell_j},$$

where $Q(z)$ is a polynomial of degree β. Upon setting

$$p(z) := \prod_{j=1}^{m}(z-z_j)^{k_j}, \quad q(z) := \prod_{j=1}^{n}(z-\zeta_j)^{\ell_j}, \quad D := B_0 - A_0,$$

one then has that

$$p(z)e^{P(z)} - q(z)e^{Q(z)} \equiv D. \tag{8.9}$$

Differentiating equation (8.9) yields

$$[p'(z) + p(z)P'(z)]e^{P(z)} - [q'(z) + q(z)Q'(z)]e^{Q(z)} \equiv 0. \tag{8.10}$$

If

$$p'(z) + p(z)P'(z) \equiv 0, \tag{8.11}$$

then

$$P'(z) \equiv -\frac{p'(z)}{p(z)}$$

has poles at the points $z_1, \dots, z_m$. Since $P(z)$ is a polynomial, this implies that $p(z) \equiv C$, which in turn implies that $P(z) \equiv C$. Since the order of f is nonzero, this last case is precluded; hence equation (8.11) is impossible. Similarly, the relationship

$$q'(z) + q(z)Q'(z) \equiv 0$$

is impossible. Thus, equation (8.10) can be rewritten as

$$\frac{p'(z) + p(z)P'(z)}{q'(z) + q(z)Q'(z)} = e^{Q(z)-P(z)}.$$

Since the function on the right is entire and nonzero, the function on the left is a rational function with no poles or zeros; hence, it is a constant. This implies that

$$Q(z) - P(z) \equiv C,$$

so that equation (8.9) can be rewritten as

$$p(z) - C_1 q(z) = C_2 e^{-P(z)},$$

where $C_1, C_2 \in \mathbb{C}\backslash\{0\}$. Since the left-hand side is a polynomial, this again implies that $P(z) \equiv C$, which is precluded.

Theorem 8.24 (Picard's little theorem)**.** *Let $f : \mathbb{C} \mapsto \mathbb{C}$ be entire with a finite nonzero order β. If $\beta \notin \mathbb{N}$, the set of A-points is infinite for all $A \in \mathbb{C}$. If $\beta \in \mathbb{N}$, there exists at most one point $A_0 \in \mathbb{C}$ such that A_0 is an A-point of finite order. The set of A-points is infinite if $A \neq A_0$.*

Example. Set

$$f(z) := \sin z - Az, \quad A \in \mathbb{C}, \tag{8.12}$$

and let us find the number of roots of $f(z) = 0$. If $A = 0$, then the solutions are given by $z = n\pi$ for $n \in \mathbb{Z}$; as a consequence, now assume that $A \neq 0$. If one sets

$$g(z) := \frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots,$$

then equation (8.12) is equivalent to finding the number of solutions to $g(z) = A$. One has that $g(z)$ is an entire function of order one; hence, by Theorem 8.24 there exists at most one point $A_0 \in \mathbb{C}\backslash\{0\}$ for which there are only a finite number of solutions. If one considers the entire function

$$h(z) := g(z^{1/2}) = 1 - \frac{z}{3!} + \frac{z^2}{5!} + \cdots,$$

then one has that $h(z)$ is of order $1/2$. As a consequence of Theorem 8.24 one has that no such point A_0 exists when solving $h(z) = A$. Since every solution to $h(z) = A$ leads to two solutions to $g(z) = A$, one can finally conclude that there exists an infinite number of solutions to $f(z) = 0$.

8.4.2. Picard's big theorem

In Theorem 8.24 it is seen that entire functions of finite order achieve every value except at most one. What can be said if the function is not of finite order?

Lemma 8.25. *If $f : \mathbb{C} \mapsto \mathbb{C}$ is nonconstant and entire, then the range of f is dense.*

Proof: Suppose that the range of f is not dense. There then exists a $P \in \mathbb{C}$ and an $r > 0$ such that there exists no $z \in \mathbb{C}$ such that $f(z) \in D(P, r)$. Set

$$g(z) := \frac{r}{f(z) - P}.$$

One has that $g(z)$ is a uniformly bounded entire function; hence, by Liouville's Theorem 3.24 $g(z)$ is constant. This yields a contradiction. □

The result of Lemma 8.25 is an implication of the work in Section 3.4, and could have been stated at that time. It is a weak result in the sense that even if a set is dense, it can still be missing an uncountable number of points. The following result is much stronger. It would have been more natural to discuss *Picard's big theorem* in Section 6.5, as the proof presented in [**11**, Chapter III.51] requires the use of normal families. It is done at this time so for the reason of conciseness. The proof will not be given here, and is left for the interested student. The following result should be contrasted with the Casorati-Weierstrass Theorem 4.5.

Theorem 8.26 (Picard's big theorem)**.** *Let $f : D(P, r)\backslash\{P\} \mapsto \mathbb{C}$ be holomorphic, and suppose that P is an essential singularity. Then $f(z)$ takes every finite value with one possible exception.*

Theorem 8.26 actually implies Theorem 8.24 in the event that $f(z)$ is of finite order. A nonconstant entire function has either a pole or essential singularity at ∞. In the first case, the function must be a polynomial (see Lemma 4.29), so that its range assumes all values. In the second case Theorem 8.26 then applies to the point at ∞, and hence the result of Theorem 8.24 is automatically satisfied.

8.5. Borel's theorem

We now wish to prove a result which is essentially the converse of the Hadamard factorization theorem. Before we do so, however, we will need a way of characterizing convergence to infinity. Let $\{a_j\} \subset \mathbb{C}\backslash\{0\}$ be an arbitrary sequence which converges to infinity, and consider the series

$$\sum_{j=1}^{\infty} \frac{1}{|a_j|^{\alpha}}, \quad \alpha \in \mathbb{R}^+. \tag{8.13}$$

The greatest lower bound of the values of α for which equation (8.13) converges, say τ, is called the *exponent of convergence* of $\{a_j\}$. It can be shown [**11**, Theorem II.10.2] that

$$\tau = \limsup_{j\to\infty} \frac{\ln j}{\ln |a_j|}. \tag{8.14}$$

For example, if $a_j = \mathrm{e}^j$, then $\tau = 0$, whereas if $a_j = j^{\ell}$, then $\tau = 1/\ell$.

Theorem 8.27 (Borel's theorem)**.** *Let $\{a_j\} \subset \mathbb{C}\backslash\{0\}$ be an arbitrary sequence which converges to infinity, let $0 \le \tau < \infty$ be the exponent of convergence of $\{a_j\}$, and let $\chi \in \mathbb{N}_0$ be the largest value for which*

$$\sum_{j=1}^{\infty} \frac{1}{|a_j|^{\chi}}$$

diverges. Let $P(z)$ be a polynomial of degree $N \in \mathbb{N}_0$. The infinite product

$$f(z) := z^m \mathrm{e}^{P(z)} \prod_{j=1}^{\infty} \left(1 - \frac{z}{a_j}\right) \exp\left(\frac{z}{a_j} + \cdots + \frac{z^{\chi}}{\chi a_j^{\chi}}\right)$$

is an entire function of finite order

$$\rho = \max(N, \tau).$$

Proof: As a consequence of Remark 7.15 and the definition of τ one has that $f(z)$ is an entire function. All that must then be done is to compute the order of $f(z)$. Let $\mu \geq \tau$ be chosen so that equation (8.13) converges. In a manner similar to that used in the proof of Lemma 7.14, it can be shown that for $|z|$ sufficiently large,

$$\left|\left(1 - \frac{z}{a_j}\right)\exp\left(\frac{z}{a_j} + \cdots + \frac{z^\chi}{\chi a_j^\chi}\right)\right| \leq \exp\left(C_1\frac{|z|^\mu}{|a_j|^\mu}\right).$$

Since $P(z)$ is a polynomial of degree N, this then implies that for $|z|$ sufficiently large,

$$|f(z)| \leq \exp\left(C_2|z|^N + C_1|z|^\mu \sum_{j=1}^{\infty} \frac{1}{|a_j|^\mu}\right).$$

Thus, $f(z)$ is of finite order ρ with

$$\rho \leq \max(N, \mu).$$

Since μ can be chosen to be arbitrarily close to τ, this then implies that

$$\rho \leq \max(N, \tau).$$

However, as a consequence of Theorem 8.12 one has that $\tau \leq \rho$, and as a consequence of Theorem 8.14 one has that $N \leq \rho$. Hence,

$$\rho \geq \max(N, \tau),$$

which yields the final result. □

The power of Borel's Theorem 8.27 is that it allows one to compute the order of an entire function via the exponent of convergence of its zeros. For example, again consider the entire function

$$f(z) := \frac{\sin z^{1/2}}{z^{1/2}}.$$

The zeros are given by the sequence $\{j^2\pi^2\}$ for $j \in \mathbb{Z}_0$, which clearly has the exponent of convergence of $\tau = 1/2$; hence, $\rho \geq 1/2$. As a consequence of the product representation of $f(z)$ one knows that $N = 0$; thus, by Theorem 8.27 one has that $\rho = 1/2$.

9. Analytic Continuation

Suppose that $f : U \mapsto \mathbb{C}$ is holomorphic. This chapter is concerned with the question of finding a holomorphic $g : V \mapsto \mathbb{C}$ with $U \subset V$ such that $g \equiv f$ on U. The function g is then a holomorphic extension of f.

9.0.1. The Schwarz reflection principle

First suppose that $U \subset \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Let $\tilde{U}$ represent the reflection of U with respect to the real axis, i.e., if $z \in U$, then $\bar{z} \in \tilde{U}$ (see Figure 6). For $z \in \tilde{U}$ set

$$\tilde{f}(z) := \bar{f}(\bar{z}),$$

i.e., if $f(z) = u(x, y) + iv(x, y)$, then $\tilde{f}(z) = u(x, -y) - iv(x, -y)$. Since u and v satisfy the Cauchy-Riemann equations, it is then a simple calculation to check that $\tilde{f}$ is holomorphic on $\tilde{U}$. The construction of $\tilde{U}$ and $\tilde{f}$ is known as the *Schwarz reflection principle*.

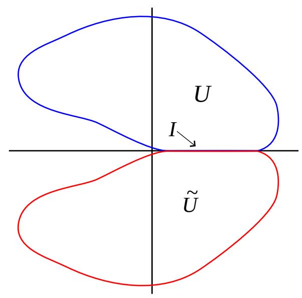


Figure 6: The reflection of the set U about the real axis.

Now suppose that the line segment $I := \{z \in \mathbb{C} \,:\, a < x < b,\, y = 0\} \subset \partial U$, and further suppose that $f(z)$ has a continuous limit as $z \to I$. Construct $\tilde{U}$ and $\tilde{f}$ as above, and set

$$g(z) := \begin{cases} f(z), & z \in U \\ \tilde{f}(z), & z \in \tilde{U}. \end{cases}$$

In order for g to have a continuous limit at I, one must clearly have that $v(x,0) \equiv 0$; hence, now assume that $f(z)$ is real-valued for $z \in I$. As a consequence, $g(z)$ is holomorphic on $U \cup \tilde{U}$, and has a continuous limit on I. A simple application of Theorem 3.1 then yields that g is holomorphic on $U \cup I \cup \tilde{U}$.

Remark 9.1. If $\mathrm{Im} f(z) \to a$ as $z \to I$, then upon setting $h(z) := f(z) - \mathrm{i}a$ one can apply the above argument to h in order to construct a holomorphic function on $U \cup I \cup \tilde{U}$. Thus, in this case a particular *translate* of $f(z)$ can be analytically extended via the Schwarz reflection principle.

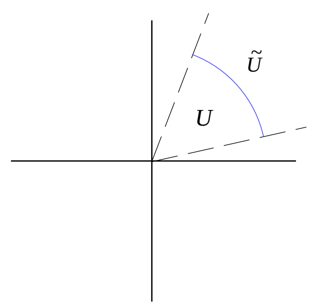


Figure 7: The reflection of the set U about $\partial D(0,R)$.

Now let us consider reflections across circular arcs. This is important to consider, as holomorphic functions can always be defined on circles via a power series. A more extensive discussion can be found in [1, Theorem 5.7.2,Section 5.8]. Suppose that for some $R > 0$,

$$U := \{z \in D(0,R) \,:\, \vartheta_1 < \arg z < \vartheta_2\}.$$

Given a point $z = r\mathrm{e}^{\mathrm{i}\vartheta} \in U$, the inverse point is given by $R^2/\bar{z} = R^2\mathrm{e}^{\mathrm{i}\vartheta}/r$. As a consequence, set

$$\tilde{U} := \{z \in \mathbb{C} \backslash D(0,R) \,:\, \vartheta_1 < \arg z < \vartheta_2\}$$

see Figure 7). If one sets

$$\tilde{f}(z) := \bar{f}\left(\frac{R^2}{\bar{z}}\right),$$

then upon applying the Cauchy-Riemann equations one can check that $\tilde{f}$ is holomorphic on $\tilde{U}$. Furthermore, on $\partial D(0, R)$ one has that

$$\tilde{f}(z) = u(x, y) - \mathrm{i}v(x, y),$$

so that a necessary condition for g to be holomorphic across $\partial D(0, R)$ is that $\mathrm{Im} f \equiv 0$ on $\partial D(0, R)$.

The following example shows that $\mathrm{Im} f \equiv 0$ on $\partial D(0, R)$ is a sufficient, but not necessary, condition to yield an analytic extension. Consider

$$f(z) := \sum_{n=0}^{+\infty} z^n \quad \left(= \frac{1}{1-z}\right).$$

It is clear that $f : D(0, 1) \mapsto \mathbb{C}$ is holomorphic. Applying the above theory yields the holomorphic reflection

$$\tilde{f}(z) = \sum_{n=0}^{+\infty} \frac{1}{z^n}, \quad \left(= -\frac{z}{1-z}\right).$$

for $z \in \mathbb{C}\backslash D(0, 1)$. Although f has a continuous limit to $\partial D(0, 1)$ except at $z = 1$, due to the fact that f is not real-valued on $\partial D(0, 1)$, the reflection does not yield an analytic extension of f; in fact, the two functions agree only at $z = -1$. However, f does have an analytic extension given by the sum

$$\tilde{f}(z) = -\frac{1}{z} \sum_{n=0}^{+\infty} \frac{1}{z^n}.$$

9.1. Definition of an analytic function element

The Schwarz reflection principle yields one way in which one can analytically extend a function. However, it has the drawback that strong restrictions on the function are necessary in order to achieve holomorphicity on the boundary.

9.1.1. The gamma function

In order to see another way of extending a function, consider the family of functions

$$\Gamma_a(z) := \int_a^{1/a} t^{z-1} \mathrm{e}^{-t} \, \mathrm{d}t, \quad a \in \mathbb{R}^+.$$

Set $U_0 := \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$. Since $|t^{z-1}| = t^{\mathrm{Re}\, z-1}$, one has that Γ_a is holomorphic on U_0. Furthermore, the Euler gamma function

$$\Gamma(z) := \int_0^{+\infty} t^{z-1} \mathrm{e}^{-t} \, \mathrm{d}t \tag{9.1}$$

is the normal limit of the family $\{\Gamma_a(z)\}$, so that $\Gamma(z)$ is holomorphic on U_0.

Now, integration by parts yields that for $z \in U_0$ one can actually write

$$\Gamma(z) = \frac{1}{z} \int_0^{+\infty} t^z \mathrm{e}^{-t} \, \mathrm{d}t. \tag{9.2}$$

The integral is well-defined for $z \in U_1 := \{z \in \mathbb{C} : \mathrm{Re}\, z > -1\}$ and defines a holomorphic function on U_1; hence, Γ can be analytically extended to be holomorphic on $U_1\backslash\{0\}$, and has a simple pole at $z = 0$. Integrating by parts again yields

$$\Gamma(z) = \frac{1}{z(z+1)} \int_0^{+\infty} t^{z+1} \mathrm{e}^{-t} \, \mathrm{d}t,$$

which, except at the poles $z = 0, -1$, is holomorphic on $U_2 := \{z \in \mathbb{C} : \mathrm{Re}\, z > -2\}$. Continuing with this process eventually yields that $\Gamma(z)$ is holomorphic on $\mathbb{C}\backslash\{0, -1, -2, \dots\}$, and has a simple pole at $z = 0, -1, -2, \dots$.

Remark 9.2. Note that equation (9.2) can be rewritten as

$$\Gamma(z+1) = z\Gamma(z). \tag{9.3}$$

Since $\Gamma(1) = 1$, an induction argument yields $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$.

Now let us find alternative representations of the gamma function. For $\operatorname{Re} z > 0$ rewrite equation (9.1) as

$$\begin{aligned}\Gamma(z) &= \int_0^1 t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t + \int_1^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t \\ &:= \phi(z) + \psi(z).\end{aligned}$$

Upon replacing e^{-t} by its power series and integrating term-by-term one sees that

$$\phi(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \int_0^1 t^{j-1+z}\,\mathrm{d}t = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(z+j)}.$$

Thus, $\phi(z)$ has simple poles at $z = -j$ for $j \in \mathbb{N}_0$ with residues $(-1)^j/j!$; otherwise, $\phi(z)$ is holomorphic. Now, upon using the functional relationship equation (9.3) one sees that

$$z(z+1)\cdots(z+j)\Gamma(z) = \Gamma(z+j+1),$$

so that upon letting $z \to -j \in \mathbb{N}_0$ one sees that

$$\lim_{z\to -j}(z+j)\Gamma(z) = \frac{(-1)^j\Gamma(1)}{j!}.$$

In other words, $\Gamma(z)$ has simple poles at $z = -j \in \mathbb{N}_0$ with residues $(-1)^j/j!$. One then has that $\Gamma(z) - \phi(z)$ is an entire function, as each function has poles at $z = -j \in \mathbb{N}_0$ with the same principal part. Since $\Gamma(z) - \phi(z) = \psi(z)$ for $\operatorname{Re} z > 0$, and $\psi(z)$ is entire (an exercise for the student), one then has that $\Gamma(z) - \phi(z) = \psi(z)$ for all $z \in \mathbb{C}$, i.e.,

$$\Gamma(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(z+j)} + \int_1^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t. \tag{9.4}$$

It can also be shown that $\Gamma(z)$ has the representation

$$\Gamma(z) = \frac{1}{z}\prod_{j=1}^{\infty}\left[\left(\frac{j+1}{j}\right)^z\left(\frac{j}{j+z}\right)\right] \tag{9.5}$$

[**8**, Proposition 15.1.6]; hence, the gamma function never vanishes. If one sets $f(z) := 1/\Gamma(z)$, then one has that $f(z)$ is entire with simple zeros for $z = -n \in \mathbb{N}_0$. As a consequence of Remark 7.15 and the Weierstrass factorization Theorem 7.18 one then has that

$$f(z) = \mathrm{e}^{g(z)} z \prod_{j=1}^{\infty}\left(1 + \frac{z}{j}\right)\mathrm{e}^{-z/j}$$

for some entire function $g(z)$. Note that

$$\mathrm{e}^{g(0)} = \lim_{z\to 0}\frac{f(z)}{z} = 1;$$

hence, without loss of generality it can be assumed that $g(0) = 0$.

The determination of a functional relationship for $g(z)$ can be found via the following argument [**13**, Section 69]. Let $f_n(z)$ be the partial product for $f(z)$, i.e.,

$$f_n(z) := \frac{1}{n!}\mathrm{e}^{(g(z)-\sum_{j=1}^{n} z/j)} z \prod_{j=1}^{n}(z+j). \tag{9.6}$$

Upon setting

$$a_n := \sum_{j=1}^{n} \frac{1}{j} - \ln(n+1),$$

one then sees that

$$\begin{aligned}\frac{zf_n(z+1)}{f_n(z)} &= (z+n+1)e^{(g(z+1)-g(z)-\sum_{j=1}^{n} 1/j)} \\ &= \left(1+\frac{z}{n+1}\right)e^{(g(z+1)-g(z)-a_n)}.\end{aligned}$$

It can be shown that $a_n \to \gamma \approx 0.5772\ldots$ (the Euler constant) [**8**, Lemma 15.1.8]. Thus, upon taking the limit $n \to +\infty$ one has that

$$\frac{zf(z+1)}{f(z)} = e^{(g(z+1)-g(z)-\gamma)}.$$

As a consequence of equation (9.3) one can then write

$$1 = e^{(g(z+1)-g(z)-\gamma)},$$

i.e.,

$$g(z+1) - g(z) = \gamma + i2k\pi, \quad k \in \mathbb{Z}. \tag{9.7}$$

Since $g(0) = 0$, this yields that

$$g(1) = \gamma + i2k\pi, \quad k \in \mathbb{Z}. \tag{9.8}$$

One solution to equation (9.7) and equation (9.8) is $g(z) = \gamma z$.

It is shown in [**8**, Proposition 15.1.9] that this is actually the desired solution. This requires an equivalent formulation, which is done in Proposition 9.7, of the gamma function representation given in equation (9.5). The following result has now been shown.

Proposition 9.3. *The gamma function is given by*

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{j=1}^{\infty} \left(1+\frac{z}{j}\right)e^{-z/j},$$

where γ is the Euler constant.

As a consequence of the representation given in Proposition 9.3, one has that $\Gamma(z)^{-1}$ is an entire function with zeros at $j \in \mathbb{N}_0$. Upon using equation (8.14) one sees that the exponent of convergence is $\tau = 1$; hence, by Borel's Theorem 8.27 one has that the order of $\Gamma(z)^{-1}$ is $\rho = 1$. By Picard's Little Theorem 8.24 one finally has that there exists at most one finite A-point such that $\Gamma(z)^{-1} = A$ has a finite number of solutions, i.e., there exists at most one $B \in \mathbb{C}$ such that $\Gamma(z) = B$ has only a finite number of solutions. In fact, as a consequence of the representation in equation (9.5) it is known that the exceptional point is $B = 0$.

Proposition 9.4. *The range of the gamma function is $\mathbb{C}\backslash\{0\}$. Furthermore, for each $A \in \mathbb{C}\backslash\{0\}$ there exists an infinite number of solutions to $\Gamma(z) = A$.*

Now let us find another relationship satisfied by the gamma function. Note that

$$\frac{1}{\Gamma(-z)} = -e^{-\gamma z} z \prod_{j=1}^{\infty} \left(1-\frac{z}{j}\right)e^{z/j};$$

this implies that

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -z^2 \prod_{j=1}^{\infty} \left(1-\frac{z^2}{j^2}\right).$$

Now, in Section 7.2 it was shown that

$$\sin(\pi z) = \pi z \prod_{j=1}^{\infty} \left(1-\frac{z^2}{j^2}\right);$$

hence, the above can be rewritten to say

$$-\frac{1}{z\Gamma(z)\Gamma(-z)} = \frac{\sin(\pi z)}{\pi}.$$

One has as a consequence of equation (9.3) that $-z\Gamma(-z) = \Gamma(1-z)$, which in turn yields

$$\frac{1}{\Gamma(z)\Gamma(1-z)} = \frac{\sin(\pi z)}{\pi}.$$

The following has now been proven.

Proposition 9.5. *The gamma function satisfies the relationship*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

Remark 9.6. Note that $\Gamma(1/2) = \sqrt{\pi}$, so that as a consequence of equation (9.3),

$$\Gamma(n+1/2) = \frac{1\cdot 3\cdots(2n-1)}{2^n}\sqrt{\pi}, \quad n \in \mathbb{N}.$$

Finally, again consider equation (9.6), where now $g(z) = \gamma z$. Setting

$$\beta_n := \gamma - \sum_{j=1}^{n}\frac{1}{j} + \ln n$$

(note that $\beta_n \to 0$ as $n \to +\infty$), one has that

$$f_n(z) = \frac{1}{n!n^z}\mathrm{e}^{\beta_n z} z\prod_{j=1}^{n}(z+j).$$

Upon taking the limit one then gets the following.

Proposition 9.7. *The gamma function satisfies*

$$\Gamma(z) = \lim_{n\to+\infty}\frac{n!n^z}{z\prod_{j=1}^{n}(z+j)}.$$

9.1.2. Multi-valued functions

Now suppose that $f_1 : R_1 \mapsto \mathbb{C}$ and $f_2 : R_2 \mapsto \mathbb{C}$ are holomorphic, with $f_1 \equiv f_2$ on the domain $R_1 \cap R_2$. One then has that $f_1 \equiv f_2$ on $R_1 \cup R_2$, and one can consider f_2 to be the analytic continuation of f_1. Similarly, if $f_3 : R_3 \mapsto \mathbb{C}$ is holomorphic with $f_3 \equiv f_2$ on the domain $R_2 \cap R_3$, then one can conclude that $f_3 \equiv f_2$ on $R_2 \cup R_3$. Suppose that $R_3 \cap R_1$ is a domain. Is $f_1 \equiv f_3$?

For example, consider the overlapping domains

$$R_j := \{z \in \mathbb{C} : \frac{3\pi}{4}j \le \arg z < \pi + \frac{3\pi}{4}j\}, \quad j = 1,\dots,3,$$

and let $f_j(z)$ be the holomorphic branch of $\ln z = \ln|z| + \mathrm{i}\arg z$ on R_j. It is clear that f_2 is the analytic continuation of f_1, and that f_3 is the analytic continuation of f_2. Furthermore, $R_3 \cap R_1$ is a domain. However,

$$f_1(-1) = \mathrm{i}\pi \ne \mathrm{i}3\pi = f_3(-1),$$

so that $f_3 \not\equiv f_1$. The difficulty is that $\ln z$ has a branch point at $z = 0$, which is contained in each domain R_j. The function f_3 is defined on a different sheet of the associated Riemann surface than is f_1.

9.3. The Monodromy theorem

As a consequence of the above discussion, it is natural to wonder the conditions for which $f_3 \equiv f_1$. Not surprisingly, all that is required is that the domain be simply connected (which is not true for a Riemann surface).

Definition 9.8. Let $U \subset \mathbb{C}$ be open and simply connected, and let $\gamma_j : [0,1] \mapsto U$ be curves for $j = 1, 2$. Assume that for $P, Q \in U$,

$$\gamma_1(0) = \gamma_2(0) = P, \quad \gamma_1(1) = \gamma_2(1) = Q.$$

γ_1 is homotopic to γ_2 if there is a continuous function $H : [0,1] \times [0,1] \mapsto U$ such that

(a) $H(0,t) \equiv \gamma_1(t)$

(b) $H(1,t) \equiv \gamma_2(t)$

(c) $H(s,0) \equiv P$

(d) $H(s,1) \equiv Q$.

One may think of a homotopy as being a continuous deformation of one curve to another with the endpoints being fixed.

