Math 355 Homework Problems #9

MATRIX ANALYSIS AND APPLIED LINEAR ALGEBRA, by C. Meyer

1. Let $\boldsymbol{A} = \begin{pmatrix} 0 & 2 \\ 6 & 1 \end{pmatrix}$.

(a) Find the spectral decomposition of $\boldsymbol{A}$.

(b) Find the spectral decomposition of $\boldsymbol{A}^{-1}$.

(c) Find the spectral decomposition of $\boldsymbol{A}^3 - 5\boldsymbol{A}^2 + 6\boldsymbol{I}_2$.

(d) Find the spectral decomposition of $e^{\boldsymbol{A}t}$.

2. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 3 \\ -2 & -6 \end{pmatrix}$.

(a) Compute the Drazin inverse, $\boldsymbol{A}^{\mathrm{D}}$.

(b) Use the Drazin inverse to find all solutions to $\boldsymbol{A}\boldsymbol{x} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$.

3. Let $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{F})$ be semi-simple with $r = \operatorname{rank}(\boldsymbol{A})$. Show that:

(a) $\mathrm{m_a}(0) = \mathrm{m_g}(0) = n - r$

(b) there are r nonzero eigenvalues (counting multiplicity)

(c) $\operatorname{Col}(\boldsymbol{A}) = \operatorname{Span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_r\}$, where $\boldsymbol{A}\boldsymbol{v}_j = \lambda_j \boldsymbol{v}_j$ for $\lambda_j \neq 0$

(d) $\mathbb{F}^n = \operatorname{Col}(\boldsymbol{A}) \oplus \operatorname{Null}(\boldsymbol{A})$.

4. Let $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{F})$ be semi-simple. Prove the following properties of the Drazin inverse, $\boldsymbol{A}^{\mathrm{D}}$:

(a) $(\boldsymbol{A}^{\mathrm{D}})^{\mathrm{D}} = \boldsymbol{A}^2\boldsymbol{A}^{\mathrm{D}} = \boldsymbol{A}$.

(b) $(\boldsymbol{A}^{\mathrm{D}})^n = (\boldsymbol{A}^n)^{\mathrm{D}}$ for any positive integer n

(c) $\boldsymbol{A}^{\mathrm{D}} = \boldsymbol{A}$ if and only if $\boldsymbol{A}^3 = \boldsymbol{A}$.

5. Let $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{F})$ be semi-simple. For each eigenvalue λ_j let $\boldsymbol{v}_j$ be an associated eigenvector, let $\boldsymbol{w}_j$ be an associated adjoint eigenvector, and suppose,

$$\langle \boldsymbol{w}_j, \boldsymbol{v}_k \rangle = \begin{cases} 0, & j \neq k \\ 1, & j = k. \end{cases}$$

For each j set $\boldsymbol{P}_j = \boldsymbol{v}_j \boldsymbol{w}_j^{\mathrm{H}}$ to be the rank one spectral projection matrix. For a given $1 \leq \ell \leq n$ set

$$\boldsymbol{Q} = \boldsymbol{P}_1 + \boldsymbol{P}_2 + \cdots + \boldsymbol{P}_\ell.$$

(a) Show that $\boldsymbol{Q}^2 = \boldsymbol{Q}$.

(b) Show that $\boldsymbol{Q}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{Q}$.

(c) Show that $\boldsymbol{Q}\boldsymbol{A}^{\mathrm{D}} = \boldsymbol{A}^{\mathrm{D}}\boldsymbol{Q}$.

(d) Find a basis for $\mathrm{Col}(\boldsymbol{Q})$ using the eigenvectors.

(e) Find a basis for $\mathrm{Null}(\boldsymbol{Q})$ using the eigenvectors.

6. In Problem **5** suppose that $\ell = \mathrm{rank}(\boldsymbol{A})$. Show that $\boldsymbol{A}\boldsymbol{A}^{\mathrm{D}} = \boldsymbol{A}^{\mathrm{D}}\boldsymbol{A} = \boldsymbol{Q}$.