Math 355 Homework Problems #8

MATRIX ANALYSIS AND APPLIED LINEAR ALGEBRA, by C. Meyer

1. For the data in each of the the files, "`LeastSquaresDataLinear.mat`" and "`LeastSquaresDataQuadratic.mat`", you are to do the following:

(a) compute the singular values

(b) determine the relative error associated with approximating the data with only the first term in the outer expansion

(c) use least-squares (line for "`...Linear.mat`", and quadratic for "`...Quadratic.mat`") to fit the data associated with the first term in the outer expansion - make sure you explicitly write down the coefficients

(d) plot the least-squares curve for the data associated with the first term in the outer expansion, along with the original data, on one plot.

2. Let $\boldsymbol{A} \in \mathcal{M}_{m\times n}(\mathbb{F})$, and suppose $\boldsymbol{A}$ has full rank. Show that the solution using the Moore-Penrose pseudo-inverse, $\boldsymbol{x} = \boldsymbol{A}^{\dagger}\boldsymbol{b}$, to the linear system, $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, is the solution to the associated normal equations.

3. Consider the matrix $\boldsymbol{A} = \begin{pmatrix} -4 & -2 & -4 & -2 \\ 2 & -2 & 2 & 1 \\ -4 & 1 & -4 & -2 \end{pmatrix}$.

(a) What is $r = \text{rank}(\boldsymbol{A})$?

(b) Compute the SVD, $\boldsymbol{A} = \boldsymbol{U}\Sigma\boldsymbol{V}^{\text{H}}$, where

$$\boldsymbol{U} \in \mathcal{M}_{3\times r}(\mathbb{F}), \quad \Sigma \in \mathcal{M}_r(\mathbb{F}), \quad \boldsymbol{V} \in \mathcal{M}_{4\times r}(\mathbb{F}).$$

(c) Using the SVD of $\boldsymbol{A}$, provide an orthonormal basis for $\text{Col}(\boldsymbol{A})$.

(d) Using the SVD of $\boldsymbol{A}$, provide an orthonormal basis for $\text{Col}(\boldsymbol{A}^{\text{H}})$.

(e) Compute the Moore-Penrose pseudo-inverse, $\boldsymbol{A}^{\dagger} \in \mathcal{M}_{4\times 3}(\mathbb{F})$.

4. Let $\boldsymbol{A} \in \mathcal{M}_{m\times n}(\mathbb{F})$ have the SVD outer product expansion,

$$\boldsymbol{A} = \sum_{j=1}^{r} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^{\text{H}}, \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0.$$

Show that the Moore-Penrose pseudo-inverse of $\boldsymbol{A}$ has the outer expansion,

$$\boldsymbol{A}^{\dagger} = \sum_{j=1}^{r} \frac{1}{\sigma_j} \boldsymbol{v}_j \boldsymbol{u}_j^{\text{H}}.$$

5. Let $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{F})$ be invertible and have the SVD outer product expansion,

$$\boldsymbol{A} = \sum_{j=1}^{n} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^{\mathrm{H}}, \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0.$$

For a given $s < n$ let $\boldsymbol{B}$ be a lower rank approximation,

$$\boldsymbol{B} = \sum_{j=1}^{s} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^{\mathrm{H}} \quad \rightsquigarrow \quad \boldsymbol{B}^{\dagger} = \sum_{j=1}^{s} \frac{1}{\sigma_j} \boldsymbol{v}_j \boldsymbol{u}_j^{\mathrm{H}}.$$

Using the norm induced by the standard inner product on $\mathbb{F}^n$, show that

$$\|\boldsymbol{A}^{-1}\boldsymbol{b} - \boldsymbol{B}^{\dagger}\boldsymbol{b}\| \geq \frac{1}{\sigma_{s+1}} \left(\sum_{j=s+1}^{n} |b_j|^2 \right)^{1/2}, \quad b_j = \langle \boldsymbol{u}_j, \boldsymbol{b} \rangle.$$

In other words, the exact solution, $\boldsymbol{A}^{-1}\boldsymbol{b}$, to the linear system is not necessarily close to the normal equations solution, $\boldsymbol{B}^{\dagger}\boldsymbol{b}$, of the approximate system. (*Hint*: use the Moore-Penrose pseudo-inverse for $\boldsymbol{A}^{-1}$)

6. Take your favorite digital image of minimal size 2 MB. Using the provided MATLAB program `DataCompressionColor.m` as a template, determine the size of your reduced image which has a relative error of 10% or less when compared to your original image. Provide supporting evidence for your conclusion. (*Hint*: use the MATLAB command `imwrite` to convert a data file to an image)