Theorem 9.9. *[Monodromy theorem] Let $U \subset \mathbb{C}$ be simply connected, and suppose that $f : U_0 \subset U \mapsto \mathbb{C}$ is holomorphic. Let $\gamma_1, \gamma_2 \subset U$ be two smooth curves which begin at $P \subset U_0$ and terminate at $Q \subset U$. If γ_1 is homotopic to γ_2, then the analytic continuation of f along γ_1 is equal to the analytic continuation of f along γ_2.*

Proof: See [8, Section 10.3]. □

9.6. The Schwarz-Christoffel transformation

Much of the following discussion can be found in [1, Section 5.6]. Let Γ be the piecewise linear boundary of a polygon in the w-plane with vertices $A_1, \dots, A_n$. Let the interior angle at A_j be denoted by $\alpha_j \pi$. It is clear that if A_j is finite, then $0 < \alpha_j \le 2$ ($\alpha_j = 2$ corresponds to the tip of a "slit"), whereas if $A_j = \infty$, then via the transformation $z = 1/t$ one has that $-2 \le \alpha_j \le 0$. Note that for a closed polygon, since the sum of the exterior angles is 2π,

$$\sum_{j=1}^{n} (1 - \alpha_j) = 2,$$

i.e.,

$$\sum_{j=1}^{n} \alpha_j = n - 2.$$

The *Schwarz-Christoffel transformation* is defined by

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \gamma (z - a_1)^{\alpha_1 - 1} \cdots (z - a_n)^{\alpha_n - 1}, \tag{9.9}$$

where $\gamma \in \mathbb{C}$ and $a_1, \dots, a_n \in \mathbb{R}$. It has the properties that:

(a) the vertex A_j is mapped to a_j

(b) Γ is mapped to the real axis

(c) the transformation conformally maps the upper half plane to the interior of the polygon.

Remark 9.10. Recall that the Riemann Mapping Theorem 6.26 guarantees the existence of such a mapping. The utility of the Schwarz-Christoffel transformation is that it tells us precisely how to do it.

Recall that for bilinear transformations the correspondence of three (and only three) points on the boundaries of the two domains can be arbitrarily prescribed. This result is actually true for any conformal map between the boundary of two simply connected domains. Thus, the three vertices $A_1, \dots, A_3$ can be associated with any three points $a_1, \dots, a_3 \in \mathbb{R}$; however, this then fixes the points $a_4, \dots, a_n$. The actual determination of these points may be difficult, although symmetry considerations are usually helpful.

The integration of equation (9.9) typically yields a multi-valued function. A single branch is chosen via the requirement

$$0 < \arg(z - a_j) < \pi, \quad j = 1, \dots, n.$$

The function $f(z)$ defined by the transformation will then be holomorphic in the upper half plane with branch points at $z = a_j$.

Equation (9.9) implicitly assumes that none of the points $a_1, \dots, a_n$ are ∞. What if this feature was desired? Consider the transformation

$$z := a_n - \frac{1}{\zeta} \quad \Longrightarrow \quad \zeta = \frac{1}{a_n - z}.$$

Under this transformation $a_n \mapsto \infty$ and

$$a_j \mapsto \zeta_j = \frac{1}{a_n - a_j}, \quad j = 1, \dots, n-1.$$

Now,

$$\frac{\mathrm{d}}{\mathrm{d}z} = \zeta^2 \frac{\mathrm{d}}{\mathrm{d}\zeta},$$

so that equation (9.9) can be rewritten as

$$\zeta^2 \frac{\mathrm{d}w}{\mathrm{d}\zeta} = \gamma \left(\frac{1}{\zeta_1} - \frac{1}{\zeta} \right)^{\alpha_1 - 1} \cdots \left(-\frac{1}{\zeta} \right)^{\alpha_n - 1}.$$

Upon getting a common denominator, setting

$$\hat{\gamma} := \gamma \prod_{j=1}^{n-1} \zeta_j^{1-\alpha_j},$$

and using the fact that

$$\zeta^{\sum_{j=1}^{n}(1-\alpha_j)} = \zeta^2,$$

one gets the new equation

$$\frac{\mathrm{d}w}{\mathrm{d}\zeta} = \hat{\gamma}(\zeta - \zeta_1)^{\alpha_1 - 1} \cdots (\zeta - \zeta_{n-1})^{\alpha_{n-1} - 1}.$$

Thus, equation (9.9) is still valid even if the point ∞ is removed. However, note that only two other vertices can now be arbitrarily prescribed, since $A_n \mapsto \infty$.

For example, consider the half strip bounded by the curves

$$\begin{aligned}
\gamma_{\mathrm{L}} &:= \{z \in \mathbb{C} : \operatorname{Re} z = -k, \operatorname{Im} z \geq 0\} \\
\gamma_{\mathrm{R}} &:= \{z \in \mathbb{C} : \operatorname{Re} z = k, \operatorname{Im} z \geq 0\} \\
\gamma_{\mathrm{B}} &:= \{z \in \mathbb{C} : |\operatorname{Re} z| \leq k, \operatorname{Im} z = 0\}
\end{aligned}$$

(see Figure 8). The vertices are given by $z = \pm k + \mathrm{i}\infty$ and $z = \pm k$. As a consequence of the above discussion, upon mapping $-k + \mathrm{i}\infty \mapsto -\infty + \mathrm{i}0$, one can arbitrarily choose the placement of two more points. Suppose that

$$\pm k \mapsto \pm 1.$$

Symmetry considerations then yield that $k + \mathrm{i}\infty \mapsto +\infty + \mathrm{i}0$. Now, one has that $\alpha_1 = \alpha_2 = 1/2$. The appropriate Schwarz-Christoffel transformation then satisfies

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \gamma(z+1)^{-1/2}(z-1)^{-1/2} = \frac{\tilde{\gamma}}{\sqrt{1-z^2}},$$

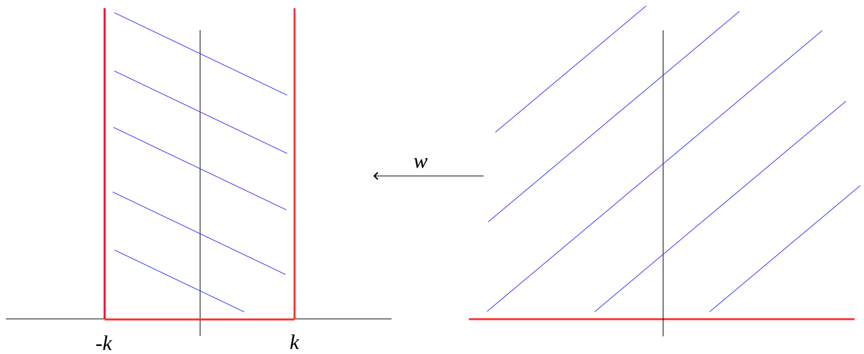


Figure 8: The mapping of the upper half plane to the half strip.

which upon integration yields

$$w = \tilde{\gamma} \sin^{-1} z + c.$$

One must have that

$$\pm k = \tilde{\gamma} \sin^{-1}(\pm 1) + c,$$

which implies that $c = 0$ and $\tilde{\gamma} = 2k/\pi$. Thus, the transformation is

$$w = \frac{2k}{\pi} \sin^{-1} z \quad \Longrightarrow \quad z = \sin\left(\frac{\pi}{2k} w\right).$$

Remark 9.11. These ideas can also be used to map the exterior of a closed polygon to the upper half plane. The mapping defined by equation (9.9) maps the boundary of the polygon to the real line. Thus, all that must be done is to map the point ∞ to the interior of the of the upper half plane, i.e., $\infty \mapsto \mathrm{i}a_0$, $a_0 \in \mathbb{R}^+$. The transformation then is modified to be

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\gamma}{(z - \mathrm{i}a_0)^2 (z + \mathrm{i}a_0)^2} (z - a_1)^{\alpha_1 - 1} \cdots (z - a_n)^{\alpha_n - 1}. \tag{9.10}$$

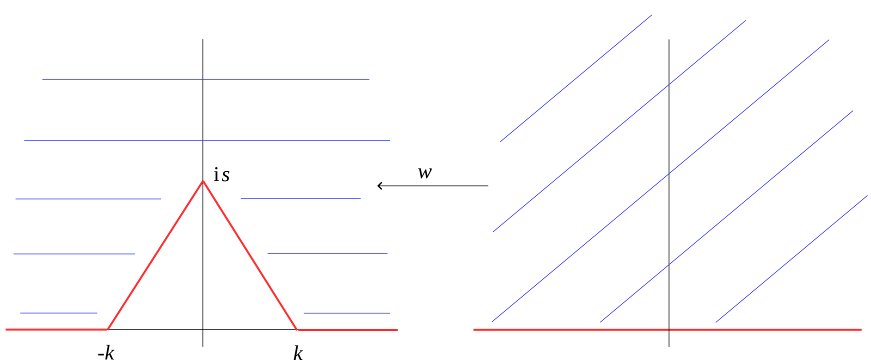


Figure 9: The mapping of the upper half plane to the exterior of the triangle.

For another example, consider the isosceles triangle with vertices at $\pm k$, $\mathrm{i}s$ for some $k, s \in \mathbb{R}^+$. The goal is to determine the transformation which maps that portion of the exterior of the triangle which resides in the upper half plane to the upper half plane (see Figure 9). Since the domain to be mapped is only a subset of the exterior of the polygon, we will use equation (9.9) in order to construct the transformation.

Set $A_1 := -k$, $A_2 := \mathrm{i}s$, and $A_3 := k$. The other two vertices are given by $z = \pm\infty + \mathrm{i}0$. Upon mapping $-\infty + \mathrm{i}0 \mapsto -\infty + \mathrm{i}0$, $A_1 \mapsto -1$, $A_2 \mapsto 0$, symmetry considerations yield that $A_3 \mapsto 1$ and $+\infty + \mathrm{i}0 \mapsto +\infty + \mathrm{i}0$. The interior angle at A_1 and A_3 is πa, and that at A_2 is $\pi(1-2a)$. Hence, the *exterior* angles are given by $\pi(1-a)$ and $\pi(1+2a)$, respectively. One sees that equation (9.9) becomes

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \gamma(z+1)^{-a}z^{2a}(z-1)^{-a} = \tilde{\gamma}\frac{z^{2a}}{(1-z^2)^a}.$$

Integration yields

$$w = \tilde{\gamma}\int_0^z \frac{\zeta^{2a}}{(1-\zeta^2)^a}\,\mathrm{d}\zeta + c.$$

Since $A_2 \mapsto 0$ and $A_3 \mapsto 1$ one has that

$$c = \mathrm{i}s, \quad k = \tilde{\gamma}\int_0^1 \frac{\zeta^{2a}}{(1-\zeta^2)^a}\,\mathrm{d}\zeta + \mathrm{i}s.$$

Now, by setting $t := \zeta^2$ one can rewrite the expression for k as

$$k = \frac{1}{2}\tilde{\gamma}\int_0^1 t^{a-1/2}(1-t)^{-a}\,\mathrm{d}t + \mathrm{i}s.$$

Upon setting $p := a + 1/2$ and $q := 1 - a$ one then gets

$$k = \frac{1}{2}\tilde{\gamma}B(p,q) + \mathrm{i}s,$$

where

$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}\,\mathrm{d}t$$

is the *beta function*. It can be shown [**8**, Proposition 15.1.13] that

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},$$

so that with the above definitions for p and q,

$$B(p,q) = \frac{\Gamma(a+1/2)\Gamma(1-a)}{\Gamma(3/2)}.$$

Since

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi},$$

one then has that

$$\tilde{\gamma} = \frac{(k-\mathrm{i}s)\sqrt{\pi}}{\Gamma(a+1/2)\Gamma(1-a)}.$$

Hence, the transformation is given by

$$w = \frac{(k-\mathrm{i}s)\sqrt{\pi}}{\Gamma(a+1/2)\Gamma(1-a)}\int_0^z \frac{\zeta^{2a}}{(1-\zeta^2)^a}\,\mathrm{d}\zeta + \mathrm{i}s.$$

An interesting limit is $k \to 0^+$, i.e., $a \to 1/2$. This domain corresponds to the exterior of a "slit". In this case the transformation becomes

$$\begin{aligned} w &= \mathrm{i}s\left(1 - \int_0^z \frac{\zeta}{(1-\zeta^2)^{1/2}}\,\mathrm{d}\zeta\right) \\ &= s\sqrt{z^2-1}. \end{aligned}$$

The interested reader should consult [**1**, Example 5.6.5] for a direct calculation.

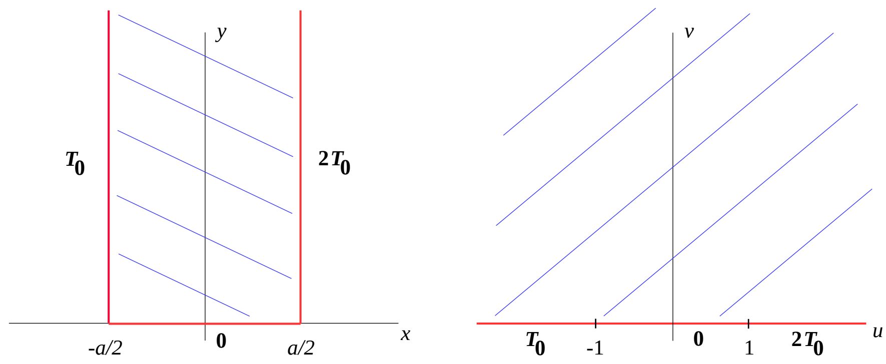


Figure 10: The boundary conditions on the two different domains.

9.6.1. Application: Heat flow

Let $f : D \mapsto D'$ be a conformal map, and suppose that $\Phi(u, v) : D' \mapsto \mathbb{R}$ is harmonic, i.e., it satisfies Laplace's equation

$$\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\Phi = 0.$$

Since f is conformal and

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = |f'(z)|^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right),$$

one then has that $\Phi(u(x, y), v(x, y)) : D \mapsto \mathbb{R}$ is harmonic. This simple fact allows one to first solve Laplace's equation on the upper-half plane, and then use a conformal map to solve it on a more complicated domain.

Consider the following example (see [**1**, Chapter 5.4] for more examples). If T denotes the temperature of a solid, it can be demonstrated that T satisfies Laplace's equation. As discussed in Section 1.5, it is then known that there exists a holomorphic Ω, known as the *complex temperature*, such that $T = \mathrm{Re}\,\Omega$.

Suppose that a semi-infinite slab has its vertical boundaries maintained at temperatures T_0 and $2T_0$, while its horizontal boundary is maintained at a temperature of zero (see Figure 10). We will determine the steady-state temperature within the slab. It is known that the transformation

$$\begin{aligned} f(z) &:= \sin\left(\frac{\pi z}{a}\right) \\ &= \sin\left(\frac{\pi x}{a}\right)\cosh\left(\frac{\pi y}{a}\right) + \mathrm{i}\cos\left(\frac{\pi x}{a}\right)\sinh\left(\frac{\pi y}{a}\right) \end{aligned} \tag{9.11}$$

is a conformal map of the semi-infinite slab to the upper-half plane U (see the discussion associated with Figure 8). Following the above idea, we will first solve Laplace's equation on U, and then use the conformal map to solve it on the original domain. Set

$$F(w) := a_1\ln(w + 1) + a\ln(w - 1) + \mathrm{i}a_3,$$

where $a_j \in \mathbb{R}$ for $j = 1, \ldots, 3$, and

$$w + 1 = r_1 e^{\mathrm{i}\vartheta_1},\ \ w - 1 = r_2 e^{\mathrm{i}\vartheta_2};\quad 0 \le \vartheta_1, \vartheta_2 \le \pi.$$

F is holomorphic on U; hence,

$$\begin{aligned} T &:= \mathrm{Im}\,F \\ &= a_1\vartheta_1 + a_2\vartheta_2 + a_3 \end{aligned}$$

is harmonic on U. The boundary conditions are such that if $\vartheta_1 = \vartheta_2 = 0$, then $T = 2T_0$, whereas if $\vartheta_1 = 0, \vartheta_2 = \pi$, then $T = 0$, and if $\vartheta_1 = \vartheta_2 = \pi$, then $T = T_0$. The solution is then given by

$$\begin{aligned} T &= \frac{T_0}{\pi}\vartheta_1 - \frac{2T_0}{\pi}\vartheta_2 + 2T_0 \\ &= \frac{T_0}{\pi}\tan^{-1}\frac{v}{u+1} - \frac{2T_0}{\pi}\tan^{-1}\frac{v}{u-1} + 2T_0. \end{aligned}$$

Using equation (9.11) with $u = \mathrm{Re} f$ and $v = \mathrm{Im} f$ yields the final result.

9.7. The Jacobian "sn" function

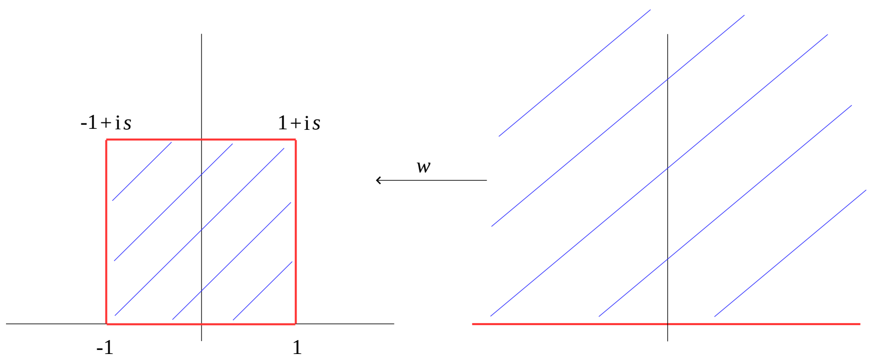


Figure 11: The mapping of the upper half plane to the interior of a rectangle.

As a final example, consider the rectangle with vertices ± 1 and $\pm 1 + \mathrm{i}s$, $s \in \mathbb{R}^+$ (see Figure 11). Let $k \in (0, 1)$ be given. If one maps

$$-1 + \mathrm{i}s \mapsto -\frac{1}{k}, \quad -1 \mapsto -1, \quad 0 \mapsto 0,$$

then by symmetry one has

$$1 \mapsto 1, \quad 1 + \mathrm{i}s \mapsto \frac{1}{k}.$$

Since the internal angles are all $\pi/2$, in this case equation (9.9) becomes

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \gamma(z^2 - 1)^{-1/2}(z^2 - 1/k^2)^{-1/2}.$$

Upon setting

$$F(z,k) := \int_0^z \frac{1}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}\,\mathrm{d}\zeta, \tag{9.12}$$

and using the fact that $0 \mapsto 0$, one has that the transformation is given by

$$w = \tilde{\gamma}F(z,k).$$

Set

$$K(k) := F(1,k) = \int_0^1 \frac{1}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}\,\mathrm{d}\zeta. \tag{9.13}$$

The function $K(k)$ is known as the *complete elliptic integral of the first kind*, and the parameter k is the *modulus of the elliptic integral*. It is straightforward to check that $K(k)$ is monotone increasing with

$$\lim_{k\to 0^+} K(k) = \frac{\pi}{2}, \quad \lim_{k\to 1^-} K(k) = +\infty.$$

Since $1 \mapsto 1$, one has that $\tilde{\gamma} = 1/K(k)$, so that the conformal map is given by

$$w = \frac{F(z,k)}{K(k)}.$$

The relationship between s and k is made explicit via

$$\begin{aligned} 1 + \mathrm{i}s &= \frac{F(1/k,k)}{K(k)} \\ &= 1 + \mathrm{i}\frac{K'(k)}{K(k)}, \end{aligned}$$

where

$$K'(k) := \int_1^{1/k} \frac{1}{\sqrt{(\zeta^2-1)(1-k^2\zeta^2)}}\,\mathrm{d}\zeta.$$

The change of variables $\zeta := (1-k'^2t^2)^{-1/2}$, where $k'^2 := 1-k^2$, yields

$$K'(k) = F(1,k') = K(\sqrt{1-k^2}).$$

In conclusion, the conformal map is given by

$$w = \frac{F(z,k)}{K(k)}, \quad s = \frac{K'(k)}{K(k)}.$$

If one defines the rectangle to have vertices $\pm K(k)$ and $\pm K(k) + \mathrm{i}K'(k)$, then the conformal map is given by $w = F(z,k)$. The Jacobi "sn" function is defined to be

$$\mathrm{sn}(w,k) := F^{-1}(w,k), \tag{9.14}$$

and it is a conformal map from the aforementioned rectangle to the upper half of the complex plane. By definition one has that $\mathrm{sn}(0,k) = 0$ for all $k \in (0,1)$; furthermore, upon using the fact that

$$\frac{\mathrm{d}}{\mathrm{d}w}\,\mathrm{sn}(0,k) = \frac{1}{F'(0,k)}$$

one gets that

$$\frac{\mathrm{d}}{\mathrm{d}z}\,\mathrm{sn}(0,k) = 1.$$

Hence, the zero is simple. Further note that as $k \to 0^+$, i.e., $s \to +\infty$, one has that the rectangle becomes an infinite strip with vertices $\pm K(0)$. Upon referring to the example associated with Figure 8, and noting the mapping of the vertices, one then sees that

$$\lim_{k\to 0^+} \mathrm{sn}(w,k) = \sin(w).$$

Hence, in this limit the Jacobi function is periodic. A more detailed examination of the Jacobi elliptic functions will be undertaken in Section 10.

10. Elliptic Functions and Applications

The primary source for the material presented in this section is [**9**]. Much of what will be presented herein is a supplement to [**8**, Chapter 10.6].

10.1. Theta functions

10.1.1. Definitions, periodicity properties, and identities

Consider the initial-boundary value problem

$$\vartheta_t = \kappa\vartheta_{zz}; \quad \vartheta(0,t) = \vartheta(\pi,t) = 0, \ \vartheta(z,0) = f(z). \tag{10.1}$$

Using the method of separation of variables, one sees that the solution is given by

$$\vartheta(z,t) = \sum_{n=1}^{\infty} b_n \mathrm{e}^{-\kappa n^2 t} \sin nz, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(z)\sin nz\,\mathrm{d}z.$$

If one considers the initial value $f(z) = \pi\delta(z - \pi/2)$, then one has that $b_n = 2\sin(n\pi/2)$, which in turn yields

$$\vartheta(z,t) = 2\sum_{n=1}^{\infty}(-1)^n \mathrm{e}^{-\kappa(2n+1)^2 t}\sin(2n+1)z.$$

Upon setting $q := \mathrm{e}^{-4\kappa t}$ (note that $|q| \le 1$), one then has that

$$\begin{aligned}\vartheta(z,t) = \vartheta_1(z,q) &:= 2\sum_{n=1}^{\infty}(-1)^n q^{(n+1/2)^2}\sin(2n+1)z \\ &= -\mathrm{i}\sum_{n=-\infty}^{+\infty}(-1)^n q^{(n+1/2)^2}\mathrm{e}^{\mathrm{i}(2n+1)z}.\end{aligned} \tag{10.2}$$

Note that $\vartheta_1(z+2\pi,q) = \vartheta_1(z,q)$, and that $\vartheta_1(z,q)$ is odd in z; furthermore, it has zeros at $z = m\pi$ for $m \in \mathbb{Z}$. Equation (10.2) defines the *first theta function*. The *second theta function* is given by

$$\begin{aligned}\vartheta_2(z,q) &:= \vartheta_1(z+\pi/2,q) \\ &= \sum_{n=-\infty}^{+\infty} q^{(n+1/2)^2}\mathrm{e}^{\mathrm{i}(2n+1)z}.\end{aligned} \tag{10.3}$$

Note that $\vartheta_2(z,q)$ is also 2π-periodic, and is even in z.

If one now considers equation (10.1) with the boundary conditions $\vartheta_z(0,t) = \vartheta_z(\pi,t) = 0$, then with $f(z) = \pi\delta(z-\pi/2)$ one gets the solution $\vartheta(z,t) = \vartheta_4(z,q)$, where

$$\begin{aligned}\vartheta_4(z,q) &:= 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nz \\ &= \sum_{n=-\infty}^{+\infty}(-1)^n q^{n^2}\mathrm{e}^{\mathrm{i}2nz}.\end{aligned} \tag{10.4}$$

Equation (10.4) defines the *fourth theta function*. Note that $\vartheta_4(z,q)$ is π-periodic, and is even in z. Finally, one gets the *third theta function* via

$$\begin{aligned}\vartheta_3(z,q) &:= \vartheta_4(z+\pi/2,q) \\ &= \sum_{n=-\infty}^{+\infty} q^{n^2}\mathrm{e}^{\mathrm{i}2nz}.\end{aligned} \tag{10.5}$$

Note that $\vartheta_3(z,q)$ is π-periodic, and is even in z.

As it will be seen later, the theta functions will be the building blocks for the elliptic functions. Consequently, their properties must be understood. Let us now show that the theta functions are entire. It will be sufficient to demonstrate this fact for $\vartheta_1(z,q)$. Let $Y > 0$ be given, and consider the series

$$\sum_{n=0}^{+\infty}(-1)^n q^{(n+1/2)^2}\mathrm{e}^{\mathrm{i}(2n+1)z}, \quad z \in \Omega_Y := \{z \in \mathbb{C} : |\operatorname{Im} z| \le Y\}.$$

One has the estimate

$$|q^{(n+1/2)^2}e^{i(2n+1)z}| \le |q|^{(n+1/2)^2}e^{-(2n+1)y} \le |q|^{(n+1/2)^2}e^{(2n+1)Y},$$

so as a consequence of the ratio test the series is uniformly convergent on Ω_Y. Hence, the sum converges to a holomorphic function on Ω_Y, and since $Y > 0$ is arbitrary, the sum is entire. A similar argument shows that the sum

$$\sum_{n=-\infty}^{-1}(-1)^n q^{(n+1/2)^2}e^{i(2n+1)z}$$

has the same properties; hence, $\vartheta_1(z,q)$ is entire.

Remark 10.1. It is interesting to note that the theta functions are entire in z for fixed q, and if z is fixed, then they are holomorphic for $q \in D(0,1)$.

Now let us locate the zeros of the theta functions. By construction one clearly has that $\vartheta_1(z,q)$ has zeros at $z = m\pi$ for $m \in \mathbb{Z}$. Does it have any others? Define τ via

$$\tau = -\frac{i}{\pi}\ln q, \quad -\pi < \arg q \le \pi, \tag{10.6}$$

so that

$$\vartheta_1(z|\tau) = -i\sum_{n=-\infty}^{+\infty}(-1)^n e^{i(n+1/2)^2\pi\tau}e^{i(2n+1)z}.$$

Simple algebraic manipulation then yields

$$\begin{aligned}\vartheta_1(z+\pi\tau|\tau) &= ie^{-i\pi\tau-2iz}\sum_{n=-\infty}^{+\infty}(-1)^{n+1}e^{i(n+3/2)^2\pi\tau}e^{i(2n+3)z}\\ &= -(qe^{2iz})^{-1}\vartheta_1(z|\tau).\end{aligned}$$

Setting $\hbar := qe^{2iz}$, one then gets that

$$\vartheta_1(z,q) = -\hbar\vartheta_1(z+\pi\tau,q). \tag{10.7}$$

Similarly, it can be shown that

$$\vartheta_2(z,q) = \hbar\vartheta_2(z+\pi\tau,q), \quad \vartheta_3(z,q) = \hbar\vartheta_3(z+\pi\tau,q), \quad \vartheta_4(z,q) = -\hbar\vartheta_4(z+\pi\tau,q). \tag{10.8}$$

One consequence of equation (10.7) is that $\vartheta_1(z,q) = 0$ for $z = m\pi + n\pi\tau,\ m,n \in \mathbb{Z}$ (see Figure 12), while as a consequence of equation (10.8) one has that for $m,n \in \mathbb{Z}$,

$$\vartheta_2(z,q) = 0, \quad z = (m+\frac{1}{2})\pi + n\pi\tau.$$

If one now considers equation (10.4), then a series of simple algebraic manipulations yields that

$$\vartheta_4(\frac{1}{2}\pi\tau,q) = -\vartheta_4(-\frac{1}{2}\pi\tau,q).$$

Since ϑ_4 is even in z, this necessarily yields that $\vartheta_4(\pi\tau/2,q) = 0$. Similarly, one finds that $\vartheta_3(\pi(1+\tau)/2,q) = 0$. As a consequence of equation (10.8) one then sees that

$$\begin{aligned}&\vartheta_3(z,q) = 0, \quad z = (m+\frac{1}{2})\pi + (n+\frac{1}{2})\pi\tau\\ &\vartheta_4(z,q) = 0, \quad z = m\pi + (n+\frac{1}{2})\pi\tau.\end{aligned} \tag{10.9}$$

Remark 10.2. Note that if $q \in (0,1)$, then by equation (10.6) $\tau \in i\mathbb{R}^+$; thus, in this case the zeros are the vertices of an infinite rectangular lattice in the complex plane (see Figure 12). If $\operatorname{Im} q \neq 0$, then the vertices will be on parallelograms.

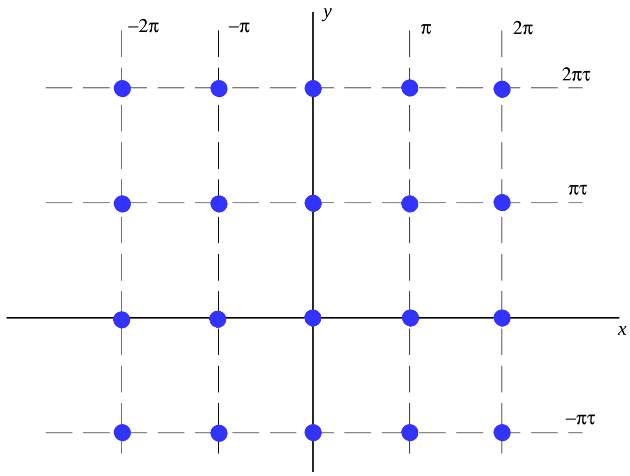


Figure 12: The zeros of the first theta function $\vartheta_1(z|\tau)$ for $\tau \in \mathrm{i}\mathbb{R}^+$.

Finally, let us derive one of the many important properties of the theta functions (see [9, Chapter 1.4] for many more). Since $\vartheta_1(z,q)$ is odd in z, one has that $\vartheta_1(0,q)=0$. Upon differentiating term-by-term one sees that

$$\vartheta_1'(0,q) = 2q^{1/4} + \mathcal{O}(|q|^{9/4}).$$

A simple evaluation yields

$$\vartheta_2(0,q) = 2q^{1/4} + \mathcal{O}(|q|^{9/4}),\quad \vartheta_3(0,q) = 1 + \mathcal{O}(|q|),\quad \vartheta_4(0,q) = 1 + \mathcal{O}(|q|).$$

Hence, one sees that

$$\lim_{q\to 0} \frac{\vartheta_1'(0,q)}{\vartheta_2(0,q)\vartheta_3(0,q)\vartheta_4(0,q)} = 1.$$

Now, it can be shown [9, Chapter 1.5] that

$$\frac{\vartheta_1'(0,q)}{\vartheta_2(0,q)\vartheta_3(0,q)\vartheta_4(0,q)} = \lim_{q\to 0} \frac{\vartheta_1'(0,q)}{\vartheta_2(0,q)\vartheta_3(0,q)\vartheta_4(0,q)}$$

for any $q \in (-1,1)$. This yields the identity

$$\vartheta_1'(0,q) = \vartheta_2(0,q)\vartheta_3(0,q)\vartheta_4(0,q). \tag{10.10}$$

10.1.2. Theta functions as infinite products

From this point forward the dependence of the theta functions on q will be implicit. Since the theta functions are entire with an infinite number of zeros, they can be represented as an infinite product. The goal of this section is to make this representation explicit. Consider

$$F(t,q) \coloneqq \prod_{n=1}^{\infty} (1+q^{2n-1}t)(1+q^{2n-1}t^{-1}),$$

where $t \in \mathbb{C}\backslash\{0\}$ and $q \in D(0,1)$. For $r \le |t| \le R$ and $q \in \overline{D}(0,\rho)$ one has that

$$\begin{aligned}|q^{2n-1}(t+t^{-1}) + q^{4n-2}| &\le |q|^{2n-1}(|t|+|t|^{-1}) + |q|^{4n-2}\\ &\le \rho^{2n-1}(R+r^{-1}) + \rho^{4n-2};\end{aligned}$$

hence, by Theorem 7.10 one has that the infinite product converges uniformly for $r \le |t| \le R$ and $q \in \overline{D}(0,\rho)$, so that it is holomorphic on $\mathbb{C}\backslash\{0\} \times D(0,1)$. Note that its zeros are given by $t = -q^{\pm(2n-1)}$ for $n \in \mathbb{N}$.

Since $F(t,q)$ is holomorphic on $\mathbb{C}\backslash\{0\}$, it can be written as the Laurent series

$$F(t,q) = \sum_{n=-\infty}^{+\infty} a_n t^n.$$

Upon noting that

$$\begin{aligned} F(q^2t,q) &= \prod_{n=1}^{\infty}(1+q^{2n+1}t)(1+q^{2n-3}t^{-1}) \\ &= \frac{1+(qt)^{-1}}{1+qt}F(t,q) \\ &= \frac{1}{qt}F(t,q), \end{aligned}$$

one gets that

$$\begin{aligned} \sum_{n=-\infty}^{+\infty} a_n t^n &= qt \sum_{n=-\infty}^{+\infty} a_n q^{2n} t^n \\ &= \sum_{n=-\infty}^{+\infty} a_{n-1} q^{2n-1} t^n. \end{aligned}$$

Equating coefficients for $n \in \mathbb{N}$ then yields the recursion relationship

$$a_n = q^{2n-1} a_{n-1},$$

i.e.,

$$a_n = q^{n^2} a_0, \quad n \in \mathbb{N}.$$

In this last step one uses the fact that

$$\sum_{j=1}^{n}(2j-1) = n^2.$$

Since $F(t,q) = F(t^{-1},q)$, one has that $a_n = a_{-n}$. Hence, one finally sees that

$$F(t,q) = a_0 \sum_{n=-\infty}^{+\infty} q^{n^2} t^n, \quad a_0 \in \mathbb{C}\backslash\{0\}.$$

Upon referencing equation (10.5) one immediately sees that

$$F(\mathrm{e}^{2\mathrm{i}z},q) = a_0\vartheta_3(z,q), \quad F(-\mathrm{e}^{2\mathrm{i}z},q) = a_0\vartheta_4(z,q); \tag{10.11}$$

in other words,

$$\begin{aligned} \vartheta_3(z,q) &= \frac{1}{a_0}\prod_{n=1}^{\infty}(1+q^{2n-1}\mathrm{e}^{2\mathrm{i}z})(1+q^{2n-1}\mathrm{e}^{-2\mathrm{i}z}) \\ \vartheta_4(z,q) &= \frac{1}{a_0}\prod_{n=1}^{\infty}(1-q^{2n-1}\mathrm{e}^{2\mathrm{i}z})(1-q^{2n-1}\mathrm{e}^{-2\mathrm{i}z}). \end{aligned} \tag{10.12}$$

It is easy to see that the other two theta functions can be written in terms of $F(\pm q\mathrm{e}^{2\mathrm{i}z},q)$, i.e.,

$$F(q\mathrm{e}^{2\mathrm{i}z},q) = \frac{a_0}{q^{1/4}}\mathrm{e}^{-\mathrm{i}z}\vartheta_2(z,q), \quad F(-q\mathrm{e}^{2\mathrm{i}z},q) = \mathrm{i}\frac{a_0}{q^{1/4}}\mathrm{e}^{-\mathrm{i}z}\vartheta_1(z,q);$$

in other words,

$$\begin{aligned}\vartheta_1(z,q) &= -\mathrm{i}\frac{1}{a_0}q^{1/4}\mathrm{e}^{\mathrm{i}z}\prod_{n=1}^{\infty}(1-q^{2n}\mathrm{e}^{2\mathrm{i}z})(1-q^{2n}\mathrm{e}^{-2\mathrm{i}z})\\ \vartheta_2(z,q) &= \frac{1}{a_0}q^{1/4}\mathrm{e}^{\mathrm{i}z}\prod_{n=1}^{\infty}(1+q^{2n}\mathrm{e}^{2\mathrm{i}z})(1+q^{2n}\mathrm{e}^{-2\mathrm{i}z}).\end{aligned}\tag{10.13}$$

Upon using the fact that ϑ_1 is odd and ϑ_2 is even in z, one finally finds that equation (10.13) can be rewritten as

$$\begin{aligned}\vartheta_1(z,q) &= \frac{2}{a_0}q^{1/4}\sin z\prod_{n=1}^{\infty}(1-q^{2n}\mathrm{e}^{2\mathrm{i}z})(1-q^{2n}\mathrm{e}^{-2\mathrm{i}z})\\ \vartheta_2(z,q) &= \frac{2}{a_0}q^{1/4}\cos z\prod_{n=1}^{\infty}(1+q^{2n}\mathrm{e}^{2\mathrm{i}z})(1+q^{2n}\mathrm{e}^{-2\mathrm{i}z}).\end{aligned}\tag{10.14}$$

It now remains to compute the constant a_0. Note that

$$F(\pm 1,q) = \prod_{n=1}^{\infty}(1\pm q^{2n-1})^2,$$

so that

$$a_0\vartheta_3(0,q) = \prod_{n=1}^{\infty}(1+q^{2n-1})^2,\quad a_0\vartheta_4(0,q) = \prod_{n=1}^{\infty}(1-q^{2n-1})^2.$$

Similar expressions relating $a_0\vartheta_1'(0,q)$ and $a_0\vartheta_2(0,q)$ with $F(\pm q,q)$ can be found. Upon using equation (10.10) it is eventually seen that

$$a_0 = 1/\prod_{n=1}^{\infty}(1-q^{2n}).$$

As a consequence, one can now write equation (10.11) as

$$\vartheta_3(z,q) = \prod_{n=1}^{\infty}(1-q^{2n})(1+2q^{2n-1}\cos 2z+q^{4n-2}),\quad \vartheta_4(z,q) = \prod_{n=1}^{\infty}(1-q^{2n})(1-2q^{2n-1}\cos 2z+q^{4n-2}).$$

Similarly,

$$\vartheta_1(z,q) = 2q^{1/4}\sin z\prod_{n=1}^{\infty}(1-q^{2n})(1-2q^{2n}\cos 2z+q^{4n}),\quad \vartheta_2(z,q) = 2q^{1/4}\cos z\prod_{n=1}^{\infty}(1-q^{2n})(1+2q^{2n}\cos 2z+q^{4n}).$$

Note that the above representations yield that all of the zeros of the theta functions are precisely captured in equation (10.9). Furthermore, note that for fixed $z\in\mathbb{C}$ that the zeros of the theta functions are dense on $\partial D(0,1)$; hence, these functions cannot be analytically extended across the boundary.

10.2. Jacobi's elliptic functions

Set

$$\begin{aligned}k &:= \frac{\vartheta_2(0)^2}{\vartheta_3(0)^2}\\ &= 4q^{1/2}\prod_{n=1}^{\infty}\left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^4,\end{aligned}\tag{10.15}$$

and

$$\begin{aligned} k' &:= \frac{\vartheta_4(0)^2}{\vartheta_3(0)^2} \\ &= \prod_{n=1}^{\infty}\left(\frac{1-q^{2n-1}}{1+q^{2n-1}}\right)^4, \end{aligned} \tag{10.16}$$

Note that by applying equation (10.12) and equation (10.14), the above definitions are valid even without the knowledge of a_0. As a consequence of the identity

$$\vartheta_3(0)^4 = \vartheta_2(0)^4 + \vartheta_4(0)^4 \tag{10.17}$$

one has that

$$k^2 + k'^2 = 1. \tag{10.18}$$

For $q \in (0,1)$ each factor associated with k' is decreasing, so that k' is uniquely defined by q. Hence, given $k \in (0,1)$, one uniquely defines $k' \in (0,1)$ via equation (10.18), and then consequently uniquely defines $q \in (0,1)$ via an inversion of equation (10.16).

Recalling equation (10.6), set

$$\begin{aligned} K &:= \frac{1}{2}\pi\vartheta_3(0)^2 \\ &= \frac{1}{2}\pi\prod_{n=1}^{\infty}(1-q^{2n})^2(1+q^{2n-1})^4, \end{aligned} \tag{10.19}$$

and

$$\begin{aligned} K' &:= -\mathrm{i}\tau K \\ &= -\frac{1}{2}\ln q\prod_{n=1}^{\infty}(1-q^{2n})^2(1+q^{2n-1})^4. \end{aligned} \tag{10.20}$$

Following the argument given above, both of these functions can be thought of as functions of $k \in (0,1)$. Note that by definition,

$$q = \mathrm{e}^{-\pi K/K'}.$$

10.2.1. Definition of Jacobi's elliptic functions

For

$$\begin{aligned} u &:= \vartheta_3(0)^2 z \\ &= \left(\prod_{n=1}^{\infty}(1-q^{2n})^2(1+q^{2n-1})^4\right) z \end{aligned} \tag{10.21}$$

define the elliptic functions

$$\operatorname{sn} u := \sqrt{\frac{1}{k}}\,\frac{\vartheta_1(z)}{\vartheta_4(z)}, \quad \operatorname{cn} u := \sqrt{\frac{k'}{k}}\,\frac{\vartheta_2(z)}{\vartheta_4(z)}, \quad \operatorname{dn} u := \sqrt{k'}\,\frac{\vartheta_3(z)}{\vartheta_4(z)} \tag{10.22}$$

Upon squaring and using the identity

$$\vartheta_4^2 = \frac{1}{k}\vartheta_1^2 + \frac{k'}{k}\vartheta_2^2,$$

one obtains

$$\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1. \tag{10.23}$$

Upon squaring and using the identities

$$\vartheta_4^2 = k\vartheta_1^2 + k'\vartheta_3^2, \quad \vartheta_4^2 = \frac{1}{k'}\vartheta_3^2 - \frac{k}{k'}\vartheta_2^2,$$

one similarly sees that

$$\operatorname{dn}^2 u + k^2\operatorname{sn}^2 u = 1, \quad \operatorname{dn}^2 u - k^2\operatorname{cn}^2 u = k'^2. \tag{10.24}$$

It is an exercise to see the form of these identities in the limit $k \to 0^+$.

10.2.2. Double periodicity of Jacobi's elliptic functions

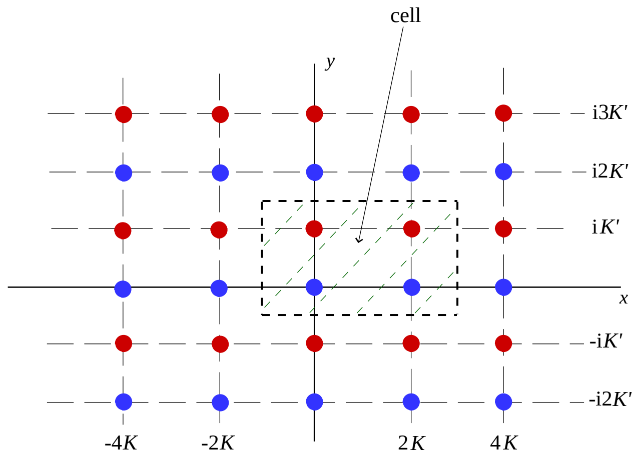


Figure 13: The zeros (blue) and poles (red) of $\operatorname{sn} u$ for $k \in (0,1)$. The "cell" and its properties are discussed in more detail in Section 10.3.

As a consequence of equation (10.9), all of the elliptic functions will have simple poles when $\vartheta_4(z) = 0$, i.e., for $m, n \in \mathbb{Z}$, when

$$\begin{aligned} u &= m\vartheta_3(0)^2\pi + (n + \frac{1}{2})\vartheta_3(0)^2\pi\tau \\ &= 2mK + \mathrm{i}(2n+1)K'. \end{aligned} \tag{10.25}$$

The zeros are found by setting $\vartheta_j(z) = 0$ for $j = 1, \dots, 3$. Upon doing so it is seen that

$$\begin{aligned} \operatorname{sn} u = 0, &\quad u = 2mK + \mathrm{i}2nK' \\ \operatorname{cn} u = 0, &\quad u = (2m+1)K + \mathrm{i}2nK' \\ \operatorname{dn} u = 0, &\quad u = (2m+1)K + \mathrm{i}(2n+1)K' \end{aligned} \tag{10.26}$$

(see Figure 13). Finally, upon using the identities in equation (10.7) and equation (10.8) it is seen that

$$\operatorname{sn} u = \operatorname{sn}(u + 2\pi\vartheta_3(0)^2), \quad \operatorname{sn} u = \operatorname{sn}(u + \pi\tau\vartheta_3(0)^2),$$

i.e.,

$$\operatorname{sn} u = \operatorname{sn}(u + 4K), \quad \operatorname{sn} u = \operatorname{sn}(u + \mathrm{i}2K'). \tag{10.27}$$

In a similar fashion,

$$\operatorname{cn} u = \operatorname{cn}(u + 4K) = \operatorname{cn}(u + 2K + \mathrm{i}2K'); \quad \operatorname{dn} u = \operatorname{dn}(u + 2K) = \operatorname{dn}(u + \mathrm{i}4K') \tag{10.28}$$

In the above, recall that if $q \in (0,1)$, then $K, K' \in \mathbb{R}^+$.

Definition 10.3. An elliptic function is any doubly periodic function in which the ratio of the periods is nonreal, and which is holomorphic except for poles.

Remark 10.4. One has that:

(a) If a function is entire and doubly periodic, then it is uniformly bounded, and hence constant.

(b) For $u \in \mathbb{R}$ the graphs of $\operatorname{sn} u$ and $\operatorname{cn} u$ are similar to those of $\sin u$ and $\cos u$, respectively, except that the zeros are now located at $u = 2mK$ and $u = (2m+1)K$, respectively. This property has been extensively exploited in the research articles [3–5], for example.

(c) It can be shown that for $k \in (0,1)$,

$$\lim_{k\to 0^+} \operatorname{sn} u = \sin u; \quad \lim_{k\to 0^+} \operatorname{cn} u = \cos u; \quad \lim_{k\to 0^+} \operatorname{dn} u = 1,$$

and

$$\lim_{k\to 1^-} \operatorname{sn} u = \tanh u; \quad \lim_{k\to 1^-} \operatorname{cn} u = \operatorname{sech} u; \quad \lim_{k\to 1^-} \operatorname{dn} u = \operatorname{sech} u.$$

Hence, the Jacobi elliptic functions can be thought of as a bridge between the trigonometric functions and the hyperbolic functions.

10.2.3. Derivatives of Jacobi's elliptic functions

As mentioned in class, and as is well-documented in [**9**, Chapters 4-5], the Jacobi elliptic functions frequently arise in applications as solutions to ordinary differential equations. It is an exercise to show that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\vartheta_1}{\vartheta_4}\right) = \vartheta_4(0)^2\frac{\vartheta_2\vartheta_3}{\vartheta_4^2}, \quad \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\vartheta_2}{\vartheta_4}\right) = -\vartheta_3(0)^2\frac{\vartheta_1\vartheta_3}{\vartheta_4^2}, \quad \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\vartheta_3}{\vartheta_4}\right) = -\vartheta_2(0)^2\frac{\vartheta_1\vartheta_2}{\vartheta_4^2}.$$

Upon using equation (10.21) and equation (10.22), one then sees that

$$\frac{\mathrm{d}}{\mathrm{d}u}\operatorname{sn} u = \operatorname{cn} u \operatorname{dn} u, \quad \frac{\mathrm{d}}{\mathrm{d}u}\operatorname{cn} u = -\operatorname{sn} u \operatorname{dn} u, \quad \frac{\mathrm{d}}{\mathrm{d}u}\operatorname{dn} u = -k^2 \operatorname{sn} u \operatorname{cn} u. \tag{10.29}$$

The identities in equation (10.23) and equation (10.24) can now be easily derived. Set $w_1 := \operatorname{sn} u$, $w_2 := \operatorname{cn} u$, and $w_3 := \operatorname{dn} u$. Equation (10.29) can then be rewritten as

$$w_1' = w_2 w_3, \ w_1(0) = 0; \quad w_2' = -w_1 w_3, \ w_2(0) = 1; \quad w_3' = -k^2 w_1 w_2, \ w_3(0) = 1.$$

Multiplying the first equation by w_1, the second by w_2, and adding yields

$$\frac{\mathrm{d}}{\mathrm{d}u}(w_1^2 + w_2^2) = 0.$$

The initial conditions then reveals that

$$w_1^2 + w_2^2 = 1,$$

which is simply equation (10.23). Similarly, one sees that

$$k^2 w_1^2 + w_3^2 = 1,$$

which is the first equation in equation (10.24). Note that these equations then imply that

$$\begin{aligned}(w_1')^2 &= w_2^2 w_3^2 \\ &= (1 - w_1^2)(1 - k^2 w_1^2),\end{aligned} \tag{10.30}$$

so that

$$u = \int_0^{w_1} \frac{1}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}\,\mathrm{d}\zeta;$$

in other words,

$$\operatorname{sn}^{-1} u = \int_0^{u} \frac{1}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}\,\mathrm{d}\zeta.$$

Comparison of the above with equation (9.14) shows the intimate relationship between the conformal map presented in Section 9.7 and the theory developed within this section. Furthermore, since as a consequence of equation (9.14) one has that $\operatorname{sn} K = 1$ (also see [**9**, equation (2.2.20)]), one recovers the result of equation (9.13), i.e.,

$$K = \int_0^{1} \frac{1}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}\,\mathrm{d}\zeta.$$

Remark 10.5. Upon making the substitutions

$$y := w_1^2 - \frac{1+k^2}{3k^2}, \quad x := ku,$$

equation (10.30) can be written as

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 4y^3 - g_2 y - g_3, \tag{10.31}$$

for appropriate constants g_2 and g_3. This ODE is important when considering the Weierstrass elliptic function (see Section 10.4).

What other differential equations are satisfied by the Jacobi elliptic functions? Differentiation of the first equation in equation (10.29) yields

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2}\,\mathrm{sn}\,u = -\,\mathrm{sn}\,u\,\mathrm{dn}^2\,u - k^2\,\mathrm{sn}\,u\,\mathrm{cn}^2\,u,$$

which upon the manipulation of the identities in equation (10.24) yields that for $y := \mathrm{sn}\,u$,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}u^2} + (1+k^2)y - 2k^2y^3 = 0.$$

In a similar fashion one sees that $y := \mathrm{cn}\,u$ solves the ODE

$$\frac{\mathrm{d}^2 y}{\mathrm{d}u^2} + (1-2k^2)y + 2k^2y^3 = 0, \tag{10.32}$$

and for $y := \mathrm{dn}\,u$,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}u^2} + (k^2-2)y + 2y^3 = 0. \tag{10.33}$$

It is interesting to note the relationship between equation (10.32) and equation (10.33). Upon setting $\tilde{y} := ky$ equation (10.32) can be rewritten as

$$\frac{\mathrm{d}^2 \tilde{y}}{\mathrm{d}u^2} + (1-2k^2)\tilde{y} + 2\tilde{y}^3 = 0,$$

which now has the solution $k\,\mathrm{cn}(u)$. When considering the ODE

$$\frac{\mathrm{d}^2 y}{\mathrm{d}u^2} + \mu y + 2y^3 = 0, \tag{10.34}$$

one then has the solutions

$$k_1\,\mathrm{cn}(u;k_1), \quad \mathrm{dn}(u;k_2),$$

where

$$k_1^2 := \frac{1}{2}(1-\mu), \quad k_2^2 := 2+\mu.$$

Note that the restriction $0 < k_j < 1$ implies that for k_1 one requires $-1 < \mu < 1$, whereas for k_2 one requires $-2 < \mu < -1$. Hence, the solutions are complementary with respect to the parameter μ. Further note that the rescalings $u := ax$ and $w := ay$ changes equation (10.34) to

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + \tilde{\mu}w + 2w^3 = 0, \quad \tilde{\mu} := a^2\mu,$$

so without loss of generality one can consider the parameter μ in equation (10.34) to be arbitrary.

Remark 10.6. The nonlinear second-order ODE

$$\frac{\mathrm{d}^2 w}{\mathrm{d}u^2} = w^3 + ew^2 + fw, \quad e,f \in \mathbb{C}$$

can be transformed to equation (10.31). First multiply by w' and integrate to get

$$\left(\frac{\mathrm{d}w}{\mathrm{d}u}\right)^2 = \frac{1}{2}w^4 + \frac{2}{3}ew^3 + fw^2 + g, \quad g \in \mathbb{C},$$

and then set $w = B(y)$, where $B(\cdot)$ is an appropriate bilinear transformation.

10.3. General properties of elliptic functions

The periods of an elliptic function $f(u)$ are often denoted by $2\omega_1$ and $2\omega_3$, with $\omega_2 := -(\omega_1+\omega_3)$. A straightforward induction argument yields that $2m\omega_1 + 2n\omega_3$ is also a period for $m, n \in \mathbb{Z}$. It will be henceforth assumed that $2\omega_1$ and $2\omega_3$ are fundamental periods, i.e., no submultiple of either is a period. Furthermore, it shall be assumed that $2\omega_1$ and $2\omega_3$ are primitive periods, i.e., any other period is a sum of multiples of these. It will be henceforth assumed any pair of primitive periods will have the property that $\mathrm{Im}(\omega_3/\omega_1) > 0$.

Example. $\mathrm{cn}\, u$ has primitive periods $2\omega_1 = 4K$ and $2\omega_3 = 2K + 2\mathrm{i}K'$, while $\mathrm{sn}\, u$ has primitive periods $2\omega_1 = 4K$ and $2\omega_3 = 2\mathrm{i}K'$. Note that in each case

$$\mathrm{Im}\left(\frac{\omega_3}{\omega_1}\right) = \frac{1}{2}\frac{K'}{K} > 0.$$

Now suppose that new periods are constructed via

$$\omega_1' = a\omega_1 + b\omega_3, \quad \omega_3' = c\omega_1 + d\omega_3,$$

where $a, b, c, d \in \mathbb{Z}$ satisfy $ad - bc = 1$. Note that

$$\omega_1 = d\omega_1' - b\omega_3', \quad \omega_3 = -c\omega_1' + a\omega_3';$$

furthermore, if $\omega_3/\omega_1 = \alpha + \mathrm{i}\beta$ with $\beta \in \mathbb{R}^+$, then via a straightforward calculation

$$\mathrm{Im}\left(\frac{\omega_3'}{\omega_1'}\right) = \frac{\beta}{(a+b\alpha)^2 + b^2\beta^2} > 0.$$

Consequently, the new periods are also primitive periods. For example, when considering $\mathrm{cn}\, u$, new primitive periods are given by

$$\omega_1' = \omega_1 - \omega_3 = K - \mathrm{i}K', \quad \omega_3' = \omega_3 = K + \mathrm{i}K'.$$

In conclusion, one has that primitive periods are not unique.

Now set $\Omega_{m,n} := 2m\omega_1 + 2n\omega_3$ for $m, n \in \mathbb{Z}$, where ω_1 and ω_3 are primitive periods. A parallelogram with vertices $\Omega_{m,n}$, $\Omega_{m+1,n}$, $\Omega_{m+1,n+1}$, and $\Omega_{m,n+1}$ is called a period parallelogram. Since $\mathrm{Im}(\omega_3/\omega_1) > 0$, the vertices are being listed in the counterclockwise sense. It is clear that once the values of an elliptic function are known in one period parallelogram, then via periodicity they are known throughout $\mathbb{C}$. Given a point $a \in \mathbb{C}\backslash\{0\}$, one can translate the period parallelogram via

$$\Omega_{i,j} \mapsto a + \Omega_{i,j}.$$

This new parallelogram will be known as a cell. By the proper choice of a, the poles and zeros of $f(u)$ can be made to lie in the interior of the cell (see Figure 13). This will facilitate much of the later analysis.

Example. Consider $\mathrm{cn}\, u$ with

$$2\omega_1 = 4K, \quad 2\omega_3 = 2K + 2\mathrm{i}K'.$$

The "base" period parallelogram has vertices

$$\Omega_{0,0} = 0, \quad \Omega_{1,0} = 4K, \quad \Omega_{1,1} = 6K + 2\mathrm{i}K', \quad \Omega_{0,1} = 2K + 2\mathrm{i}K'.$$

The poles and zeros are given in equation (10.25) and equation (10.26). If one sets $a := -\mathrm{i}K'/3$, then all of the zeros and poles will be contained within the interior of the "base" cell.

The poles of the Jacobi elliptic functions are given in equation (10.25). Since the zeros of $\vartheta_4(z)$ are simple, one has that the poles are simple. It is natural then to wonder as to the residue associated with each pole. This question is specifically addressed in [9, Chapter 2.8]. It is a relatively straightforward, but tedious calculation. Instead, we will consider more general questions. For example, for a given elliptic function and given $a \in \mathbb{C}$, how many solutions exist to $f(u) = a$ for u in a given cell?

Definition 10.7. The order of an elliptic function is given by the number of poles within a particular cell.

Remark 10.8. The Jacobi elliptic functions are each of order two. Since each pole is simple, differentiation of a Jacobi elliptic function N times will yield an elliptic function of order $2N$. We will later consider in detail an elliptic function which is of order three.

Let $f(z)$ be an elliptic function $f(z)$ of order N, with the poles within a particular cell being denoted by $z = b_j$ for $j = 1, \dots, N$. Let Ω represent the interior of a cell, and let the vertices be denoted by $ABCD$. Consider the integral

$$\oint_{\partial\Omega} f(z)\,\mathrm{d}z.$$

Since the values of $f(z)$ are the same on both AB and DC, one has that

$$\begin{aligned}\oint_{AB} f(z)\,\mathrm{d}z &= \oint_{DC} f(z)\,\mathrm{d}z\\ &= -\oint_{CD} f(z)\,\mathrm{d}z.\end{aligned}$$

Similarly,

$$\oint_{BC} f(z)\,\mathrm{d}z = -\oint_{DA} f(z)\,\mathrm{d}z.$$

As a consequence, one has that

$$\oint_{\partial\Omega} f(z)\,\mathrm{d}z = 0,$$

which implies by the residue theorem that

$$\sum_{j=1}^{N} \mathrm{Res}_f(b_j) = 0. \tag{10.35}$$

As a consequence of the above argument, one has the following result.

Lemma 10.9. *An elliptic function cannot have only one simple pole within a cell.*

Proof: If there is only one simple pole at $z = b_j$ within a cell, then one clearly has that

$$\frac{1}{2\pi\mathrm{i}} \oint_{\partial\Omega} f(z)\,\mathrm{d}z = \mathrm{Res}_f(b_j) \neq 0.$$

However, this clearly violates the conclusion of equation (10.35). □

Now let $a \in \mathbb{C}$ be given, and consider $g(z) := f(z) - a$. As a consequence of the argument principal Lemma 5.5 one has that

$$\frac{1}{2\pi\mathrm{i}} \oint_{\partial\Omega} \frac{g'(z)}{g(z)}\,\mathrm{d}z = M - N,$$

where M is the number of zeros of $g(z)$ within Ω and N is the number of poles within Ω (counting multiplicities). Using a periodicity argument as above, one can first conclude that

$$\oint_{\partial\Omega} \frac{g'(z)}{g(z)}\,\mathrm{d}z = 0.$$

Hence, $M = N$, so that $f(z)$ takes the value a the order N times within a cell. In particular, the number of zeros within a cell is equal to the order of the function.

Lemma 10.10. *Let $f(z)$ be an elliptic function of order N, and let $a \in \mathbb{C}$ be given. There exist N solutions (counting multiplicity) to $f(z) = a$ in each cell.*

Now consider

$$h(z) := \frac{zf'(z)}{f(z)},$$

along with the integral

$$\frac{1}{2\pi\mathrm{i}} \oint_{\partial\Omega} h(z)\,\mathrm{d}z.$$

Let the zeros of $f(z)$ be denoted by a_j, and let each have multiplicity m_j, for $j = 1, \dots, q$. A simple calculation shows that

$$\mathrm{Res}_h(a_j) = m_j a_j.$$

If each of the poles b_ℓ have residues n_ℓ for $j = 1, \dots, p$, then one has that

$$\mathrm{Res}_h(b_j) = -n_j b_j.$$

Thus, by the residue theorem one has that

$$\frac{1}{2\pi\mathrm{i}} \oint_{\partial\Omega} h(z)\,\mathrm{d}z = \sum_{j=1}^{q} m_j a_j - \sum_{\ell=1}^{p} n_\ell b_\ell.$$

Let the vertices be denoted by ζ, $\zeta + 2\omega_1$, $\zeta + 2\omega_1 + 2\omega_3$, and $\zeta + 2\omega_3$. One then has that

$$\begin{aligned}\oint_{\partial\Omega} h(z)\,\mathrm{d}z = {}& \omega_1 \int_0^2 h(\zeta + t\omega_1)\,\mathrm{d}t + \omega_3 \int_0^2 h(\zeta + 2\omega_1 + t\omega_3)\,\mathrm{d}t \\ & - \omega_1 \int_0^2 h(\zeta + 2\omega_3 + t\omega_1)\,\mathrm{d}t - \omega_3 \int_0^2 h(\zeta + t\omega_3)\,\mathrm{d}t.\end{aligned}$$

Upon using the periodicity properties of $f(z)$ one then has that

$$\begin{aligned}\oint_{\partial\Omega} h(z)\,\mathrm{d}z &= 2\omega_1\omega_3 \int_0^2 \frac{f'(\zeta + t\omega_3)}{f(\zeta + t\omega_3)}\,\mathrm{d}t - 2\omega_1\omega_3 \int_0^2 \frac{f'(\zeta + t\omega_1)}{f(\zeta + t\omega_1)}\,\mathrm{d}t \\ &= \left(2\omega_1 \ln\{f(\zeta + t\omega_3)\} - 2\omega_3 \ln\{f(\zeta + t\omega_1)\}\right)\big|_0^2 \\ &= 2\pi\mathrm{i}(2r\omega_1 + 2s\omega_3), \quad r, s \in \mathbb{Z}.\end{aligned}$$

The last line follows from periodicity, and the fact that logarithms of functions can differ only by integer multiples of $2\pi\mathrm{i}$. Combining the above calculations then yields that

$$\sum_{j=1}^{q} m_j a_j - \sum_{\ell=1}^{p} n_\ell b_\ell = 2r\omega_1 + 2s\omega_3; \tag{10.36}$$

note that the right-hand side is simply a period of the elliptic function. Equation (10.36) provides a restriction on the placement of zeros and poles for elliptic functions.

Lemma 10.11. *Let $f(z)$ be an elliptic function of order N with periods $2\omega_1$ and $2\omega_3$. For a given cell Ω, let $a_1, \dots, a_N \in \Omega$ be the zeros, and let $b_1, \dots, b_N \in \Omega$ be the poles. One then has that*

$$\sum_{j=1}^{N} a_j - \sum_{j=1}^{N} b_j = 2r\omega_1 + 2s\omega_3$$

for some $r, s \in \mathbb{Z}$.

10.4. Weierstrass's elliptic function

Set

$$\tau := \frac{\omega_3}{\omega_1}, \ \mathrm{Im}\,\tau > 0; \quad u := 2\pi\omega_1 z; \quad \eta_1 := -\frac{\pi^2}{12\omega_1} \frac{\vartheta_1'''(0)}{\vartheta_1'(0)},$$

and define

$$\sigma(u) := \frac{2\omega_1}{\pi\vartheta_1'(0)} \mathrm{e}^{\eta_1 u^2/2\omega_1} \vartheta_1(z). \tag{10.37}$$

Equation (10.37) defines the *Weierstrass sigma function*. One has that the choice of the constants yields

$$\sigma(0) = \sigma''(0) = \sigma'''(0) = 0, \quad \sigma'(0) = 1.$$

As a consequence of the properties of the theta function, one has that $\sigma(u)$ is odd, entire, and has zeros at $u = 2m\omega_1 + 2n\omega_3$, $m, n \in \mathbb{Z}$. Finally, the quasi-periodicity of the theta function given in equation (10.7) yields that

$$\sigma(u \pm 2\omega_1) = -\mathrm{e}^{2\eta_1(u\pm\omega_1)}\sigma(u), \quad \sigma(u \pm 2\omega_3) = -\mathrm{e}^{2\eta_3(u\pm\omega_3)}\sigma(u), \tag{10.38}$$

where η_3 is defined via the relationship

$$\eta_1\omega_3 - \eta_3\omega_1 = \frac{\pi}{2}\mathrm{i}.$$

Now define the *Weierstrass zeta function* via

$$\zeta(u) := \frac{\mathrm{d}}{\mathrm{d}u}\ln\sigma(u) \quad \left(= \frac{\sigma'(u)}{\sigma(u)}\right).$$

Differentiating equation (10.38) yields

$$\sigma'(u + 2\omega_j) = -\mathrm{e}^{2\eta_j(u+\omega_j)}(\sigma'(u) + 2\eta_j\sigma(u))$$

for $j = 1, 3$; hence,

$$\zeta(u + 2\omega_j) = \zeta(u) + 2\eta_j. \tag{10.39}$$

Since $\sigma(u)$ is odd, $\zeta(u)$ is also odd. Upon putting $u = -\omega_j$ one sees that

$$\zeta(\omega_j) = \zeta(-\omega_j) + 2\eta_j,$$

i.e., $\zeta(\omega_j) = \eta_j$. Since the zeros of $\sigma(u)$ are simple, $\zeta(u)$ has simple poles at $u = 2m\omega_1 + 2n\omega_3$, $m, n \in \mathbb{Z}$, with the Laurent expansion at $u = 0$ being given by

$$\zeta(u) = \frac{1}{u} + \sum_{j=0}^{\infty} a_j u^{2j+1}. \tag{10.40}$$

If one differentiates equation (10.39) with respect to u, then the resulting function will be periodic with periods $2\omega_1$ and $2\omega_3$; furthermore, it will have poles of order two at $u = 2m\omega_1 + 2n\omega_3$, $m, n \in \mathbb{Z}$. Hence, the resulting function is an elliptic function. Since $\zeta(u)$ is odd, the resulting function will be even.

Definition 10.12. The *Weierstrass elliptic function* is given by

$$\wp(u) := -\frac{\mathrm{d}^2}{\mathrm{d}u^2}\ln\sigma(u).$$

Remark 10.13. One has that:

(a) The Laurent expansion at the pole $u = 0$ is given by

$$\wp(u) = \frac{1}{u^2} + \sum_{j=0}^{\infty} a_j u^{2j}. \tag{10.41}$$

This expansion is valid on $D(0, \min(2|\omega_1|, 2|\omega_3|))\backslash\{0\}$.

(b) Since $\wp(u)$ is an elliptic function of order two, it has two zeros within each cell.

(c) Differentiation of $\wp(u)$ N times yields an elliptic function of order $2 + N$; furthermore, this new function will have a pole of order $2 + N$ at $u = 0$.

(d) An alternative definition is given via

$$\wp(u) = \left(\frac{\pi}{2\omega_1}\right)^2\left(\frac{1}{3}\frac{\vartheta_1'''(0)}{\vartheta_1'(0)} - \frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\vartheta_1(z)\right).$$

10.4.1. Differential equation satisfied by $\wp(u)$

Define the other sigma functions via

$$\sigma_j(u) := \mathrm{e}^{-\eta_j u}\frac{\sigma(u+\omega_j)}{\sigma(\omega_j)}, \tag{10.42}$$

where ω_2 and η_2 are defined by the relations

$$\sum_{j=1}^{3}\omega_j = 0, \quad \sum_{j=1}^{3}\eta_j = 0.$$

Now set

$$e_j := \wp(\omega_j), \quad j = 1,\dots,3.$$

One has that

$$e_1 = \frac{\pi^2}{6\omega_1^2}, \quad e_2 = e_3 = -\frac{\pi^2}{12\omega_1^2}$$

[9, Chapter 6.6]. It can be shown [9, Chapter 6.7] that

$$\wp(u) - \wp(v) = \frac{\sigma(u+v)\sigma(v-u)}{\sigma^2(u)\sigma^2(v)}. \tag{10.43}$$

Upon making use of equation (10.42) one sees that

$$\wp(u) - e_j = \frac{\sigma_j^2(u)}{\sigma^2(u)}, \quad j = 1,\dots,3,$$

which implies that

$$\sqrt{(\wp(u)-e_1)(\wp(u)-e_2)(\wp(u)-e_3)} = \frac{\sigma_1(u)\sigma_2(u)\sigma_3(u)}{\sigma^3(u)}.$$

It can be shown [9, equation (6.4.4)] that

$$\sigma(2u) = 2\sigma(u)\sigma_1(u)\sigma_2(u)\sigma_3(u),$$

which allows one to rewrite the above in the more compact form

$$\sqrt{(\wp(u)-e_1)(\wp(u)-e_2)(\wp(u)-e_3)} = \frac{\sigma(2u)}{\sigma^4(u)}.$$

Finally, if in equation (10.43) one divides by $(v-u)$ and uses the fact that $\sigma'(0) = 1$, then one sees that

$$\wp'(u) = -\frac{\sigma(2u)}{\sigma^4(u)}.$$

Thus, one has that $\wp(u)$ satisfies the first-order ODE

$$y' = -2\sqrt{(y-e_1)(y-e_2)(y-e_3)}. \tag{10.44}$$

Note that as a consequence of equation (10.44) one has that $\wp'(\omega_j) = 0$ for $j = 1,\dots,3$. As a consequence of equation (10.41), $u = 0$ is a pole of order three for $\wp'(u)$; furthermore, $u = 0$ is the only pole within the cell. Since $\wp'(u)$ is an elliptic function of order three, it is then known that $z = \omega_j$ are the only zeros in the particular cell.

Squaring both sides of equation (10.44) yields the equivalent ODE

$$(y')^2 = 4y^3 - g_1y^2 - g_2y - g_3, \tag{10.45}$$

where

$$g_1 := 4(e_1+e_2+e_3), \quad g_2 := -4(e_1e_2+e_2e_3+e_3e_1), \quad g_3 := 4e_1e_2e_3. \tag{10.46}$$

Upon considering the Laurent expansion of equation (10.41), one has that

$$\wp'(u) = -\frac{2}{u^3} + \sum_{j=1}^{\infty} 2ja_j u^{2j-1},$$

i.e.,

$$(\wp'(u))^2 = \frac{4}{u^6}(1 - 2a_1u^4 - 4a_2u^6 + \cdots).$$

Since,

$$(\wp(u))^3 = \frac{1}{u^6}(1 + 3a_1u^4 + 3a_2u^6 + \cdots), \quad (\wp(u))^2 = \frac{1}{u^4}(1 + 2a_1u^4 + 2a_2u^6 + \cdots),$$

substitution of these expressions into equation (10.45) and equating coefficients yields the relationships

$$g_1 = 0, \quad a_1 = \frac{1}{20}g_2, \quad a_2 = \frac{1}{28}g_3.$$

Thus,

$$e_1 + e_2 + e_3 = 0, \quad \wp(u) = \frac{1}{u^2} + \frac{1}{20}g_2u^2 + \frac{1}{28}g_3u^4 + O(u^6), \tag{10.47}$$

and $\wp(u)$ satisfies the ODE

$$(y')^2 = 4y^3 - g_2y - g_3. \tag{10.48}$$

Note that e_j are the roots of the cubic equation

$$4y^3 - g_2y - g_3 = 0.$$

The quantities g_2 and g_3 are known as the *invariants* of $\wp(u)$.

Remark 10.14. Since the Jacobi elliptic functions also satisfy equation (10.48) (see equation (10.31)), one expects that there is a direct relationship between them and the Weierstrass elliptic function. Indeed there is, and it is given by

$$\operatorname{sn} u = (\wp(u) - e_3)^{-1/2}, \quad \operatorname{cn} u = \left(\frac{\wp(u) - e_1}{\wp(u) - e_3}\right)^{1/2}, \quad \operatorname{dn} u = \left(\frac{\wp(u) - e_2}{\wp(u) - e_3}\right)^{1/2}$$

[9, Chapter 6.9]. Thus, the Weierstrass elliptic function can be thought of as a building block for the Jacobi elliptic functions.

10.4.2. Partial fraction expansion of $\wp(u)$

Let $\omega_1, \omega_3 \in \mathbb{C}$ be given with $\mathrm{Im}(\omega_3/\omega_1) > 0$, and for $r, s \in \mathbb{Z}$ set $\Omega_{rs} := 2r\omega_1 + 2s\omega_3$. Consider the function

$$f(u) := -2\sum_{r,s} \frac{1}{(u - \Omega_{rs})^3}.$$

It will first be shown that $f(u)$ is holomorphic except at the poles $u = \Omega_{rs}$. For $n \in \mathbb{N}_0$ let C_n be a period parallelogram with vertices $\pm(2n+1)(\omega_1 + \omega_3)$ and $\pm(2n+1)(\omega_1 - \omega_3)$. The points Ω_{rs} lying between C_{n-1} and C_n for $n \in \mathbb{N}$ all lie on a parallelogram which shall be denoted Γ_n. Suppose that $u \in D$, where $D \subset \mathbb{C}$ is compact. There then exists an N such that $D \subset \mathrm{int}(\Gamma_n)$ for $n \geq N$. For $\hbar \in (0, 1)$ and $u \in D$ consider the circle

$$\begin{aligned} C_{\hbar}(u) :=& \{z \in \mathbb{C} : |u - z| = \hbar|z|\} \\ =& \{z \in \mathbb{C} : |\frac{1}{1-\hbar^2}u - z| = \frac{\hbar}{1-\hbar^2}|u|\}. \end{aligned}$$

Since the circle collapses onto the point u as $\hbar \to 0^+$, there exists a $\hbar > 0$ such that $C_{\hbar}(u) \in \mathrm{int}\,\Gamma_n$ for any $n \geq N$ and any $u \in D$. In particular, this implies that

$$|u - \Omega_{rs}| > \hbar|\Omega_{rs}|, \quad \Omega_{rs} \in \Gamma_n, \; n \geq N. \tag{10.49}$$

Let $p > 0$ be such that $D(0, p) \subset C_0$ and $\partial D(0, p) \cap C_0 \neq \emptyset$. One has that for any $\Omega_{rs} \in \Gamma_n$, $|\Omega_{rs}| \geq 2np$. There are $8n$ such lattice points, so that upon summing over these points alone

$$\sum_{rs} |\Omega_{rs}|^{-3} < 8n(2np)^{-3} = \frac{1}{p^3}\frac{1}{n^2}.$$

Thus, upon summing over the lattice points on Γ_n for $n \geq N$ and using equation (10.49) one has

$$\sum_{rs} |u - \Omega_{rs}|^{-3} < \beta^{-3} \sum_{rs} |\Omega_{rs}|^{-3} < \frac{1}{(\beta p)^3} \sum_{n=N}^{\infty} \frac{1}{n^2} < \infty.$$

Since only a finite number of terms are being excluded in the above estimate, one then has that the series associated with $f(u)$ converges uniformly on D. Hence, $f(u)$ is holomorphic except at the poles $u = \Omega_{rs}$. Also note that $f(u)$ is odd, and that $f(u + 2\omega_1) = f(u + 2\omega_3) = f(u)$ (simply rearrange the ordering of the summation), so that $f(u)$ actually defines an elliptic function of order three.

Now consider

$$G(u) := f(u) - \wp'(u),$$

which is a doubly periodic function. Recalling the Laurent series for $\wp(u)$ given in equation (10.47), one has that $G(u)$ has no poles, and is therefore uniformly bounded; hence, $G(u)$ is a constant. Appealing to equation (10.47), and using the fact that

$$\sum_{r,s}{}' \frac{1}{\Omega_{rs}^3} = 0,$$

where $\sum_{r,s}'$ indicates a summation which excludes $(r, s) = (0, 0)$, yields that $G(0) = 0$. Hence, one has that

$$\wp'(u) = -2 \sum_{r,s} \frac{1}{(u - \Omega_{rs})^3}. \tag{10.50}$$

Now, as the convergence is uniform the series in equation (10.50) can be integrated term-by-term. Upon doing so and evaluating at the end points 0 and u one gets that

$$\wp(u) := \frac{1}{u^2} + \sum_{r,s}{}' \left(\frac{1}{(u - \Omega_{rs})^2} - \frac{1}{\Omega_{rs}^2} \right) + A$$

for some $A \in \mathbb{C}$. Following the same argument as in the preceding paragraph yields that $A = 0$. The following result has now been proven.

Lemma 10.15. *Let $\omega_1, \omega_3 \in \mathbb{C}$ be given with $\mathrm{Im}(\omega_3/\omega_1) > 0$, and for $r, s \in \mathbb{Z}$ set $\Omega_{rs} := 2r\omega_1 + 2s\omega_3$. The Weierstrass elliptic function is given by*

$$\wp(u) = \frac{1}{u^2} + \sum_{r,s}{}' \left(\frac{1}{(u - \Omega_{rs})^2} - \frac{1}{\Omega_{rs}^2} \right).$$

10.4.3. Invariants expressed in terms of the periods

Unfortunately, the expression for $\wp(u)$ given in Lemma 10.15 is not practical for computations, as the series converges too slowly. However, recall that in Section 10.4.1 $\wp(u)$ is defined as a solution to the ODE equation (10.48). In order to use this formulation, one needs to know the invariants g_2 and g_3. In particular, it would be beneficial to have them expressed in terms of the periods.

Recall the Laurent expansion given in equation (10.47). From that expansion one sees that

$$g_2 = 10 \lim_{u \to 0} \frac{\mathrm{d}^2}{\mathrm{d}u^2} \left(\wp(u) - \frac{1}{u^2} \right).$$

Now, from Lemma 10.15 one has that

$$\wp(u) - \frac{1}{u^2} = \sum_{r,s}{}' \left(\frac{1}{(u - \Omega_{rs})^2} - \frac{1}{\Omega_{rs}^2} \right),$$

so that

$$\frac{d^2}{du^2}\left(\wp(u)-\frac{1}{u^2}\right)=6\sum_{r,s}{}'\frac{1}{(u-\Omega_{rs})^4}.$$

Evaluating the above at $u=0$ and substituting above then gives

$$g_2=60\sum_{r,s}{}'\frac{1}{\Omega_{rs}^4}. \tag{10.51}$$

Similarly, one has that

$$g_3=\frac{7}{6}\lim_{u\to 0}\frac{d^4}{du^4}\left(\wp(u)-\frac{1}{u^2}\right), \tag{10.52}$$

which upon following the above procedure yields

$$g_3=140\sum_{r,s}{}'\frac{1}{\Omega_{rs}^6}.$$

Remark 10.16. Using the above idea, it can be shown that the Laurent expansion of $\wp(u)$ at $u=0$ is given by

$$\wp(u)=\frac{1}{u^2}+\sum_{j=1}^{\infty}c_{2j}u^{2j},$$

where

$$c_{2j}=(2j+1)\sum_{r,s}{}'\frac{1}{\Omega_{rs}^{2j+2}}.$$

10.4.4. Expansions for $\zeta(u)$ and $\sigma(u)$

Recalling that

$$\wp(u)=-\frac{d}{du}\zeta(u),$$

one can use Lemma 10.15, and the fact that $\zeta(u)$ is odd with

$$\lim_{u\to 0}\left(\zeta(u)-\frac{1}{u}\right)=0,$$

(see equation (10.40)) to get that

$$\zeta(u)=\frac{1}{u}+\sum_{r,s}{}'\left(\frac{1}{u-\Omega_{rs}}+\frac{1}{\Omega_{rs}}+\frac{u}{\Omega_{rs}^2}\right).$$

Recalling that

$$\zeta(u)=\frac{d}{du}\ln\sigma(u)$$

then yields that

$$\ln\sigma(u)=\ln u+\sum_{r,s}{}'\left(\ln\left(1-\frac{u}{\Omega_{rs}}\right)+\frac{u}{\Omega_{rs}}+\frac{u^2}{2\Omega_{rs}^2}\right),$$

i.e.,

$$\sigma(u)=u\prod_{r,s}{}'\left(1-\frac{u}{\Omega_{rs}}\right)\exp\left(\frac{u}{\Omega_{rs}}+\frac{u^2}{2\Omega_{rs}^2}\right).$$

10.5. Representation of general elliptic functions

10.5.1. Representation with theta functions

Let $F(u)$ be an elliptic function of order N, and for a given cell let its zeros be given by u_j and its poles be given by v_j for $j = 1, \dots, N$. As a consequence of Lemma 10.11 one has that

$$\sum_{j=1}^{N}(u_j - v_j) = 2r\omega_1 + 2s\omega_3, \quad r, s \in \mathbb{Z}.$$

It is possible to choose the zeros and poles by congruent points in the lattice so that $r = s = 0$; henceforth, it will be assumed that this has been done. Set $z := \pi u/2\omega_1$, so that now $F(z)$ will have the periods π and $\tau\pi$, where $\tau := \omega_3/\omega_1$. Let the zeros now be given by z_j, and the poles by p_j, and note that $\sum z_j = \sum p_j$.

Now set

$$\Theta(z) := \prod_{j=1}^{N} \frac{\vartheta_1(z - z_j)}{\vartheta_1(z - p_j)}.$$

Since $\vartheta_1(0) = 0$, one has that $\Theta(z)$ has zeros at z_j and poles at p_j. Furthermore, as a consequence of equation (10.7), and using the fact that $\vartheta_1(z)$ is π-periodic, one has that

$$\begin{aligned}\Theta(z + \pi) &= \Theta(z)\\ \Theta(z + \tau\pi) &= \exp\{2\mathrm{i}(\sum z_j - \sum p_j)\}\Theta(z)\\ &= \Theta(z).\end{aligned}$$

The last line follows from the fact that $\sum z_j = \sum p_j$. Hence, $\Theta(z)$ is an elliptic function with the same zeros, poles, and periods as $F(2\omega_1 z/\pi)$. Finally, set

$$\Phi(z) := \frac{F(2\omega_1 z/\pi)}{\Theta(z)}.$$

One has that $\Phi(z)$ is an elliptic function with no poles, and hence it must be a constant. Thus, one has that

$$F(2\omega_1 z/\pi) = A\Theta(z), \quad A \in \mathbb{C}.$$

For a first example, consider $F(u) = \operatorname{sn} u$. One has that $\omega_1 = 2K$ and $\omega_3 = \mathrm{i}K'$, and the zeros and poles are given by $0, 2K$ and $\mathrm{i}K', 2K - \mathrm{i}K'$, respectively. Thus,

$$z_1 = 0,\ z_2 = \frac{1}{2}\pi; \quad p_1 = \frac{1}{2}\tau\pi,\ p_2 = \frac{1}{2}\pi(1 - \tau),$$

so that

$$\operatorname{sn} u = A\frac{\vartheta_1(z)\,\vartheta_1(z - \frac{1}{2}\pi)}{\vartheta_1(z - \frac{1}{2}\tau\pi)\,\vartheta_1(z - \frac{1}{2}\pi(1 - \tau))}.$$

The constant A can be determined by using the facts that

$$\frac{\mathrm{d}}{\mathrm{d}u}\operatorname{sn} u|_{u=0} = 1, \quad \frac{\mathrm{d}u}{\mathrm{d}z} = \frac{4K}{\pi},$$

which yields

$$A = -\frac{4K}{\pi}\frac{\vartheta_1(\frac{1}{2}\tau\pi)\,\vartheta_1(\frac{1}{2}\pi(1 - \tau))}{\vartheta_1'(0)\,\vartheta_1(\frac{1}{2}\pi)}.$$

For a second example, consider $F(u) = \wp'(u)$. One has a pole of order three at $u = 0$, and simple zeros at ω_1, ω_3, and $-(\omega_1 + \omega_3)$. Consequently,

$$z_1 = \frac{\pi}{2},\ z_2 = \frac{\pi}{2}\tau,\ z_3 = -\frac{\pi}{2}(1 + \tau); \quad p_1 = p_1 = p_3 = 0,$$

so that

$$\wp'(u) = A\frac{\vartheta_1(z-\pi/2)\,\vartheta_1(z-\pi\tau/2)\,\vartheta_1(z+\pi(1+\tau)/2)}{\vartheta_1^3(z)}$$
$$= \frac{\vartheta_1\left[\pi(u-\omega_1)/2\right]\,\vartheta_1\left[\pi(u-\omega_3)/2\right]\,\vartheta_1\left[\pi(u+\omega_1+\omega_3)/2\right]}{\vartheta_1^3\left[\pi u/2\right]}.$$

10.5.2. Representation in terms of $\wp(u)$

Before we look at the representation of elliptic functions in terms of the Weierstrass functions, we must first collect a few facts about even and odd elliptic functions. This may appear to be a restriction to special cases. However, upon recalling that any function $G(u)$ can be written as

$$G(u) = G_{\mathrm{e}}(u) + G_{\mathrm{o}}(u), \tag{10.53}$$

where

$$G_{\mathrm{e}}(u) := \frac{1}{2}(G(u)+G(-u)), \quad G_{\mathrm{o}}(u) := \frac{1}{2}(G(u)-G(-u)),$$

are such that $G_{\mathrm{e}}(u)$ is even and $G_{\mathrm{o}}(u)$ is odd, one immediately sees that the special cases actually take into account all functions.

Suppose that an elliptic function $F(u)$ is odd, and suppose that $u = \omega_1$ is not a pole. One then has that $F(-\omega_1) = -F(\omega_1)$; furthermore, since $2\omega_1$ is a period one has that $F(-\omega_1) = F(\omega_1)$. Hence $F(\omega_1) = 0$; similarly $F(\omega_2) = F(\omega_3) = 0$. One then can conclude that the midlattice and lattice points of an odd elliptic function are zeros if they are not poles. If $a \in \mathbb{C}$ is a lattice or midlattice point which is a zero of order m, then it is not a zero of the elliptic function $F^{(m)}(u)$. By the above reasoning one concludes that $F^{(m)}(u)$ cannot be an odd elliptic function, which implies that m must be odd. Similarly, if one considers $a \in \mathbb{C}$ to be a pole of order m, than by examining the odd elliptic function $1/F(u)$ one immediately concludes that m is again odd. Thus, all poles and zeros at midlattice and lattice points must be of odd order.

Suppose that $F(u)$ is an even elliptic function. The midlattice and lattice points must no longer necessarily be zeros or poles. However, if one assumes that they are, then by arguing in a manner similar to above one sees that they must be a zero or pole of even order. Choose a cell Ω for which $F(u)$ has no zeros or poles on $\partial\Omega$. If $a \in \Omega$ is a zero, then since $F(u)$ is even one has that $-a$ is a zero. By periodicity one than has that the point congruent to $-a$ within Ω, i.e., $-a \mapsto 2r\omega_1 + 2s\omega_3 - a$ for some $r, s \in \mathbb{Z}$, is also a zero. Thus, the zeros within Ω can be arranged in pairs, which will be distinct unless $a = 2r\omega_1 + 2s\omega_3 - a$. In this case one has that $a = r\omega_1 + s\omega_3$, which means that a is either a midlattice point or lattice point. From the above it is known that this zero must be of even order. Hence, the total number of zeros within a cell is even, which implies that $F(u)$ is an elliptic function of even order.

Proposition 10.17. *If $F(u)$ is an even elliptic function, then it is of even order.*

Remark 10.18. Proposition 10.17 has no analogue for odd elliptic functions. For example, $\operatorname{sn} u$ is an odd elliptic function of order two, whereas $\wp'(u)$ is an odd elliptic function of odd order.

Supposes that $F(u)$ is even, of order $2N$, and has no zero or pole at a lattice point. Following the above reasoning, for a given cell choose an irreducible set of zeros u_j and poles v_j for $j = 1, \dots, N$, and consider the function

$$G(u) := \prod_{j=1}^{N}\frac{\wp(u)-\wp(u_j)}{\wp(u)-\wp(v_j)}.$$

One has that $G(u)$ is periodic with periods $2\omega_1$ and $2\omega_3$, and has no pole at any lattice point. Furthermore, by supposition $G(u)$ is nonzero at any lattice point. Now, recall that $\wp(u)$ is even and of order two; hence, if u_j is not a midlattice point, $\wp(u) - \wp(u_j)$ has a simple zero at $\pm u_j$; otherwise, the function has a double zero at u_j. Consequently, $G(u)$ has the same zeros as $F(u)$ with the same multiplicities. Similarly, it has the identical number of poles as $F(u)$ with the same multiplicities. Thus, $F(u)/G(u)$ is an elliptic function with no poles, and is hence constant. One then has that

$$F(u) = A\prod_{j=1}^{N}\frac{\wp(u)-\wp(u_j)}{\wp(u)-\wp(v_j)}.$$

Now suppose that $F(u)$ has a zero of order $2n$ at each lattice point. Upon setting

$$G(u) := \frac{\prod_{j=1}^{N-n} \wp(u) - \wp(u_j)}{\prod_{j=1}^{N} \wp(u) - \wp(v_j)},$$

one sees that $G(u)$ has a zero of order $2n$ at each lattice point. Arguing as above, one then sees that

$$F(u) = A\frac{\prod_{j=1}^{N-n} \wp(u) - \wp(u_j)}{\prod_{j=1}^{N} \wp(u) - \wp(v_j)}. \tag{10.54}$$

Finally, if $F(u)$ has a pole of order $2n$ at each lattice point, then by a similar reasoning one finally sees that

$$F(u) = A\frac{\prod_{j=1}^{N} \wp(u) - \wp(u_j)}{\prod_{j=1}^{N-n} \wp(u) - \wp(v_j)}.$$

In all cases, one has that

$$F(u) = P(\wp(u)),$$

where $P(\cdot)$ is a rational function.

If one supposes that $F(u)$ is odd, then one has that the function $\tilde{F}(u) := F(u)/\wp'(u)$ is even. Arguing as above one then sees that

$$F(u) = \wp'(u)Q(\wp(u)),$$

where $Q(\cdot)$ is a rational function. Finally, suppose that $F(u)$ is an arbitrary elliptic function. Upon using equation (10.53) one sees that

$$F(u) = P(\wp(u)) + \wp'(u)Q(\wp(u)).$$

The following has now been proved.

Lemma 10.19. *Suppose that $F(u)$ is an elliptic function with periods $2\omega_1$ and $2\omega_3$. There exist rational functions P and Q such that*

$$F(u) = P(\wp(u)) + \wp'(u)Q(\wp(u)).$$

For an example, consider $F(u) = \operatorname{sn} u$. The periods are given by

$$\omega_1 = 2K, \quad \omega_3 = \mathrm{i}K';$$

hence, the lattice points are given by $2r\omega_1 + 2s\omega_3$, $r, s \in \mathbb{Z}$. Modulo the lattice points, an irreducible set of simple zeros is $\{0, \omega_1\}$, and an irreducible set of simple poles is $\{\omega_3, \omega_1 + \omega_3\}$. Now, $\wp'(u)$ has a triple pole at $u = 0$, and simple zeros at the midlattice points ω_1, ω_3, and $\omega_1 + \omega_3$. It then follows that the even function $\operatorname{sn} u/\wp'(u)$ has a zero of order four at $u = 0$, and a pair of double poles at $v_1 := \omega_3$ and $v_2 := \omega_1 + \omega_3$. Application of equation (10.54) then yields

$$\frac{\operatorname{sn} u}{\wp'(u)} = \frac{A}{(\wp(u) - \wp(v_1))(\wp(u) - \wp(v_2))}.$$

Upon setting

$$e_1 := \wp(\omega_1), \quad e_2 := \wp(\omega_1 + \omega_3), \quad e_3 := \wp(\omega_3),$$

the above can be rewritten as

$$\frac{\operatorname{sn} u}{\wp'(u)} = \frac{A}{(\wp(u) - e_2)(\wp(u) - e_3)}.$$

Upon applying equation (10.44) and simplifying one then gets that

$$\operatorname{sn} u = \tilde{A}\frac{\wp(u) - e_1}{\wp'(u)}.$$

Upon expanding about $u = 0$ and using the facts that

$$\operatorname{sn} u = u + \cdots; \quad \wp(u) = \frac{1}{u^2} + \cdots,$$

one sees that $\tilde{A} = -2$; hence,

$$\operatorname{sn} u = -2\frac{\wp(u) - e_1}{\wp'(u)}.$$

10.6. Applications

Here we will consider problems which naturally arise in physical applications, and demonstrate the manner in which the above theory yields solutions to the governing equations.

10.6.1. The simple pendulum

The equation governing the motion of a simple planar pendulum is given by

$$\ddot{\theta} + \sin\theta = 0.$$

Here time has been appropriately normalized in order to absorb the gravitational constant g and the length of the pendulum ℓ. The system has a first integral given by

$$E(\theta, \dot{\theta}) := -\frac{1}{2}\dot{\theta}^2 + \cos\theta,$$

i.e., $\dot{E} = 0$ [**12**, Chapter 2]. Upon setting

$$x := \frac{1}{k}\sin\frac{\theta}{2}, \quad k^2 := \frac{1}{2}(1 - E)$$

it is eventually seen that

$$\dot{x}^2 = (1 - x^2)(1 - k^2x^2).$$

Comparison with equation (10.30) then yields the solution to be $x(t) = \mathrm{sn}(t; k)$, i.e.,

$$\theta(t) = 2\mathrm{Sin}^{-1}(k\,\mathrm{sn}(t; k)),$$

with the period being given by $4K$, where

$$K = \int_0^1 \frac{1}{\sqrt{(1 - \zeta^2)(1 - k^2\zeta^2)}}\,\mathrm{d}\zeta.$$

Note that the period depends upon the energy, and that

$$\lim_{E\to -1^+} K = +\infty, \quad \lim_{E\to 1^-} K = \frac{\pi}{2}.$$

In the second case the motion of the pendulum is well approximated by the linear system

$$\ddot{\theta} + \theta = 0.$$

10.6.2. The spherical pendulum

The following example can be found in [**11**, Chapter III.5.27]. Using cylindrical coordinates, the total energy associated with a spherical pendulum is given by

$$E := \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) + gz,$$

where g represents the acceleration due to gravity. If ℓ represents the length of the pendulum, then one has that

$$r^2 = \ell^2 - z^2.$$

The angular momentum about the z-axis must be conserved, which implies that

$$r^2\dot{\theta} = C, \quad C \geq 0. \tag{10.55}$$

If $C = 0$, then the motion becomes that of a simple planar pendulum, which was studied in the previous subsection. As such, it will be henceforth assumed that $C > 0$. Substitution of the above two identities into that for E yields

$$\frac{z^2\dot{z}^2}{\ell^2 - z^2} + \frac{C^2}{\ell^2 - z^2} + \dot{z}^2 = -2gz + h, \quad h := 2E,$$

i.e.,

$$\ell^2\dot{z}^2 = (h - 2gz)(\ell^2 - z^2) - C^2 := q(z), \tag{10.56}$$

where $q(z)$ is a polynomial of degree three.

If $z_0 \in (-\ell, \ell)$ is the coordinate of the pendulum at $t = 0$, then one must have that $q(z_0) \geq 0$; otherwise, the initial velocity would not be real-valued. Since

$$q(\pm\infty) = \pm\infty, \quad q(\pm\ell) = -C^2 < 0, \quad q(z_0) \geq 0,$$

one can conclude that the zeros of $q(z)$ are all real. Assuming that $q(z_0) > 0$, each zero is necessarily distinct. Label the zeros z_j such that

$$-\ell < z_3 < z_0 < z_2 < \ell < z_1.$$

By uniqueness of solutions to ODEs the coordinate z now satisfies the constraint $z \in [z_3, z_2]$ for all time.

Now set

$$z := au + b; \qquad a := \frac{2\ell^2}{g}, \quad b := \frac{h}{6g}. \tag{10.57}$$

Upon using this transformation equation (10.56) becomes

$$\dot{u}^2 = 4u^3 - g_2u - g_3, \tag{10.58}$$

and the zeros are now given by

$$e_j := \frac{z_j - b}{a}, \quad j = 1, \dots, 3,$$

and satisfy $e_1 > e_2 > e_3$. Moreover, $u \in [e_3, e_2]$ for all time.

Consider the ODE

$$(y')^2 = 4y^3 - g_2y - g_3, \quad ' := \frac{\mathrm{d}}{\mathrm{d}\tau},$$

where g_2 and g_3 are as above. It was seen in equation (10.48) that the solution is given by the Weierstrass elliptic function $\wp(\tau)$ with fundamental periods $2\omega_1$ and $2\omega_3$, where g_2 and g_3 are related to the periods via equation (10.51) and equation (10.52), respectively. Since the zeros are real and distinct, it can be shown that $\omega_1 = \alpha$ and $\omega_3 = \mathrm{i}\beta$, where $\alpha, \beta \in \mathbb{R}^+$ are given by

$$\alpha = \int_{e_1}^{+\infty} \frac{1}{\sqrt{4x^3 - g_2x - g_3}}\,\mathrm{d}x, \quad \beta = \int_{-\infty}^{e_3} \frac{1}{\sqrt{-4x^3 + g_2x + g_3}}\,\mathrm{d}x$$

[**11**, Chapter III.5.24].

If one sets $\tau = t + \mathrm{i}\beta$, where $t \in \mathbb{R}$, then one has that

$$u(t) = \wp(t + \mathrm{i}\beta)$$

is a solution to equation (10.58) which is of period 2α. Note that as a consequence of [**11**, Lemma III.22.2] that this solution is real-valued for all $t \in \mathbb{R}$. Furthermore, this solution has no singularities for $t \in \mathbb{R}$, as its poles are given by $t_\mathrm{p} = 2r\alpha + \mathrm{i}(2s + 1)\beta$ for $r, s, \in \mathbb{Z}$. Finally, as is seen in [**11**, Theorem 5.6] one has that

$$u(0) = e_3, \quad u(\alpha) = e_2.$$

Thus, the solution $u(t)$ is 2α-periodic and achieves its maximum values at $t = (2j + 1)\alpha$ and its minimum values at $t = 2j\alpha$ for $j \in \mathbb{Z}$ (see [**11**, Figure III.5.4]). Upon using equation (10.57) one finally sees that

$$z(t) = au(t) + b = a\wp(t + \mathrm{i}\beta) + b, \tag{10.59}$$

i.e., the solution to equation (10.56) is given by an elliptic function.

Now, $r(t) = \sqrt{\ell^2 - z^2(t)}$ is also an elliptic function of real period $2a$. In order to solve for $\partial(t)$, one uses equation (10.55) to get

$$\dot{\partial} = \frac{C}{r^2} = \frac{C}{\ell^2 - z^2}.$$

In other words,

$$\partial(t) = \partial_0 + C\int_0^t \frac{1}{\ell^2 - z^2(s)}\,\mathrm{d}s.$$

It is periodic with period $2a$ if and only if

$$\partial_{\text{prec}} := C\int_0^{2a} \frac{1}{\ell^2 - z^2(s)}\,\mathrm{d}s = 0 \qquad (\text{mod } 2\pi);$$

otherwise,

$$\partial(t + 2a) = \partial(t) + \partial_{\text{prec}}.$$

It can eventually be shown that

$$\mathrm{e}^{2\mathrm{i}(\partial - \partial_0)} = \mathrm{e}^{\mathrm{i}\mu t}F(t),$$

where $\mu \in \mathbb{C}$ is constant and $F(t)$ is an elliptic function with periods $2a$ and $2\mathrm{i}\beta$ (see [11, equation (III.5.79)]). Generically one will have that

$$\mu \neq 0 \quad (\text{mod } \frac{\pi}{a}),$$

so that $\partial_{\text{prec}} \neq 0$.

11. Asymptotic evaluation of integrals

The source material for this section is [1, Chapter 6]. A good supplemental source is [2]. One motivation for understanding the material in this text is that in practice one often must evaluate integrals of the form

$$I(\epsilon) := \int_{-\infty}^{+\infty} f(x)\mathrm{e}^{\mathrm{i}x/\epsilon}\,\mathrm{d}x, \quad 0 < \epsilon \ll 1.$$

It is known via the Riemann-Lebesgue lemma that under the condition that $f \in L^1(\mathbb{R})$, which is defined below, then

$$\lim_{\epsilon \to 0^+} I(\epsilon) = 0.$$

However, the result does not indicate the size of $I(\epsilon)$ as $\epsilon \to 0^+$. For example, it can be shown that

$$\begin{aligned}
\int_{-\infty}^{+\infty} \sin((x - \xi)/\epsilon)\,\mathrm{sech}^2 x\,\mathrm{d}x &= -\frac{\pi}{\epsilon\sinh(\pi/2\epsilon)}\sin(\xi/\epsilon) \\
&= -\frac{2\pi}{\epsilon}\mathrm{e}^{-\pi/2\epsilon}\sin(\xi/\epsilon) + \mathcal{O}(\mathrm{e}^{-\pi/\epsilon});
\end{aligned}$$

hence, the integral has an expansion which is in composed only of exponentially small terms. Furthermore, the fact that the error terms in the second line can be uniformly bounded arises from the fact an explicit expression is known for the integral. The above integral was computed using standard contour integration techniques. It will be of interest to compute $I(\epsilon)$ when the techniques are no longer applicable.

11.1. *Fourier and Laplace transforms*

Let

$$L^p(\mathbb{R}) := \{f \in C^0(\mathbb{R}) : \|f\|_p^p := \int_{\mathbb{R}} |f(x)|^p\,\mathrm{d}x < \infty\}.$$

For $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $k \in \mathbb{R}$ define the Fourier transform of $f(x)$ to be

$$\hat{f}(k) := \int_{-\infty}^{+\infty} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x. \tag{11.1}$$

It can be shown that the inverse Fourier transform satisfies

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(k)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k.$$

It is clear that for each $k \in \mathbb{R}$

$$|\hat{f}(k)| \le \|f\|_1;$$

hence, the Fourier transform is well-defined.

The utility of the Fourier transform in solving differential equations follows from the following set of facts. First, suppose that $f \in L^2(\mathbb{R})$, and that $\lim_{|x|\to\infty} |f(x)| = 0$. Set

$$\hat{f}_j(k) := \int_{-\infty}^{+\infty} f^{(j)}(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x.$$

First note that upon integrating by parts,

$$\begin{aligned}
\hat{f}_1(k) &= \int_{-\infty}^{+\infty} f'(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\
&= f(x)\mathrm{e}^{-\mathrm{i}kx}\Big|_{-\infty}^{+\infty} + \mathrm{i}k\int_{-\infty}^{+\infty} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\
&= \mathrm{i}k\hat{f}(k).
\end{aligned}$$

An induction argument then immediately yields that

$$\hat{f}_j(k) = (\mathrm{i}k)^j\hat{f}(k). \tag{11.2}$$

This formal argument can be made rigorous (see [7, Theorem 4.3.2]).

As a consequence of linearity one has that for

$$g(x) := \sum_{j=0}^{N} a_j\frac{\mathrm{d}^j f}{\mathrm{d}x^j}, \quad a_j \in \mathbb{C},$$

the associated Fourier transform satisfies

$$\hat{g}(k) = \left(\sum_{j=0}^{N} a_j(\mathrm{i}k)^j\right)\hat{f}(k).$$

Upon applying the Fourier transform to the differential equation

$$\sum_{j=0}^{N} a_j\frac{\mathrm{d}^j f}{\mathrm{d}x^j} = h(x),$$

one sees that the Fourier transform of the solution satisfies

$$\hat{f}(k) = \frac{\hat{h}(k)}{p(k)}, \quad p(k) := \sum_{j=0}^{N} a_j(\mathrm{i}k)^j.$$

The polynomial $p(\cdot)$ is known as the *symbol* associated with the differential equation.

Now consider the convolution product

$$(f * g)(x) := \int_{-\infty}^{+\infty} f(s)g(x-s)\,\mathrm{d}s.$$

Applying the Fourier transform to the convolution yields

$$\begin{aligned}\int_{-\infty}^{+\infty}(f*g)(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x &= \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}f(s)g(x-s)\,\mathrm{d}s\right)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x\\ &= \int_{-\infty}^{+\infty}f(s)\mathrm{e}^{-\mathrm{i}ks}\int_{-\infty}^{+\infty}g(x-s)\mathrm{e}^{-\mathrm{i}k(x-s)}\,\mathrm{d}s\,\mathrm{d}x\\ &= \hat{f}(k)\hat{g}(k);\end{aligned}$$

hence, the Fourier transform of a convolution is the product of the Fourier transforms. This then implies that the solution to the above ODE is given by

$$f(x) = (p*h)(x),$$

where

$$p(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{p(k)}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k.$$

Now, note that the inverse transform of the product satisfies

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(k)\hat{g}(k)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k = \int_{-\infty}^{+\infty}f(s)g(x-s)\,\mathrm{d}s.$$

If one sets $g(x) := \overline{f}(-x)$ in the above, then upon evaluating at $x = 0$ one sees that

$$\int_{-\infty}^{+\infty}|f(s)|^2\,\mathrm{d}s = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(k)\hat{g}(k)\,\mathrm{d}k.$$

Since

$$\begin{aligned}\hat{g}(k) &= \int_{-\infty}^{+\infty}\overline{f}(-x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x\\ &= \int_{-\infty}^{+\infty}\overline{f}(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x\\ &= \overline{\hat{f}(k)},\end{aligned}$$

the above can be refined to read

$$\int_{-\infty}^{+\infty}|f(s)|^2\,\mathrm{d}s = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|\hat{f}(k)|^2\,\mathrm{d}k. \tag{11.3}$$

Equation (11.3) is the *Parseval formula*.

A particular extension of the Fourier transform, known as the Laplace transform, can be constructed in the following manner. Suppose that $f(x) \equiv 0$ for $x < 0$. For $c \in \mathbb{R}^+$ one has that

$$\mathrm{e}^{-cx}f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_0^{+\infty}\mathrm{e}^{-ct}f(t)\mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t\right)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k,$$

so that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\int_0^{+\infty}\mathrm{e}^{-(c+\mathrm{i}k)t}f(t)\,\mathrm{d}t\right)\mathrm{e}^{(c+\mathrm{i}k)x}\,\mathrm{d}k.$$

Setting $s := c + \mathrm{i}k$ then yields

$$f(x) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\left(\int_0^{+\infty}\mathrm{e}^{-st}f(t)\,\mathrm{d}t\right)\mathrm{e}^{sx}\,\mathrm{d}s.$$

The Laplace transform is then given by

$$\hat{f}(s) := \int_0^{+\infty}\mathrm{e}^{-sx}f(x)\,\mathrm{d}x, \tag{11.4}$$

and the inverse Laplace transform is

$$f(x) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \hat{f}(s)\mathrm{e}^{sx}\,\mathrm{d}s.$$

The utility of the Laplace transform in solving differential equations is that

$$\int_0^{+\infty} \mathrm{e}^{-sx} f^{(n)}(x)\,\mathrm{d}x = s^n\hat{f}(s) - f^{(n-1)}(0) - sf^{(n-2)}(0) - \cdots - s^{n-1}f(0);$$

hence, in a manner similar to that of the Fourier transform, a differential equation can be transformed into an algebraic equation for the transform $\hat{f}(s)$. From this perspective, the primary difference between the two transforms is that the Laplace transform requires the initial data in order to solve the differential equation, whereas the Fourier transform does not. It is also useful to know that for

$$h(x) := \int_0^x f(s)g(x-s)\,\mathrm{d}s,$$

the Laplace transform is given by

$$\hat{h}(s) = \hat{f}(s)\hat{g}(s).$$

This last fact will be useful when solving nonhomogeneous ordinary differential equations via the Laplace transform.

11.2. Applications of transforms to differential equations

The heat equation is given by

$$\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2}, \quad \phi(x,0) = h(x).$$

Let $\hat{\phi}$ represent the Fourier transform of the solution. One then has that for each $k \in \mathbb{R}$, $\hat{\phi}$ satisfies the ODE

$$\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}t} = -k^2\hat{\phi}, \quad \hat{\phi}(k,0) = \hat{h}(k).$$

The solution is given by

$$\hat{\phi}(k,t) = \mathrm{e}^{-k^2 t}\hat{h}(k).$$

Letting $G(x,t)$ represent the inverse Fourier transform of $\mathrm{e}^{-k^2 t}$, one then has that the solution is given by

$$\phi(x,t) = G(x,t) * h(x) = \int_{-\infty}^{+\infty} G(x-\xi,t)h(\xi)\,\mathrm{d}\xi.$$

Now let us determine $G(x,t)$, which is known as the *Green's function*. Upon using the definition one has that

$$\begin{aligned} G(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-k^2 t}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2/4t}\mathrm{e}^{-(k-\mathrm{i}x/2t)^2 t}\,\mathrm{d}k \\ &= \frac{1}{2\sqrt{\pi t}}\mathrm{e}^{-x^2/4t}. \end{aligned}$$

The last line follows from the fact that $\int_{\mathbb{R}} \mathrm{e}^{-u^2}\,\mathrm{d}u = \sqrt{\pi}$. Hence, the solution is given by

$$\phi(x,t) = \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty} \mathrm{e}^{-(x-\xi)^2/4t}h(\xi)\,\mathrm{d}\xi.$$

For example, if $h(x) = \delta(x - x_0)$, where $\delta(\cdot)$ represents the Dirac delta function, then

$$\varphi(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-(x-x_0)^2/4t}.$$

Thus, even though the initial data is nonzero at only one point, for every $t > 0$ the solution is nonzero for all $x \in \mathbb{R}$. The heat equation then supports an *infinite* speed of propagation of initial data.

Remark 11.1. The above solution can also be found via the Laplace transform. The interested student is directed to [1, Chapter 4.6] for the details.

Now consider the wave equation

$$\frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = f(x)e^{i\omega t}.$$

Here $c > 0$ represents the speed of propagation of the unforced wave, and the forcing is assumed to be periodic in time with a constant frequency. Note that a real solution is obtained by taking the real part of φ, and that this simply corresponds to the forcing $f(x)\cos\omega t$. Setting $\varphi(x,t) := \Phi(x)e^{i\omega t}$ yields that

$$\frac{d^2\Phi}{dx^2} + \left(\frac{\omega}{c}\right)^2 \Phi = f(x),$$

which upon using the Fourier transform has the solution

$$\Phi(x) = H(x, \omega/c) * f(x),$$

where

$$H(x,\omega/c) = -\frac{1}{2\pi}\,\mathrm{P.V.}\int_{-\infty}^{+\infty} \frac{e^{ikx}}{k^2 - (\omega/c)^2}\,dk.$$

In computing the above integral, the standard choice is to specify that the portion of the contour on the real axis be indented below the pole $k^- := -\omega/c$ and above the pole $k^+ := +\omega/c$. Upon noting that

$$\mathrm{Res}_f(k^\pm) = \frac{1}{2}\frac{e^{ik^\pm x}}{k^\pm},$$

one then gets that

$$H(x,\omega/c) = i\frac{e^{-i\omega|x|/c}}{2(\omega/c)}.$$

In conclusion one then has that the solution is given by

$$\varphi(x,t) = \mathrm{Re}\left(\frac{i}{2(\omega/c)}\int_{-\infty}^{+\infty} e^{i\omega(t-|x-\xi|/c)} f(\xi)\,d\xi\right).$$

For example, if $f(x) = \delta(x - x_0)$, then the solution is given by

$$\varphi(x,t) = -\frac{1}{2(\omega/c)}\sin\omega(t - |x - x_0|/c).$$

Thus, a forcing at only one point yields a sinusoidal response from the medium, with the solution being constant along the rays

$$ct - |x - x_0| = C, \quad C \in \mathbb{R}.$$

Remark 11.2. It is an exercise for the student to determine the relationship between the function $H(x,\omega/c)$ and the different choices for the contour of integration.

As a final example, consider

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} = 0, \quad u(x,0) = f(x).$$

Here $u(x,t)$ represents the small amplitude vibrations of a continuous medium such as water waves. Letting $\hat{u}(k,t)$ represent the Fourier transform of $u(x,t)$, one sees that

$$\frac{\partial \hat{u}}{\partial t} - ik^3\hat{u} = 0, \quad \hat{u}(k,0) = \hat{f}(k),$$

which has the solution

$$\hat{u}(k, t) = \hat{f}(k)e^{ik^3 t}.$$

Upon inversion one then has that

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(kx+k^3 t)} \hat{f}(k) \, dk.$$

The above solution can be viewed as a superposition of waves of the form

$$e^{i(kx - \omega(k)t)}, \quad \omega(k) := -k^3.$$

Here $\omega(k)$ is known as the dispersion relation associated with the PDE. Unfortunately, one cannot find an explicit representation for the solution $u(x, t)$ beyond what is given above. However, one can use asymptotic methods, such as the method of stationary phase and the method of steepest descent, to approximate the solution as $t \to +\infty$. It turns out that the major contribution to the integral defining the solution is found near the location where for the phase $\Psi := kx - \omega(k)t$ one has $\partial\Psi/\partial k = 0$, i.e., $\omega'(k) = x/t$. The quantity $\omega'(k)$ is known as the group velocity, and it represents the speed of a packet of waves centered around the wave number k. For $x/t > 0$ one has that the solution decays exponentially. For $x/t < 0$ one can show that for particular constants $c, \phi_j \in \mathbb{R}$,

$$u(x, t) \approx \frac{c}{\sqrt{t}} \left(\sum_{j=1}^{2} \frac{\hat{f}(k_j)}{\sqrt{|k_j|}} e^{i(k_j x - \omega(k_j)t + \phi_j)} \right); \qquad k_1 := \sqrt{-\frac{x}{3t}}, \ k_2 := -\sqrt{-\frac{x}{3t}}.$$

Hence, the solution still decays, except now it does so only at the rate $t^{-1/2}$.

Finally, consider the limit $x/t \to 0$, i.e., the transition region between the above two solution behaviors. The above solution can be rearranged and put into the self-similar form

$$u(x, t) \approx \frac{d}{(3t)^{1/3}} A(x/(3t)^{1/3}),$$

where $A(\cdot)$ is the solution to Airy's equation

$$\frac{d^2 A}{dz^2} - zA = 0, \quad \lim_{z \to +\infty} A(z) = 0.$$

The solution to Airy's equation is given by

$$A(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(kz + k^3/3)} \, dk.$$

It can be shown that $A(z)$ decays exponentially fast as $z \to +\infty$, and has oscillatory behavior and decays like $|z|^{-1/4}$ as $z \to -\infty$ (see [**1**, Figure 4.6.2] for a depiction of the wave form).

Remark 11.3. One interpretation of the above asymptotic solution is that the solution decays exponentially fast when travelling faster than the group velocity, and oscillates and decays very slowly when travelling slower than the group velocity. When travelling at the speed of group velocity, the solution is governed by a self-similar solution to Airy's equation.

11.3. Laplace type integrals

Recalling the definition of the Laplace transform in equation (11.4), we shall first consider

$$\lim_{k \to +\infty} I(k) := \int_a^b f(t) e^{-k\phi(t)} \, dt, \tag{11.5}$$

where $f, \phi \in C^1(\mathbb{R})$.

11.3.1. Integration by parts

In order to better understand the relevant issues regarding the asymptotic expansions of integrals, consider

$$I(x) := x\mathrm{e}^x E(x), \quad E(x) := \int_x^{+\infty} \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t,$$

for $x \in \mathbb{R}^+$. It is clear that $I(x)$ is well-defined for each $x \in \mathbb{R}^+$; furthermore, upon applying L'Hospital's rule one sees that

$$\lim_{x\to 0^+} I(x) = 0, \quad \lim_{x\to +\infty} I(x) = 1.$$

It turns out that $I(x)$ has the expansion

$$I(x) = x\mathrm{e}^x\left(-\ln x - \gamma + \sum_{j=1}^{\infty}(-1)^{j+1}\frac{x^j}{j\cdot j!}\right),$$

where γ is the Euler constant [2, equation (1.1.5)]. The expansion is clearly valid for all $x \in \mathbb{R}^+$. Unfortunately, it the rate of convergence for $x \ge 10$ is very slow; for example, in order to compute $I(10)$ accurate to three significant figures, one must take over 40 terms in the expansion. Hence, this expansion is not of much practical use in computing $I(x)$ for $x \gg 1$.

As a consequence, it is natural to seek an expansion about $x = +\infty$. If one integrates by parts N times, then one sees that

$$I(x) = \sum_{j=1}^{N-1}(-1)^j\frac{j!}{x^j} + E(x,N),$$

where

$$E(x,N) := (-1)^N N! x\mathrm{e}^x \int_x^{+\infty} \frac{\mathrm{e}^{-t}}{t^{N+1}}\,\mathrm{d}t.$$

It is natural to first attempt to write

$$I(x) = \sum_{j=1}^{\infty}(-1)^j\frac{j!}{x^j}.$$

However, it is quickly seen that the sum diverges for all finite values of x, and hence it is of no value. Therefore, one can approximate $I(x)$ only with the finite sum, with the understanding that there will be some error with this approximation. Since

$$|E(x,N)| \le \frac{N!}{x^N},$$

one has that for fixed N, $E(x,N) \to 0$ as $x \to +\infty$. Note that the control of the error is not uniform. Indeed, if $\epsilon > 0$ is given, then one requires that $x > (N!/\epsilon)^{1/N}$ in order to achieve $|E(x,N)| < \epsilon$. Since

$$\lim_{N\to+\infty}(N!)^{1/N} = +\infty,$$

in order to achieve the desired accuracy one must carefully exploit the interplay between the size of ϵ and the size of N. For example, in order to compute $I(10)$ accurate to three significant figures using the above formulation, one needs that $N = 10$. However, no value of N will yield a result accurate to four significant figures! For another example, in order to compute $I(100)$ accurate to five significant figures, one needs only that $N = 4$.

In conclusion, it is seen that by using the method of integration by parts to derive an asymptotic expansion, one derives a divergent series. However, the partial sums are still quite useful in approximating the original function. All that must be kept in mind is that the error associated with the approximation is not uniform in x; hence, if one wishes to choose a large value of N, then one must choose $x > x^*(N)$, where $x^*(N) \to +\infty$ as $N \to +\infty$.

Upon using the above example as a mathematical guide, we have the following general lemma concerning the method of integration by parts. Note that we first consider $I(k)$ defined in equation (11.5) with $\varphi(t) \equiv t$.

Lemma 11.4. *Consider $I(k)$ defined in equation (11.5) with $\phi(t) \equiv t$. If $b = +\infty$, then suppose that $|f(t)| = O(\mathrm{e}^{at})$ for some $a \in \mathbb{R}^+$. If $f \in C^{N+2}(\mathbb{R})$, then*

$$I(k) \sim \mathrm{e}^{-ka} \sum_{j=0}^{N} \frac{f^{(j)}(a)}{k^{j+1}}, \quad k \to +\infty.$$

Proof: An integration by parts yields

$$I(k) = \mathrm{e}^{-ka} \sum_{j=0}^{N} \frac{f^{(j)}(a)}{k^{j+1}} - \mathrm{e}^{-kb} \sum_{j=0}^{N} \frac{f^{(j)}(b)}{k^{j+1}} + E_N(k),$$

where

$$E_N(k) := \frac{1}{k^{N+1}} \int_a^b \mathrm{e}^{-kt} f^{(N+1)}(t)\,\mathrm{d}t.$$

Since $b > a$, the second sum is negligible in the estimate as it is of $O(\mathrm{e}^{-(k-a)b})$; hence, it can be ignored. Concerning the error term one has that upon another integration by parts,

$$E_N(k) = \frac{1}{k^{N+2}}(\mathrm{e}^{-ka} f^{(N+1)}(a) - \mathrm{e}^{-kb} f^{(N+1)}(b)) + E_{N+1}(k).$$

Thus, as $k \to +\infty$ one has that $|E_N(k)| = O(\mathrm{e}^{-(k-a)a}/k^{N+2})$, which again is asymptotically negligible. In conclusion one has that

$$I(k) = \mathrm{e}^{-ka} \sum_{j=0}^{N} \frac{f^j(a)}{k^{j+1}} + O\left(\frac{\mathrm{e}^{-(k-a)a}}{k^{N+2}}\right) + O\left(\mathrm{e}^{-(k-a)b}\right),$$

which is the desired result. □

For an example, consider

$$\lim_{\epsilon \to 0^+} I(\epsilon) := \int_0^{+\infty} (1 + \epsilon t)^{-1} \mathrm{e}^{-t}\,\mathrm{d}t.$$

Note that the integral is well-defined for each $\epsilon \ge 0$. If one sets $\tau := \epsilon t$, then one sees that

$$I(\epsilon) = \frac{1}{\epsilon} \int_0^{+\infty} (1 + \tau)^{-1} \mathrm{e}^{-\tau/\epsilon}\,\mathrm{d}\tau,$$

which is in the framework of Lemma 11.4 with $f(\tau) = (1 + \tau)^{-1}$. Since $f^{(j)}(0) = (-1)^j j!$ for each $j \in \mathbb{N}_0$, one has that

$$I(\epsilon) \sim \sum_{j=0}^{N} (-1)^j j! \epsilon^j, \quad \epsilon \to 0^+.$$

Note that the series is divergent as $N \to +\infty$, even thought $f(\cdot)$ is holomorphic for $\operatorname{Re} z > -1$.

11.3.2. Watson's lemma

If $f(t)$ is integrable at $t = a$, but is not sufficiently smooth, i.e., $f(t) \approx (t - a)^{-1/2}$ for $0 < t - a \ll 1$, then Lemma 11.4 is not applicable. Thus, we need another method to evaluate integrals with integrands of this type. In what follows we need to recall that for $z \in D(0, 1)$,

$$(1 + z)^a = \sum_{j=0}^{\infty} c_j(a) z^j, \quad c_j(a) := \frac{\Gamma(1 + a)}{\Gamma(1 + j)\Gamma(1 + a - j)}. \tag{11.6}$$

For example,

$$\frac{1}{\sqrt{t^2 + 2t}} = \frac{1}{\sqrt{2t}} \sum_{j=0}^{\infty} c_j(-\frac{1}{2}) \left(\frac{t}{2}\right)^j, \quad t \in D(0, 2). \tag{11.7}$$

Lemma 11.5 (Watson's Lemma). *Consider $I(k)$ defined in equation (11.5) with $\alpha = 0$ and $\phi(t) \equiv t$. Let $f \in C(\mathbb{R}^+)$ satisfy $|f(t)| = O(e^{ct})$ for some $c \in \mathbb{R}^+$. Furthermore, suppose that for $t \in D(0, R)$ one has that*

$$f(t) \sim t^a \sum_{j=0}^{\infty} a_j t^{\beta j}; \quad a > -1, \; \beta > 0.$$

Then

$$I(k) \sim \sum_{j=0}^{\infty} a_j \frac{\Gamma(1 + a + \beta j)}{k^{1+a+\beta j}}, \quad k \to +\infty.$$

Proof: Set $I(k) = I_1(k) + I_2(k)$, where

$$I_1(k) := \int_0^R f(t) e^{-kt} \, dt, \quad I_2(k) := \int_R^b f(t) e^{-kt} \, dt.$$

Under the assumption on $f(t)$ one can easily compute that

$$I_2(k) = O(\frac{e^{-kR}}{k}), \quad k \to +\infty;$$

hence, the contribution from this term is negligible. Now consider the term $I_1(k)$. First, by the definition of the gamma function one has that

$$\int_0^{+\infty} t^{a+\beta j} e^{-kt} \, dt = \frac{\Gamma(1 + a + \beta j)}{k^{1+a+\beta j}};$$

thus,

$$\begin{aligned} \int_0^R t^{a+\beta j} e^{-kt} \, dt &= \int_0^{+\infty} t^{a+\beta j} e^{-kt} \, dt - \int_R^{+\infty} t^{a+\beta j} e^{-kt} \, dt \\ &= \frac{\Gamma(1 + a + \beta j)}{k^{1+a+\beta j}} + O(\frac{e^{-kR}}{k}), \quad k \to +\infty. \end{aligned}$$

The estimate on the second integral follows from an integration by parts. Since

$$I_1(k) = \int_0^R \left\{ \sum_{j=0}^{N} a_j t^{a+\beta j} + O(t^{a+\beta(N+1)}) \right\} e^{-kt} \, dt,$$

and

$$\begin{aligned} \int_0^R t^{a+\beta(N+1)} e^{-kt} \, dt &\le \int_0^{+\infty} t^{a+\beta(N+1)} e^{-kt} \, dt \\ &= \frac{\Gamma(1 + a + \beta(N + 1))}{k^{1+a+\beta(N+1)}}, \end{aligned}$$

one has that for each $N \in \mathbb{N}$,

$$I_1(k) = \sum_{j=0}^{N} a_j \frac{\Gamma(1 + a + \beta j)}{k^{1+a+\beta j}} + O(k^{-(1+a+\beta(N+1))}), \quad k \to +\infty.$$

The result now follows. □

Remark 11.6. Note that the asymptotic behavior of $I(k)$ is again determined by the behavior of $f(t)$ at $t = 0$.

For an example, consider the modified Bessel equation of order p, which is given by

$$k^2 \frac{d^2 w}{dk^2} - k \frac{dw}{dk} - (k^2 + p^2) w = 0.$$

It can be shown that a solution with $p = 0$ is given by $w(k) = \mathrm{e}^{-k} I(k)$, where

$$I(k) = \int_0^{+\infty} \frac{1}{\sqrt{t^2 + 2t}} \mathrm{e}^{-kt} \, \mathrm{d}t.$$

Upon using equation (11.7) and the result of Watson's lemma one has that

$$\begin{aligned} I(k) &\sim \sum_{j=0}^{\infty} \frac{c_j(-\frac{1}{2})}{2^{1/2+j}} \frac{\Gamma(1/2+j)}{k^{1/2+j}} \\ &= \sum_{j=0}^{\infty} \frac{\sqrt{\pi}}{j! 2^{1/2+j}} \frac{\Gamma(1/2+j)}{\Gamma(1/2-j)} \frac{1}{k^{1/2+j}} \\ &= \sqrt{\frac{\pi}{2}}\, k^{-1/2} + \mathcal{O}(k^{-3/2}), \quad k \to +\infty. \end{aligned}$$

The solution to the modified Bessel equation with $p = 0$ is then given by

$$\begin{aligned} w(k) &\sim \mathrm{e}^{-k} \sum_{j=0}^{\infty} \frac{\sqrt{\pi}}{j! 2^{1/2+j}} \frac{\Gamma(1/2+j)}{\Gamma(1/2-j)} \frac{1}{k^{1/2+j}} \\ &= \mathrm{e}^{-k}\Big(\sqrt{\frac{\pi}{2}}\, k^{-1/2} + \mathcal{O}(k^{-3/2})\Big), \quad k \to +\infty. \end{aligned}$$

11.3.3. Laplace's method

Now consider equation (11.5) in the case that $\phi(t) \not\equiv t$. First suppose that $\phi \in C^1(R)$ with $\phi'(t) \neq 0$. Upon setting

$$\tau := \phi(t) - \phi(a),$$

equation (11.5) becomes

$$I(k) = \mathrm{e}^{-k\phi(a)} \int_0^{\phi(b)-\phi(a)} \frac{f \circ \phi^{-1}}{\phi' \circ \phi^{-1}} (\tau + \phi(a)) \mathrm{e}^{-k\tau} \, \mathrm{d}\tau.$$

Hence, in this case the integral can be transformed into a type that has already been studied. Note that as a consequence of Watson's lemma, in order to evaluate $I(k)$ as $k \to +\infty$ one must only determine the behavior of

$$\frac{f \circ \phi^{-1}}{\phi' \circ \phi^{-1}} (\tau + \phi(a)) \quad \left(= \frac{f'(t)}{\phi'(t)} \right)$$

near $\tau = 0$, $(t = a)$.

Now suppose that there exists a $c \in (a, b)$ such that $\phi'(c) = 0$; furthermore, suppose that $\phi''(c) > 0$, and that $\phi'(t) \neq 0$ for all $t \in [a, b] \backslash \{c\}$. One can then write $I(k) = I_a(k) + I_b(k)$, where $I_a(k)$ and $I_b(k)$ are the integrals to be evaluated on $[a, c)$ and $(c, b]$, respectively. Consider $I_b(k)$. Upon using the above ideas one can write

$$I_b(k) = \mathrm{e}^{-k\phi(c)} \int_0^{\phi(b)-\phi(c)} \frac{f \circ \phi^{-1}}{\phi' \circ \phi^{-1}} (\tau + \phi(c)) \mathrm{e}^{-k\tau} \, \mathrm{d}\tau,$$

which can be evaluated upon using Watson's Lemma 11.5. Since

$$\phi(t) = \phi(c) + \frac{1}{2}\phi''(c)(t-c)^2 + \frac{1}{6}\phi'''(c)(t-c)^3 + \mathcal{O}((t-c)^4),$$

one has that

$$\tau = \frac{1}{2}\phi''(c)(t-c)^2 + \frac{1}{6}\phi'''(c)(t-c)^3 + \mathcal{O}((t-c)^4).$$

Solving the above equation recursively yields

$$t - c = \sqrt{\frac{2}{\phi''(c)}}\, \tau^{1/2} \left(1 - \frac{\phi'''(c)}{3\sqrt{2}\, \phi''(c)^{3/2}} \tau^{1/2} + \mathcal{O}(\tau) \right).$$

Upon using a Taylor expansion for $f(t)$ and $\varphi'(t)$ about $t = c$ and substituting the above expression for $t - c$ one finds that for $t \approx c$,

$$\frac{f(t)}{\varphi'(t)} = a_0 \tau^{-1/2} + a_1 + \mathcal{O}(\tau^{1/2}),$$

where

$$a_0 := \frac{f(c)}{\sqrt{2\varphi''(c)}}, \quad a_1 := \frac{f'(c)}{\varphi''(c)} - \frac{f(c)\varphi'''(c)}{3\varphi''(c)^2}. \tag{11.8}$$

Hence, as a consequence of Watson's Lemma 11.5 one can now write

$$I_b(k) \sim \mathrm{e}^{-k\varphi(c)}(a_0\sqrt{\pi}\, k^{-1/2} + a_1 k^{-1} + \mathcal{O}(k^{-3/2})), \quad k \to +\infty.$$

In a similar fashion, it can be shown that

$$I_a(k) \sim \mathrm{e}^{-k\varphi(c)}(a_0\sqrt{\pi}\, k^{-1/2} - a_1 k^{-1} + \mathcal{O}(k^{-3/2})), \quad k \to +\infty.$$

Adding then gives the following result.

Lemma 11.7 (Laplace's method)**.** *Consider equation (11.5), and suppose that $\varphi \in C^4(\mathbb{R})$ and $f \in C^2(\mathbb{R})$. If $c \in (a, b)$ is such that $\varphi'(c) = 0$ and $\varphi''(c) > 0$, and if $\varphi'(t) \neq 0$ for $t \in [a, b] \backslash \{c\}$, then*

$$I(k) \sim \mathrm{e}^{-k\varphi(c)} f(c) \sqrt{\frac{2\pi}{\varphi''(c)}}\, k^{-1/2}, \quad k \to +\infty.$$

Remark 11.8. One has that:

(a) if $c \in \{a, b\}$, then

$$I(k) \sim \frac{1}{2}\mathrm{e}^{-k\varphi(c)} f(c) \sqrt{\frac{2\pi}{\varphi''(c)}}\, k^{-1/2}, \quad k \to +\infty.$$

(b) if $\varphi''(c) < 0$, then

$$I(k) \sim \mathrm{e}^{k\varphi(c)} f(c) \sqrt{\frac{2\pi}{-\varphi''(c)}}\, k^{-1/2}, \quad k \to +\infty.$$

As an example, consider the gamma function

$$\Gamma(k+1) = \int_0^{+\infty} t^k \mathrm{e}^{-t}\, \mathrm{d}t.$$

Setting $s := t/k$ yields

$$\Gamma(k+1) = k^{k+1} \int_0^{+\infty} \mathrm{e}^{-k\varphi(s)}\, \mathrm{d}s, \quad \varphi(s) = s - \ln s.$$

Since $s = 1$ is a global minimum for $\varphi(s)$ with $\varphi(1) = \varphi''(1) = 1$ and $\varphi'(1) = 0$, upon applying Lemma 11.7 one has that

$$\Gamma(k+1) \sim \sqrt{2\pi k}\left(\frac{k}{\mathrm{e}}\right)^k.$$

This result is known as *Stirling's formula*.

For another example, consider

$$I(n) := \sum_{k=0}^{n} \frac{n!}{(n-k)!} n^{-k}.$$

Since

$$\int_0^{+\infty} x^k \mathrm{e}^{-nx}\, \mathrm{d}x = \Gamma(k+1) n^{-(k+1)} = k!\, n^{-(k+1)},$$

one can rewrite the above as

$$I(n) = \int_0^{+\infty} n \left(\sum_{k=0}^{n} \binom{n}{k} x^k \right) \mathrm{e}^{-nx}\, \mathrm{d}x.$$

Due to the fact that

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k,$$

one finally has that

$$I(n) = n\int_0^{+\infty} (1+x)^n e^{-nx}\,dx = n\int_0^{+\infty} e^{n(\ln(1+x)-x)}\,dx.$$

One can now evaluate $I(n)$ via Laplace's method with $\phi(x) := \ln(1+x) - x$. Note that $\phi(x)$ has a maximum at $x = 0$ with $\phi(x) = -x^2/2 + \mathcal{O}(x^3)$, so that for large n the primary contribution to the integral occurs at $x = 0$. For large n one then has that

$$I(n) \sim n\int_0^{+\infty} e^{-nx^2/2}\,dx = \sqrt{\frac{n}{2}}\int_0^{+\infty} x^{-1/2}e^{-x}\,dx = \sqrt{\frac{\pi}{2}}\,n^{1/2}, \quad n \to +\infty.$$

11.4. Fourier type integrals

Recalling the definition of the Fourier transform in equation (11.1), we shall consider

$$\lim_{k\to+\infty} I(k) := \int_a^b f(t)e^{ik\phi(t)}\,dt, \tag{11.9}$$

where $f, \phi \in C^1(\mathbb{R})$. If $\phi'(t) \neq 0$, then one can change variables as in Section 11.3.3 to get that

$$I(k) = \int_0^{\phi(b)-\phi(a)} \tilde{f}(t)e^{ikt}\,dt,$$

for a suitably defined $\tilde{f}(t)$. As it will be seen in Section 11.4.1, one will than have that $I(k) \sim \mathcal{O}(k^{-1})$ as $k \to +\infty$. Consequently, one expects that the major contribution to the integral will be at those points for which $\phi'(t) = 0$. Before this can be explored more fully, some preliminary results are needed.

11.4.1. Integration by parts

First suppose that $\phi(t) \equiv t$, and that $f \in C^{N+2}(\mathbb{R})$. Upon following the proof of Lemma 11.4, it can then be shown that

$$I(k) \sim \sum_{j=0}^{N} \frac{(-1)^j}{(ik)^{j+1}}(f^{(j)}(b)e^{ikb} - f^{(j)}(a)e^{ika}), \quad k \to +\infty.$$

Note that unlike the result of Lemma 11.4, the value of $I(k)$ depends upon the evaluation of $f(t)$ at both endpoints. For an example, consider

$$\lim_{\epsilon\to 0^+} I(\epsilon) := \int_0^{1/\epsilon} (1+\epsilon t)^{-1}e^{it}\,dt.$$

Note that the integral is well-defined for each $\epsilon \geq 0$. If one sets $\tau := \epsilon t$, then one sees that

$$I(\epsilon) = \frac{1}{\epsilon}\int_0^1 (1+\tau)^{-1}e^{i\tau/\epsilon}\,d\tau,$$

which is in the framework of the above result with $f(\tau) = (1+\tau)^{-1}$. Since

$$f^{(j)}(0) = (-1)^j j!, \quad f^{(j)}(1) = (-1)^j \frac{j!}{2^{j+1}}$$

for each $j \in \mathbb{N}_0$, one has that

$$I(\epsilon) \sim e^{i/\epsilon}\left(\frac{1}{2i} + \cdots + \frac{j!}{(2i)^{j+1}}\epsilon^j + \cdots\right) - \left(\frac{1}{i} + \cdots + \frac{j!}{(i)^{j+1}}\epsilon^j + \cdots\right), \quad \epsilon \to 0^+;$$

in particular, one has that

$$I(\epsilon) \sim \mathrm{i}(1 - \frac{1}{2}\mathrm{e}^{\mathrm{i}/\epsilon}) + (1 - \frac{1}{4}\mathrm{e}^{\mathrm{i}/\epsilon})\epsilon, \quad \epsilon \to 0^+.$$

For a comparison with Laplace type integrals, recall that in the example following Lemma 11.4 it was seen that

$$\lim_{\epsilon\to 0^+} \int_0^{1/\epsilon} (1 + \epsilon t)^{-1}\mathrm{e}^{-t}\,\mathrm{d}t \sim 1 - \epsilon.$$

11.4.2. Analogue of Watson's lemma

Now assume that integration by parts fails, i.e., $f(t) = at^\gamma + \mathrm{o}(t^\gamma)$ as $t \to 0^+$ for some $\gamma > -1$. If one further assumes that $f(t)$ and all of its derivatives vanish at $t = b$, then one may expect that for

$$I(k) = \int_0^b f(t)\mathrm{e}^{\mathrm{i}\mu k t}\,\mathrm{d}t, \quad \mu \in \mathbb{R}\backslash\{0\},$$

one has

$$I(k) \sim a\int_0^{+\infty} t^\gamma \mathrm{e}^{\mathrm{i}\mu k t}\,\mathrm{d}t \quad k \to +\infty.$$

Although the estimates leading to this conclusion will not be shown herein, it can indeed be justified. One then needs to evaluate the above integral.

First suppose that $\mu > 0$. Since the integrand is holomorphic in the first quadrant, as an application of the residue theorem one can rotate the contour to the positive imaginary axis. This yields

$$\begin{aligned} I(k) &\sim a\int_0^{+\mathrm{i}\infty} t^\gamma \mathrm{e}^{\mathrm{i}\mu k t}\,\mathrm{d}t \\ &= a\mathrm{e}^{\mathrm{i}\pi(1+\gamma)/2}\int_0^{+\infty} t^\gamma \mathrm{e}^{-\mu k t}\,\mathrm{d}t \\ &= a\frac{\mathrm{e}^{\mathrm{i}\pi(1+\gamma)/2}}{(\mu k)^{1+\gamma}}\Gamma(1+\gamma). \end{aligned}$$

If $\mu < 0$, then upon rotating the contour to the negative imaginary axis one finds that

$$I(k) \sim a\frac{\mathrm{e}^{-\mathrm{i}\pi(1+\gamma)/2}}{(-\mu k)^{1+\gamma}}\Gamma(1+\gamma).$$

Thus, one can conclude that

$$I(k) \sim a\frac{\mathrm{e}^{\mathrm{sign}(\mu)\mathrm{i}\pi(1+\gamma)/2}}{(|\mu|)^{1+\gamma}}\Gamma(1+\gamma)k^{1+\gamma}.$$

Remark 11.9. More generally, it can be shown that

$$\int_0^{+\infty} t^\gamma \mathrm{e}^{\mathrm{i}\mu t^p}\,\mathrm{d}t = \left(\frac{1}{|\mu|}\right)^{(1+\gamma)/p}\frac{\Gamma((1+\gamma)/p)}{p}\mathrm{e}^{\mathrm{sign}(\mu)\mathrm{i}\pi(1+\gamma)/2p}, \quad p \in \mathbb{N},$$

[**1**, Example 6.3.3].

11.4.3. The stationary phase method

Now consider equation (11.9) for a general $\phi(t) \in C^2(\mathbb{R})$. It is expected that a major contribution to $I(k)$ will be at those points c which satisfy $\phi'(c) = 0$, i.e., those points for which the phase is *stationary*. At such a point

$$\phi(t) = \phi(c) + \frac{1}{2}\phi''(c)(t-c)^2 + \mathcal{O}((t-c)^2),$$

and

$$I(k) \sim \int_{c-\delta}^{c+\delta} f(c) \exp\left[\mathrm{i}k(\phi(c) + \frac{1}{2}\phi''(c)(t-c)^2)\right] \mathrm{d}t, \quad 0 < \delta \ll 1.$$

In order to evaluate this integral, for $\mu := \operatorname{sign}(\phi''(c))$ set

$$\mu\tau^2 = \frac{1}{2}k\phi''(c)(t-c)^2,$$

so that

$$I(k) \sim f(c)\mathrm{e}^{\mathrm{i}\phi(c)k}\sqrt{\frac{2}{|\phi''(c)|k}}\int_{-\delta\sqrt{|\phi''(c)|k/2}}^{\delta\sqrt{|\phi''(c)|k/2}} \mathrm{e}^{\mathrm{i}\mu\tau^2}\,\mathrm{d}\tau.$$

Upon letting $k \to +\infty$ the integral can be evaluated as

$$\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\mu\tau^2}\,\mathrm{d}\tau = 2\int_0^{+\infty} \mathrm{e}^{\mathrm{i}\mu\tau^2}\,\mathrm{d}\tau = \sqrt{\pi}\mathrm{e}^{\mathrm{i}\mu\pi/4}.$$

Thus, one expects that

$$I(k) \sim f(c)\mathrm{e}^{\mathrm{i}\phi(c)k}\sqrt{\frac{2\pi}{|\phi''(c)|}}\mathrm{e}^{\mathrm{i}\mu\pi/4}k^{-1/2}. \tag{11.10}$$

This argument can be made rigorous if $f(t)$ and all of its derivatives vanish at the end points $t = a$ and $t = b$ [**1**, Lemma 6.3.3].

Remark 11.10. Note that the contribution at the point $t = c$ is of $O(k^{-1/2})$, whereas it was seen in Section 11.4.1 that any contribution from the end points is expected to be of $O(k^{-1})$.

For an example, consider

$$I(k) := \int_{1/2}^{2} (1+t)\mathrm{e}^{\mathrm{i}k\phi(t)}\,\mathrm{d}t, \quad \phi(t) := -t + \frac{1}{3}t^3, \quad k \to +\infty.$$

Using the method of stationary phase, the dominant contribution will occur at those points for which $\phi'(t) = 0$, i.e., $t = 1$. Since $\phi(1) = -2/3$ and $\phi''(1) = 2$, by equation (11.10) one sees that

$$I(k) \sim 2\sqrt{\pi}\,\mathrm{e}^{-\mathrm{i}(2k/3-\pi/4)}k^{-1/2}, \quad k \to +\infty.$$

Now suppose that $f(t) \sim \beta(t-c)^\gamma + O((t-c)^\gamma)$ for some $\gamma > -1$. In this case the above argument must be modified. Let $\phi(t) - \phi(c) \sim \alpha(t-c)^2 + O(t-c)^2)$, and set

$$\mu\tau := \phi(t) - \phi(c), \quad \mu = \operatorname{sign}(\alpha),$$

i.e.,

$$t - c \sim (\tau/|\alpha|)^{1/2}.$$

Using the idea behind the proof of Laplace's method and splitting the integral into the two parts $I_a(k)$ and $I_b(k)$ yields

$$I_b(k) = \mathrm{e}^{\mathrm{i}k\phi(c)}\int_0^{|\phi(b)-\phi(c)|} F(\tau)\mathrm{e}^{\mathrm{i}\mu k\tau}\,\mathrm{d}\tau, \quad F(\tau) := \mu\frac{f(t)}{\phi'(t)}.$$

It is expected that the dominant contribution will occur at $\tau = 0$. Since

$$F(\tau) \sim \frac{\beta}{2|\alpha|}(t-c)^{\gamma-1} \sim \frac{\beta}{2|\alpha|^{(1+\gamma)/2}}\tau^{(\gamma-1)/2},$$

as a consequence of the results in Section 11.4.2 one can conclude that

$$I_b(k) = \frac{1}{2}\beta\mathrm{e}^{\mathrm{i}k\phi(c)}\Gamma\left(\frac{1}{2}(1+\gamma)\right)(k|\alpha|)^{-(1+\gamma)/2}\mathrm{e}^{\mathrm{i}\mu\pi(1+\gamma)/4} + O(k^{-(1+\gamma)/2}), \quad k \to +\infty.$$

The term $I_a(k)$ is computed in the same manner, and one finds that to leading order $I_a(k) = I_b(k)$. Hence,

$$I(k) \sim \beta\mathrm{e}^{\mathrm{i}k\phi(c)}\Gamma\left(\frac{1}{2}(1+\gamma)\right)(|\alpha|)^{-(1+\gamma)/2}\mathrm{e}^{\mathrm{i}\mu\pi(1+\gamma)/4}k^{-(1+\gamma)/2} + o(k^{-(1+\gamma)/2}), \quad k \to +\infty.$$

Remark 11.11. Note that if $\gamma \geq 1$, then as a consequence of the discussion in Section 11.4.1 one expects that the dominant terms will arise via an integration by parts.

11.5. The method of steepest descent

Now consider

$$I(k) := \oint_C f(z)\mathrm{e}^{k\varphi(z)}\,\mathrm{d}z, \quad k \to +\infty, \tag{11.11}$$

where $C \subset \mathbb{C}$ is a given contour, and $f(z)$ and $\varphi(z)$ are holomorphic. The basic idea in evaluating $I(k)$ is to deform the contour C to a new contour C', on which one can more easily perform calculations and make estimates. In particular, if one writes $\varphi = u + \mathrm{i}v$, suppose that $v \equiv \beta$ on C'. One then has that

$$I(k) = \mathrm{e}^{\mathrm{i}k\beta} \oint_{C'} f(z)\mathrm{e}^{ku(z)}\,\mathrm{d}z, \quad k \to +\infty,$$

so that $I(k)$ is a Laplace type integral. However, since the curve will no longer generally coincide with the line $\operatorname{Im} z = 0$, one has to be careful in making the necessary estimates.

Remark 11.12. In general, Laplace type integrals are preferred to Fourier type integrals, as in the former case one can often derive an infinite asymptotic expansion, whereas in the latter case one often only finds the first term in the expansion.

Now, on the curve C' one has that $\nabla v = (v_x, v_y)$ points in the direction for which v changes most rapidly. As a consequence of the Cauchy-Riemann equations one has that $\nabla v = (-u_y, u_x)$, i.e., ∇v is orthogonal to ∇u. Hence, the curve C' is the curve on which u changes most rapidly. For this reason the above deformation of the original contour C is known as the *method of steepest descent*. The question to next be answered is: how does one choose the constant β? In Lemma 11.7 it was seen that the dominant contribution is determined at those points $z_0 \in \mathbb{C}$ for which $\varphi'(z_0) = 0$. Such a point is known as a saddle point, and it is known as a saddle point of order N if $\varphi^{(j)}(z_0) = 0$ for $j = 1, \dots, N$. If $N = 1$, then z_0 is known as a simple saddle point. Thus, one would like to deform C to the curve $v(x, y) = v(x_0, y_0)$, where $z_0 = x_0 + \mathrm{i}y_0$ is a saddle point. However, this may be problematic, for it may not be possible due to the presence of singularities. In this case the endpoints or singularities will yield the dominant contribution. Furthermore, if there are multiple saddle points, then one must take care to determine which yields the dominant contribution.

Suppose that $z_0 \in \mathbb{C}$ is a saddle point of order N, and that $\varphi^{(N+1)}(z_0) = a\mathrm{e}^{\mathrm{i}\alpha}$, where $a \in \mathbb{R}^+$. If one sets $z - z_0 = \rho\mathrm{e}^{\mathrm{i}\partial}$, then near the point $z = z_0$ one has

$$\begin{aligned}\varphi(z) - \varphi(z_0) &\sim \frac{(z-z_0)^{N+1}}{(N+1)!}\varphi^{(N+1)}(z_0) \\ &= a\frac{\rho^{N+1}}{(N+1)!}\mathrm{e}^{\mathrm{i}(\alpha+(N+1)\partial)}.\end{aligned}$$

Since the directions of steepest descent are defined by $v(z) = v(z_0)$, one must then minimally have that $\sin(\alpha + (N+1)\partial) = 0$. In order to determine which directions are those corresponding to descent, one must also include the condition that $\cos(\alpha + (N+1)\partial) < 0$. Thus, the steepest descent directions are given by

$$\partial = \frac{-\alpha + (2m+1)\pi}{N+1}, \quad m = 0, \dots, N. \tag{11.12}$$

Note that for a simple saddle point the directions of steepest descent are given by

$$\partial = \frac{-\alpha + \pi}{2}, \quad \partial = \frac{-\alpha + 3\pi}{2}.$$

For example, suppose that $\varphi(z) = z - z^3/3$. One has that there exist two simple saddle points at $z_0 = \pm 1$. For $z_0 = -1$ one has that $\varphi''(-1) = 2$, so that $\alpha = 0$. The directions are then given by $\partial = \pi/2$ and $\partial = 3\pi/2$. for $z_0 = 1$ one has that $\varphi''(1) = 2\mathrm{e}^{\mathrm{i}\pi}$, so that the directions of steepest descent are $\partial = 0$ and $\partial = \pi$. The situation is depicted in Figure 14:

11.5.1. Laplace's method for complex contours

Let $z_0 \in \mathbb{C}$ be a saddle point, and let C_s be a steepest descent curve which passes through z_0. If C_1 is a descent curve which coincides with C_s only for some finite length near C_s, i.e., C_1 is asymptotically equivalent

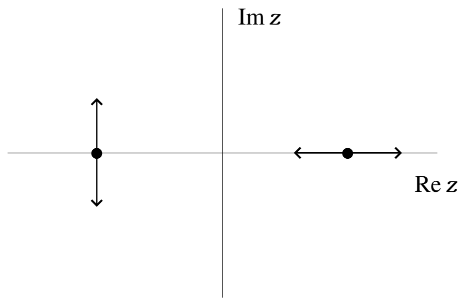


Figure 14: Steepest descent directions for $\varphi(z) = z - z^3/3$.

to C_s, then it can rigorously be shown that the integral over C_1 differs from that over C_s by an exponentially small quantity. Hence, one has that only the asymptotically equivalent contours are important. This may be useful in certain applications in which it is not clear as to how one should deform the original contour to the steepest descent contour.

Assume that the original contour C can be deformed onto the steepest descent contour C_s which passes through the saddle point z_0 which is of order $n-1$. Since $\mathrm{Im}(\varphi(z) - \varphi(z_0)) = 0$ along C_s, one can write for $t \in \mathbb{R}^+$,

$$-t = \varphi(z) - \varphi(z_0) = (z - z_0)^n \hat{\varphi}(z), \tag{11.13}$$

where

$$\hat{\varphi}(z) = \sum_{j=0}^{\infty} \hat{\varphi}_j (z - z_0)^j, \quad \hat{\varphi}_j := \frac{\varphi^{(n+j)}(z_0)}{(n+j)!}.$$

There are n roots to this equation. As an application of the Implicit Function Theorem one can, if desired, write the series expansion in powers of $t^{1/n}$ for each of these roots. In practice, however, one often only needs the first term. In this case one then has

$$-t \sim \frac{\varphi^{(n)}(z_0)}{n!}(z - z_0)^n,$$

which yields

$$|z - z_0| \sim \left(\frac{n!}{|\varphi^{(n)}(z_0)|}\right)^{1/n} t^{1/n}.$$

Note that

$$z - z_0 = |z - z_0| \mathrm{e}^{\mathrm{i}\vartheta},$$

where the angle ϑ is given in equation (11.12).

Upon using the change of variables defined by equation (11.13), one has the the integral in equation (11.11) along C_s becomes

$$I_s(k) \sim -\mathrm{e}^{k\varphi(z_0)} \int_0^{+\infty} \frac{f(z)}{\varphi'(z)} \, \mathrm{e}^{-kt} \, \mathrm{d}t.$$

The upper limit of integration is replaced by $+\infty$, as from Watson's Lemma 11.5 one knows that the dominant contribution to the integral comes from the neighborhood of the origin. Now,

$$\varphi'(z) \sim \varphi^{(n)}(z_0) \frac{(z - z_0)^{n-1}}{(n-1)!}.$$

If near the saddle point one assumes that

$$f(z) \sim f_0 (z - z_0)^{\beta - 1}, \quad \mathrm{Re}\, \beta > 0,$$

then one has that

$$\begin{aligned}\frac{f(z)}{\phi'(z)} &\sim (n-1)!\frac{f_0}{\phi^{(n)}(z_0)}(z-z_0)^{\beta-n}\\ &= -\frac{f_0(n!)^{\beta/n}\mathrm{e}^{\mathrm{i}\beta\vartheta}}{n|\phi^{(n)}(z_0)|^{\beta/n}}t^{\beta/n-1}.\end{aligned}$$

In the second step one implicitly uses the facts that $\phi^{(n)}(z_0) = |\phi^{(n)}(z_0)|\mathrm{e}^{\mathrm{i}\alpha}$ and $\alpha + n\vartheta = (2j+1)\pi$. In conclusion, one has that along the curve C_s,

$$\begin{aligned}I_s(k) &\sim \mathrm{e}^{k\phi(z_0)+\mathrm{i}\beta\vartheta}\frac{f_0(n!)^{\beta/n}}{n|\phi^{(n)}(z_0)|^{\beta/n}}\int_0^{+\infty} t^{\beta/n-1}\mathrm{e}^{-kt}\,\mathrm{d}t\\ &= \mathrm{e}^{\mathrm{i}\beta\vartheta}\frac{f_0(n!)^{\beta/n}}{n}\frac{\Gamma(\beta/n)}{|\phi^{(n)}(z_0)|^{\beta/n}}\mathrm{e}^{k\phi(z_0)}k^{-\beta/n}.\end{aligned} \tag{11.14}$$

It must be kept in mind here that the above result holds only only *one* of the steepest descent curves. Since there are n such curves, one must carefully consider the contribution to $I(k)$ of each curve. This will be illustrated in some of the following examples.

First consider the Hankel function

$$H_\nu^{(1)}(k) := \frac{1}{\pi}\oint_C \mathrm{e}^{\mathrm{i}k\cos z}\mathrm{e}^{\mathrm{i}\nu(z-\pi/2)}\,\mathrm{d}z,\quad k\to+\infty,$$

where the contour C is given in Figure 15. $H_\nu^{(1)}(k)$ is a solution to Bessel's equation,

$$\frac{\mathrm{d}^2u}{\mathrm{d}k^2}+\frac{1}{k}\frac{\mathrm{d}u}{\mathrm{d}k}+\left(1-\frac{\nu^2}{k^2}\right)u=0,$$

which decays for $\nu\in\mathbb{R}$ as $k\to+\infty$. A derivation of this fact is given in [6, Chapter 7.2.1]. Let us first determine the leading order term in the asymptotic expansion of $H_\nu^{(1)}(k)$. Since $\phi(z) := \mathrm{i}\cos z$, one has that the simple saddle points are given by $z_n := n\pi$ for $n\in\mathbb{Z}$. Given the choice of the contour C, it is clear that the only relevant saddle point is $z_0 = 0$. Since $\phi''(0) = \mathrm{e}^{-\mathrm{i}\pi/2}$, upon using equation (11.12) one sees that the steepest descent directions are $\vartheta = 3\pi/4$ (depicted by $\hat{C}_s^1$ in Figure 15) and $\vartheta = 7\pi/4$ (depicted by $\hat{C}_s^2$ in Figure 15). Now, to leading order one has that

$$\oint_C \sim \oint_{\hat{C}_s^2} - \oint_{\hat{C}_s^1}.$$

Upon evaluating each of the integrals on the right hand side via equation (11.14) with

$$f_0 = \frac{\mathrm{e}^{-\mathrm{i}\nu\pi/2}}{\pi},\quad \beta = 1,$$

one finally sees that

$$H_\nu^{(1)}(k) \sim \sqrt{\frac{2}{\pi}}\,\mathrm{e}^{\mathrm{i}(k-\nu\pi/2-\pi/4)}k^{-1/2},\quad k\to+\infty. \tag{11.15}$$

Now let us determine the higher-order asymptotic corrections to the integral. It will then be necessary to achieve better than a linear approximation to C_s at the saddle point. The steepest descent curve is globally defined by $\operatorname{Im}\phi(z) = \operatorname{Im}\phi(0)$, i.e.,

$$\cos x\cosh y = 1.$$

Here the identity $\cos(x+\mathrm{i}y) = \cos x\cosh y - \mathrm{i}\sin x\sinh y$ was used. Using the Taylor expansion about $z_0 = 0$ yields

$$\left(1-\frac{1}{2}x^2+\cdots\right)\left(1+\frac{1}{2}y^2+\cdots\right)=1,$$

i.e.,

$$\frac{1}{2}(y-x)(y+x)\sim 0.$$

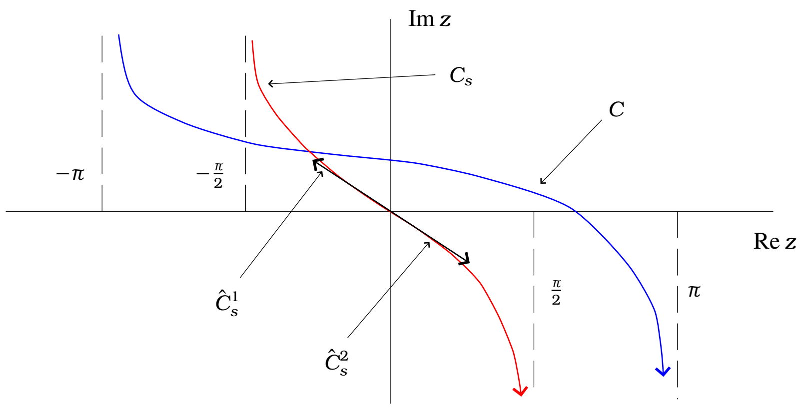


Figure 15: Contours for the Hankel function. The contour C is the original contour, C_s is the global steepest descent contour, and $\hat{C}_s^1 \cup \hat{C}_s^2$ is the local steepest descent contour.

Thus, one recovers the local portion of the steepest descent curve, $\arg z = 3\pi/4$. As $|y| \to +\infty$ one has that $\cos x \sim 2e^{-|y|}$, so that $x \sim \pi/2$ as $y \to -\infty$ and $x \sim -\pi/2$ as $y \to +\infty$. This qualitative information yields that we can deform C onto C_s, as depicted in Figure 15.

The steepest descent transformation is given by

$$-t = \phi(z) - \phi(z_0),$$

which near the saddle point yields

$$\frac{1}{2}z^2 - \frac{1}{4!}z^4 + \cdots = e^{-i\pi/2}t.$$

Solving the above recursively then gives that near the saddle point,

$$z = \sqrt{2}\,e^{-i\pi/4}t^{1/2} + \frac{\sqrt{2}}{12}e^{-i3\pi/4}t^{3/2} + \cdots.$$

Asymptotically one then has that

$$\begin{aligned} H_\nu^{(1)}(k) &\sim \frac{e^{i(k-\nu\pi/2)}}{\pi}\oint_{C_s} e^{-kt}e^{i\nu z}\frac{dz}{dt}\,dt \\ &\sim \frac{e^{i(k-\nu\pi/2)}}{\pi}\oint_{C_s} e^{-kt}\left(1 + i\nu z + \frac{1}{2}(i\nu z)^2 + \cdots\right)\frac{dz}{dt}\,dt. \end{aligned}$$

Upon substituting the expression for z into the above, collecting terms, and using Watson's lemma, one eventually sees that as $k \to +\infty$,

$$H_\nu^{(1)}(k) \sim \frac{2e^{i(k-\nu\pi/2)}}{\pi}\left(\frac{\sqrt{2\pi}}{2}e^{-i\pi/4}k^{-1/2} + \nu k^{-1} + \frac{\sqrt{2\pi}}{4}(\frac{1}{4} - \nu^2)e^{-3i\pi/4}k^{-3/2} + \frac{1}{3}i\nu(\nu^2 - 1)k^{-2} + \cdots\right).$$

The intermediate details are given in [**1**, Example 6.4.3]. Note that the above result agrees to leading order with that of equation (11.15).

For the second example, let us determine the full asymptotic expansion for

$$I(k) := \int_0^1 e^{ikt^2}\,dt, \quad k \to +\infty. \tag{11.16}$$

Note that the stationary phase method outlined in Section 11.4.3 yields that to leading order,

$$I(k) \sim \frac{\sqrt{\pi}}{2}e^{i\pi/4}k^{-1/2}, \quad k \to +\infty.$$

Since $\phi(z) := \mathrm{i}z^2$, one has that $z_0 = 0$ is a simple saddle point. Furthermore, the steepest descent and ascent paths are given by $\operatorname{Im}\phi(z) = \beta \in \mathbb{R}$, i.e., $x^2 - y^2 = \beta$, where $x := \operatorname{Re} z$ and $y := \operatorname{Im} z$. Thus, the steepest descent and ascent paths through the end points of the integral are given by $x = \pm y$ and $x = \sqrt{1+y^2}$. It is clear that $\mathrm{e}^{\phi(z)}$ is increasing along the path $x = -y$; hence, the steepest descent path is given by $x = y$. Since $\operatorname{Im}\phi(0) \neq \operatorname{Im}\phi(1)$, there is no continuous contour joining $z = 0$ and $z = 1$ on which $\operatorname{Im}\phi(z)$ is constant. Hence, the two steepest descent contours will be joined by the contour C_2 (see Figure 16).

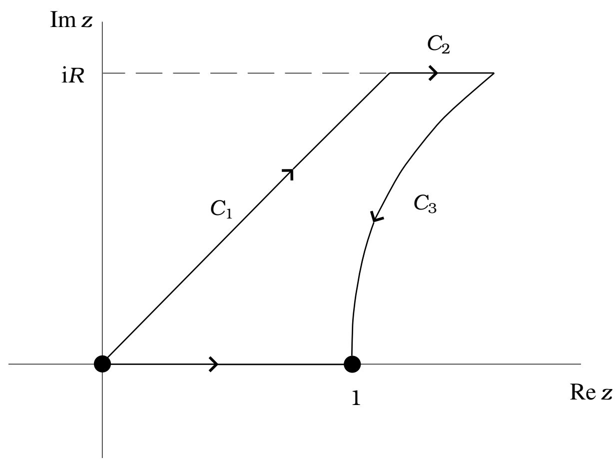


Figure 16: Steepest descent contours for equation (11.16).

As a consequence of Cauchy's theorem one has that

$$I(k) = \oint_{C_1} \mathrm{e}^{\mathrm{i}kz^2}\,\mathrm{d}z + \oint_{C_2} \mathrm{e}^{\mathrm{i}kz^2}\,\mathrm{d}z + \oint_{C_3} \mathrm{e}^{\mathrm{i}kz^2}\,\mathrm{d}z.$$

It is clear that

$$\lim_{R\to+\infty} \oint_{C_2} \mathrm{e}^{\mathrm{i}z^2}\,\mathrm{d}z = 0,$$

so that

$$I(k) = \lim_{R\to+\infty}\left(\oint_{C_1} \mathrm{e}^{\mathrm{i}kz^2}\,\mathrm{d}z + \oint_{C_3} \mathrm{e}^{\mathrm{i}kz^2}\,\mathrm{d}z\right).$$

A simple change of variables yields that

$$\lim_{R\to+\infty} \oint_{C_1} \mathrm{e}^{\mathrm{i}kz^2}\,\mathrm{d}z = \mathrm{e}^{\mathrm{i}\pi/4}\int_0^{+\infty} \mathrm{e}^{-kr^2}\,\mathrm{d}r = \frac{\sqrt{\pi}}{2}\mathrm{e}^{\mathrm{i}\pi/4}k^{-1/2},$$

so all that is left to do is evaluate the second integral. Along C_3 one has that the contour is given by $z = \sqrt{1+y^2} + \mathrm{i}y$, so that $z^2 = 1 + \mathrm{i}2y\sqrt{1+y^2}$. Setting $s := 2y\sqrt{1+y^2}$ then yields that

$$\lim_{R\to+\infty} \oint_{C_3} \mathrm{e}^{\mathrm{i}kz^2}\,\mathrm{d}z = -\frac{1}{2}\mathrm{i}\mathrm{e}^{\mathrm{i}k}\int_0^{+\infty} \frac{\mathrm{e}^{-ks}}{\sqrt{1+\mathrm{i}s}}\,\mathrm{d}s.$$

By equation (11.6) one has that for $s \in D(0,1)$,

$$(1+\mathrm{i}s)^{-1/2} = \sum_{j=0}^{\infty} \mathrm{i}^j c_j s^j, \quad c_j := \frac{\sqrt{\pi}}{j!\,\Gamma(1/2-j)},$$

so that as a consequence of Watson's Lemma 11.5 one has that

$$\int_0^{+\infty} \frac{\mathrm{e}^{-ks}}{\sqrt{1+\mathrm{i}s}}\,\mathrm{d}s \sim \sum_{j=0}^{\infty} \mathrm{i}^j \frac{\sqrt{\pi}}{\Gamma(1/2-j)} k^{-(1+j)}, \quad k \to +\infty.$$

In conclusion, one then has that

$$I(k) \sim \frac{\sqrt{\pi}}{2}\left(e^{i\pi/4}k^{-1/2} - e^{ik}\sum_{j=0}^{\infty}\frac{i^{1+j}}{\Gamma(1/2-j)}k^{-(1+j)}\right), \quad k \to +\infty.$$

Note that, as expected, the dominant contribution to the evaluation of $I(k)$ is due to the saddle point, and is captured via the method of stationary phase.

As a final example, consider

$$I(k) := \int_{-\infty}^{+\infty}\frac{e^{ik(t+t^3/3)}}{t^2+t_0^2}\,dt, \quad k \to +\infty, \tag{11.17}$$

where $t_0 \in \mathbb{R}^+$. The method of stationary phase is not immediately applicable for this problem, as the stationary points of $\phi(z) := i(z+z^3/3)$ are not real-valued. Hence, one must use the method of steepest descent. It will now be assumed that $t_0 \neq 1$. The case of $t_0 = 1$ will be left to the interested student.

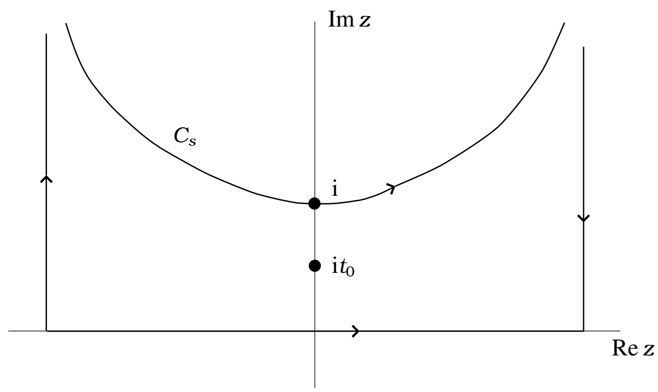


Figure 17: The integration contour for equation (11.17). In this case $t_0 \in (0,1)$.

The saddle points are given by $z_0 := \pm i$, with $\phi''(\pm i) = \mp 2$. Recalling equation (11.12), one has that for $z_0 = i$ the steepest descent directions are $\vartheta = 0, \pi$, whereas for $z_0 = -i$ the directions are $\vartheta = \pi/2, 3\pi/2$. Since

$$\phi(z) = \frac{1}{3}y(y^2 - 3y - 3x^2) + i\frac{1}{3}x(x^2 - 3y^2 + 3),$$

one has that the steepest descent curves through the saddle points are given by

$$x(3y^2 - x^2 - 3) = 0.$$

One must determine which of the two curves to use. Upon noting that the integral does not converge on the curve $\mathrm{Re}\,z \equiv 0$, one immediately sees that the only possible relevant curve is $3y^2 - x^2 - 3 = 0$. Since $e^{\phi(z)}$ decays for $\arg z \in (0, \pi/3)$ and $\arg z \in (2\pi/3, \pi)$, and since these sectors contain the steepest descent curve (note that the curves are asymptotic to $\arg z = \pi/6$ for $\mathrm{Re}\,z > 0$ and $\arg z = 5\pi/6$ for $\mathrm{Re}\,z < 0$), one can safely deform onto this curve.

Consider the contour given in Figure 17. The leading order steepest descent contribution is given in equation (11.14). As in the computation of the asymptotics for the Hankel function, one must subtract the computation of $\vartheta = \pi$ from that of $\vartheta = 0$. Upon doing so one sees that to leading order,

$$\oint_{C_s}\frac{e^{ik(t+t^3/3)}}{t^2+t_0^2}\,dt \sim \frac{\sqrt{\pi}}{t_0^2-1}e^{-2k/3}k^{-1/2}, \quad k \to +\infty.$$

Upon using Cauchy's theorem and calculating the pole contribution one then gets that as $k \to +\infty$,

$$I(k) \sim \frac{\sqrt{\pi}}{t_0^2-1}e^{-2k/3}k^{-1/2} + \frac{\pi}{t_0}e^{-k(t_0-t_0^3/3)}H(1-t_0),$$

where $H(\cdot)$ represents the Heaviside function. Note that for $t_0 \in (0,1)$, the pole contribution dominates that from the saddle point.

11.6. Applications

First consider the linear Schrödinger equation

$$\mathrm{i}\frac{\partial q}{\partial t}+\frac{\partial^2 q}{\partial x^2}=0,\quad q(x,0)=q_0(x).$$

Upon using the Fourier transform as in Section 11.2 it can be shown that the solution is given by

$$q(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{q}_0(k)\mathrm{e}^{\mathrm{i}(kx-k^2t)}\,\mathrm{d}k,$$

where $\hat{q}_0(k)$ is the Fourier transform of $q_0(x)$. Let us determine the large t behavior of the solution for x/t fixed. Set

$$\phi(k):=\frac{x}{t}k-k^2,$$

and rewrite the solution as

$$q(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{q}_0(k)\mathrm{e}^{\mathrm{i}t\phi(k)}\,\mathrm{d}k.$$

Since

$$\phi'(k)=\frac{x}{t}-2k,\quad \phi''(k)=-2,$$

one has that the dominant asymptotic result is achieved along the line

$$\frac{x}{t}=2k.$$

The quantity $c_g(k):=2k$ is called the *group velocity*. One can now use the method of stationary phase and the result of equation (11.10) to conclude that as $t\to+\infty$, and along the line $x=2kt$,

$$q(x,t)\sim\sqrt{\pi}\,\mathrm{e}^{-\mathrm{i}\pi/4}\hat{q}_0(k)\frac{\mathrm{e}^{\mathrm{i}k^2t}}{t^{1/2}}.$$

Note that in addition to decaying, the solution also oscillates.

Remark 11.13. The significance of this velocity is that for $t\gg 1$, each wave number k dominates the solution in the region $x\sim c_g(k)t$. In general, if the solution to a particular dispersive PDE is given by

$$q(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{q}_0(k)\mathrm{e}^{\mathrm{i}(kx-\omega(k)t)}\,\mathrm{d}k,$$

where $\omega(k)\in\mathbb{R}$ is the dispersion relationship, then the group velocity is given by $c_g(k)=\omega'(k)$. As above, each wave number k dominates the solution in the region $x\sim\omega'(k)t$.

The linear Korteweg-de Vries (KdV) equation,

$$\frac{\partial u}{\partial t}+\frac{\partial^3 u}{\partial x^3}=0,\quad u(x,0)=u_0(x),\tag{11.18}$$

was considered in Section 11.2. Therein it was shown that the solution is given by

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{u}_0(k)\mathrm{e}^{\mathrm{i}(kx+k^3t)}\,\mathrm{d}k,$$

where $\hat{u}_0(k)$ is the Fourier transform of $u_0(x)$. Let us now derive the solution estimates given at that time.

First, it will be assumed that $\hat{u}_0(k)$ is entire. Upon checking the definition of the Fourier transform, it is clear that this condition will hold if $u_0(x)=\mathcal{O}(\mathrm{e}^{-\delta x^2})$ for some $0<\delta\ll 1$. This step is taken to ensure that Cauchy's theorem can be used whenever one wishes to do so. Rewrite the solution as

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{u}_0(k)\mathrm{e}^{t\phi(k)}\,\mathrm{d}k,\quad \phi(k):=\mathrm{i}\left(\frac{x}{t}k+k^3\right).$$

The stationary points satisfy

$$\frac{x}{t} = -3k^2,$$

so that they are given by

$$k_{\pm} := \pm\sqrt{-\frac{x}{3t}}.$$

Let $\epsilon > 0$ be given, and first assume that $x/t \le -\epsilon$. Upon using the method of stationary phase one sees that to leading order

$$u(x,t) \sim \frac{1}{2\sqrt{\pi k_+ t}}\left(\hat{u}_0(-k_+)\mathrm{e}^{\mathrm{i}(2k_+^3 t-\pi/4)} + \hat{u}_0(k_+)\mathrm{e}^{-\mathrm{i}(2k_+^3 t-\pi/4)}\right).$$

Upon using the fact that $u_0(x) \in \mathbb{R}$ one immediately sees that $\hat{u}_0(-k_+) = \overline{\hat{u}_0}(k_+)$; hence, upon writing $\hat{u}_0(k_+) = \rho(k_+)\mathrm{e}^{\mathrm{i}\psi(k_+)}$ one gets that

$$u(x,t) \sim \frac{\rho(k_+)}{2\sqrt{\pi k_+}}\frac{\cos(2k_+^3 t - \pi/4 - \psi(k_+))}{t^{1/2}}. \tag{11.19}$$

Note that for fixed k_+ one has that $u(x,t) = \mathcal{O}(t^{-1/2})$, with the decay possessing an oscillatory nature.

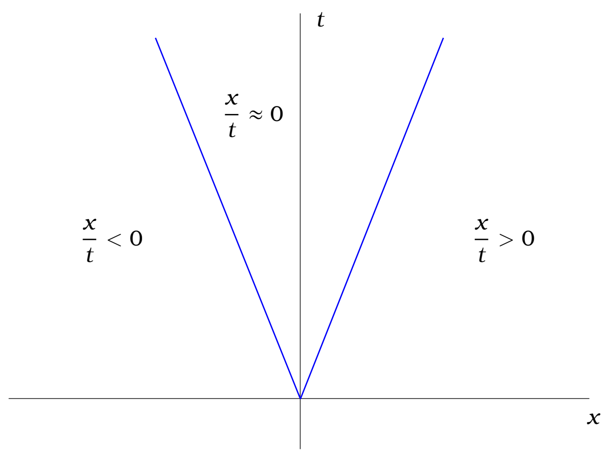


Figure 18: Regions in which the asymptotic formulas for the solution to equation (11.18) are valid. The solution behaves as in equation (11.19) for $x/t < 0$, as in equation (11.20) for $x/t > 0$, and as in equation (11.21) for $x/t \sim 0$.

Now assume that $x/t \ge \epsilon$. It is here that we will use the fact that $\hat{u}_0(k)$ is entire. In this case one has that the stationary points are $k_\pm \in \mathrm{i}\mathbb{R}$, so that the method of stationary phase is no longer applicable. Hence, we must use the method of steepest descent. Since $\mathrm{e}^{t\phi(k)}$ grows for $\operatorname{Im} k < 0$, one cannot use the point k_-. When considering the simple saddle point k_+, upon using the fact that $\phi''(k_+) = -6|k_+|$ one has that the steepest descent directions are given by $\vartheta = 0, \pi$. Appealing to equation (11.14) yields that to leading order,

$$u(x,t) \sim \frac{1}{2\sqrt{\pi|k_+|}}\hat{u}_0(k_+)\frac{\mathrm{e}^{-2|k_+|^3 t}}{t^{1/2}}. \tag{11.20}$$

Note that for fixed a one has that $u(x,t) = \mathcal{O}(t^{-1/2}\mathrm{e}^{-2|k_+|^3 t})$, i.e., the decay is exponentially fast.

Finally, consider the case that $|x/t| < \epsilon$, so that the above asymptotics are no longer valid. Define the similarity variables

$$\xi := k(3t)^{1/3}, \quad \eta := \frac{x}{(3t)^{1/3}},$$

so that the solution can be rewritten as

$$u(x,t) = \frac{1}{2\pi(3t)^{1/3}} \int_{-\infty}^{+\infty} \hat{u}_0\left(\frac{\xi}{(3t)^{1/3}}\right) e^{i(\eta\xi+\xi^3/3)}\,d\xi.$$

For large t one then has that

$$\begin{aligned} u(x,t) &\sim \frac{\hat{u}_0(0)}{2\pi(3t)^{1/3}} \int_{-\infty}^{+\infty} e^{i(\eta\xi+\xi^3/3)}\,d\xi \\ &= \frac{\hat{u}_0(0)}{(3t)^{1/3}} \mathrm{Ai}(\eta), \end{aligned} \tag{11.21}$$

where $\mathrm{Ai}(\eta)$ is the integral representation of the *Airy function*, which is the solution of the differential equation

$$\frac{d^2 v}{d\eta^2} - \eta v = 0; \quad v(0) = \frac{3^{-2/3}}{\Gamma(2/3)}, \; v'(0) = -\frac{3^{-4/3}}{\Gamma(4/3)}. \tag{11.22}$$

The relevant asymptotic formulas are

$$\mathrm{Ai}(\eta) \sim \begin{cases} \frac{1}{\sqrt{\pi}}\eta^{-1/4}\sin(2|\eta|^{3/2}/3+\pi/4), & \eta \to -\infty \\ \frac{1}{2\sqrt{\pi}}\eta^{-1/4}e^{-2\eta^{3/2}/3}, & \eta \to +\infty \end{cases}$$

[**1**, Section 6.7]. Note that the solution decays exponentially fast as $\eta \to +\infty$, which matches with the result of equation (11.20), and oscillates and decays as $\eta \to -\infty$, which matches with the result of equation (11.19).

Remark 11.14. If one substitutes

$$u(x,t) = (3t)^{-1/3}A(\eta)$$

into equation (11.18), then one finds that

$$\frac{d}{d\eta}(A_{\eta\eta} - \eta A) = 0.$$

The boundary condition $A(\eta) \to 0$ as $\eta \to +\infty$ then yields equation (11.22). Hence, it is not unexpected that the Airy function plays a role in the solution of equation (11.18).

For the final example, consider a variant of the Klein-Gordon equation,

$$c^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} - b^2 u = \delta(x)e^{-i\omega_0 t}, \quad u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0,$$

where $u(x,t) \equiv 0$ for $t < 0$ (also considered in Section 11.2). It is assumed that there is no forcing for $t \le 0$. The solution asymptotics will now be determined without explicitly determining the Green's function. It will be assumed that $\omega_0 > b$. This condition ensures that under the change of variables $v := ue^{i\omega_0 t}$, the time-independent solutions $v(x)$ will be oscillatory in space. Many of the intermediate details in the subsequent calculations can be found in [**2**, Chapter 7.5]. Before continuing, the following preliminary result is necessary.

Proposition 11.15. *For $\omega_0 \in \mathbb{R}^+$ consider*

$$f(t) := \begin{cases} e^{-i\omega_0 t}, & t \ge 0 \\ 0, & t < 0. \end{cases}$$

The Fourier transform is given by $\hat{f}(k) = i/(k-\omega_0)$.

Proof: For each $\epsilon > 0$ set $f_\epsilon(t) := e^{-\epsilon t}f(t)$. It is clear that $f_\epsilon(t) \to f(t)$ pointwise (but not uniformly), and that $f_\epsilon \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for each $\epsilon > 0$. A straightforward calculation yields that

$$\hat{f}_\epsilon(k) = \frac{i}{k-\omega_0+i\epsilon}.$$

The desired result is achieved upon letting $\epsilon \to 0^+$. □

Remark 11.16. One has that:

(a) It is interesting to note that for $f(t) = \mathrm{e}^{-\mathrm{i}\omega_0 t}$, $t \in \mathbb{R}$, one has that the Fourier transform is given by $\hat{f}(k) = \delta(k - \omega_0)$. This is seen upon the application of distribution theory.

(b) One may be troubled by the fact that in the above proof $f_\epsilon \not\to f$ uniformly, although for $|k - \omega| \geq \delta$, $0 < \delta \ll 1$, one has that the Fourier transform does converge uniformly to its limit. The difficulty is that $f \notin L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. However, it is straightforward to check that the above proof yields a valid way of defining the Fourier transform for those functions $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Hence, one can then be comfortable in using it above.

(c) The above approach is similar to one in which one first defines the Fourier transform for $\operatorname{Im} k > 0$, and then takes the limit $\operatorname{Im} k \to 0^+$.

Upon using the Fourier transform with respect to time, and using Proposition 11.15, one sees that $\hat{u}(x, k)$ satisfies

$$c^2 \frac{\partial^2 \hat{u}}{\partial x^2} + (k^2 - b^2)\hat{u} = \mathrm{i}\frac{\delta(x)}{k - \omega_0}.$$

Note that in Section 11.2 the transform was taken with respect to space. This ODE has the solution

$$\hat{u}(x, k) = \frac{\mathrm{e}^{\mathrm{i}(k^2-b^2)^{1/2}|x|/c}}{2c(k - \omega_0)(k^2 - b^2)^{1/2}},$$

where the branch cuts for $(k^2 - b^2)^{1/2}$ are taken vertically downward from the points $k = \pm b$. Furthermore, $(k^2 - b^2)^{1/2} \in \mathbb{R}^+$ for $k > b$, while $(k^2 - b^2)^{1/2} \in \mathbb{R}^-$ for $k < -b$. Thus, the solution satisfies the "outgoing" radiation condition. Upon inversion one has that

$$u(x, t) = \frac{1}{4c\pi} \oint_C \frac{\mathrm{e}^{\mathrm{i}[(k^2-b^2)^{1/2}|x|/c - kt]}}{(k - \omega_0)(k^2 - b^2)^{1/2}}\, \mathrm{d}k,$$

where C is the contour which passes above the three singularities (see Figure 19).

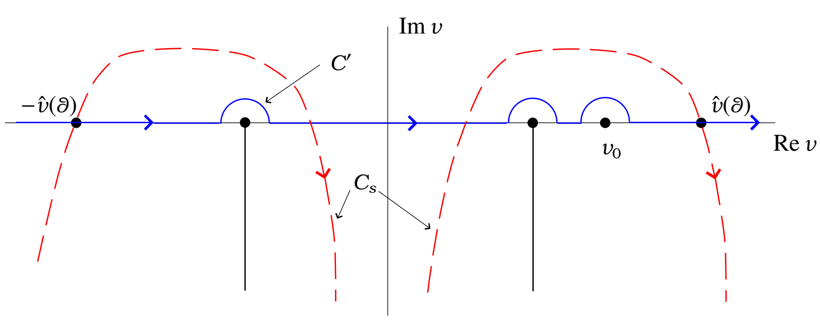


Figure 19: The path C' is the path of integration for equation (11.23), whereas C_s is the steepest descent path. The other parameters are labelled in the text. Note that in this figure $\nu_0 < \hat{\nu}(\partial)$.

The goal is to determine the asymptotics as $t \to +\infty$. In order to facilitate this calculation, set

$$\hbar := bt, \quad \nu := \frac{k}{b}, \quad \partial := \frac{|x|}{ct}.$$

Upon setting $U(\hbar; \partial) := u(x, t)$, one sees that

$$U(\hbar; \partial) = \frac{1}{4bc\pi} \oint_{C'} \frac{\mathrm{e}^{\hbar\varphi(\nu;\partial)}}{(\nu - \nu_0)(\nu^2 - 1)^{1/2}}\, \mathrm{d}\nu, \tag{11.23}$$

where C' is simply a rescaled version of C and

$$\phi(\nu;\partial) := \mathrm{i}(\partial(\nu^2-1)^{1/2}-\nu), \quad \nu_0 := \frac{\omega_0}{b}.$$

Note that it is here that the nonresonant condition $\omega_0 > b$ is crucial, for if it were to fail the pole singularity would coincide with the branch point.

In order to determine the asymptotics as $\jmath \to +\infty$, one must use the method of steepest descents. First,

$$\frac{\partial}{\partial\nu}\phi(\nu;\partial) = \mathrm{i}\left(\frac{\partial\nu}{(\nu^2-1)^{1/2}}-1\right), \quad \frac{\partial^2}{\partial\nu^2}\phi(\nu;\partial) = -\mathrm{i}\frac{\partial}{(\nu^2-1)^{3/2}},$$

so that a simple saddle point exists at the points

$$\nu = \pm\hat{\nu}(\partial), \quad \hat{\nu}(\partial) := (1-\partial^2)^{-1/2}.$$

Note that $\hat{\nu}(\partial) \to +\infty$ as $\partial \to 1^-$, and that at $\partial = 0$ the branch points coincide with the saddle points. Hence, in these two limits one anticipates complications. This is covered in detail in [**2**, Chapter 9].

First assume that $\partial > 1$, so that $\hat{\nu}(\partial) \notin \mathbb{R}$. Upon using the fact that $(\nu^2-1)^{1/2} = \nu(1+\mathcal{O}(|\nu|^{-1})$ for $|\nu| \gg 1$, upon closing the contour of integration and using Cauchy's theorem, and finally upon using the result of Lemma 4.36, one can show that

$$\frac{1}{4bc\pi}\,\mathrm{P.V.}\oint_{C'}\frac{\mathrm{e}^{\jmath\phi(\nu;\partial)}}{(\nu-\nu_0)(\nu^2-1)^{1/2}}\,\mathrm{d}\nu = \frac{\mathrm{i}}{4bc}\frac{\mathrm{e}^{\mathrm{i}(\partial(\nu_0^2-1)^{1/2}-\nu_0)\jmath}}{(\nu_0^2-1)^{1/2}}.$$

The details leading to the result are left for the interested student. Hence, if one defines $U(\jmath;\partial)$ via the principal value integral, then $U(\jmath;\partial) = \mathcal{O}(1)$ for $\partial > 1$. This is equivalent to $u(x,t) = \mathcal{O}(1)$ for $|x| > ct$, which is not physical. Hence, one cannot define $U(\jmath;\partial)$ via the principal value integral. Note that this argument also forces the curve to go above the pole $\nu = \nu_0$. If one does not use the principal value integral, then for $\partial > 1$ one has that $U(\jmath;\partial) \equiv 0$, i.e., $u(x,t) \equiv 0$ for $|x| > ct$. This result is simply a reflection of the fact that signals travel at the characteristic speed c, and no faster.

Now assume that $0 < \partial < 1$. At the saddle points one has

$$\phi(\pm\hat{\nu}(\partial)) = \mp\mathrm{i}(1-\partial^2)^{1/2}, \quad \phi''(\pm\hat{\nu}(\partial)) = \mp\mathrm{i}\frac{(1-\partial^2)^{3/2}}{\partial^2}.$$

Upon using equation (11.12) one sees that at $\hat{\nu}(\partial)$ the steepest descent directions are $-\pi/4,\ 3\pi/4$, while at $\hat{\nu}(\partial)$ the steepest descent directions are $\pi/4,\ -3\pi/4$. In order to obtain further qualitative information about the paths of steepest descent, first note that for $|\nu| \gg 1$, $\phi(\nu;\partial) = -\mathrm{i}(1\mp\partial)\nu+\mathcal{O}(|\nu|^{-1})$. Here the minus sign is taken for $|\mathrm{Re}\,\nu| > 1$, whereas the plus sign is taken for $|\mathrm{Re}\,\nu| < 1$. In either case, one sees that $\mathrm{Im}\,\nu \to -\infty$ on the path of steepest descent. Finally, since $\mathrm{Re}(\pm\hat{\nu}(\partial)) = 0$, one has that $\mathrm{Re}(\phi) < 0$ on all descent paths away from the saddle points. Thus, the steepest descent paths are as depicted in Figure 19. While it is not needed, it can be shown that the vertical asymptotes are given by $\mathrm{Re}\,\nu = \pm\sqrt{(1+\partial)/(1-\partial)}$ and $\mathrm{Re}\,\nu = \pm\sqrt{(1-\partial)/(1+\partial)}$.

If $\nu_0 < \hat{\nu}(\partial)$, i.e., $\partial > (\nu_0^2-1)^{1/2}/\nu_0$, then the deformation of C' to a curve passing through the saddle points does not pass through a pole of the integrand. Hence, upon applying Cauchy's theorem and adding up the leading order contributions from each saddle point via equation (11.14), one eventually sees that to leading order,

$$U(\jmath;\partial) \sim \frac{1}{2bc\sqrt{2\pi}(1-\partial^2)^{1/4}}\left(\frac{\mathrm{e}^{\mathrm{i}((1-\partial^2)^{1/2}\jmath+\pi/4)}}{\nu_0+(1-\partial^2)^{-1/2}}-\frac{\mathrm{e}^{-\mathrm{i}((1-\partial^2)^{1/2}\jmath+\pi/4)}}{\nu_0-(1-\partial^2)^{-1/2}}\right)\jmath^{-1/2}. \tag{11.24}$$

If $\nu_0 > \hat{\nu}(\partial)$, i.e., $\partial < (\nu_0^2-1)^{1/2}/\nu_0$, then the deformation of C' to a curve passing through the saddle points passes through the simple pole of the integrand. Upon applying the residue theorem one sees that to leading order,

$$U(\jmath;\partial) \sim -\frac{\mathrm{i}}{2bc}\frac{\mathrm{e}^{\mathrm{i}(\partial(\nu_0^2-1)^{1/2}-\nu_0)\jmath}}{(\nu_0^2-1)^{1/2}}. \tag{11.25}$$

Remark 11.17. If $\vartheta = (v_0^2 - 1)^{1/2}/v_0$, then the saddle point and pole coincide. One can again use equation (11.14) to cover this case. The revised asymptotic formula is given in [**2**, equation (7.5.24)].

Now consider the physical significance of the above results. If $\vartheta < (v_0^2-1)^{1/2}/v_0$, i.e., $|x| < (v_0^2-1)^{1/2}ct/v_0$, then the signal is given by equation (11.25) and is of $O(1)$. It oscillates at the source frequency v_0. Thus, one must move at a sufficiently slow speed in order to "see" a sustained signal from the source. Outside of this region, i.e., for $(v_0^2 - 1)^{1/2}ct/v_0 < |x| < ct$, the wave is given by equation (11.24) and is of $O(t^{-1/2})$. Hence, the observer does not see the main signal, but rather an algebraically damped oscillatory wave.

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