Math 355 Homework Problems #7

MATRIX ANALYSIS AND APPLIED LINEAR ALGEBRA, by C. Meyer

1. From the shared class Google Drive folder there are three files entitled `MatrixMarket*.mat`. You are to solve the linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ using the GMRES algorithm using the MATLAB command `gmres` for one of those three files. The data file you choose depends on the first letter of your last name:

- A-H: `MatrixMarket1.mat`
- I-Q: `MatrixMarket2.mat`
- S-Z: `MatrixMarket3.mat`

(a) How many iterates are required to achieve a relative error of 10^{-3}?

(b) How many iterates are required to achieve a relative error of 10^{-4}?

(c) How many iterates are required to achieve a relative error of 10^{-5}?

Provide a plot of the relative residual vs. the number of iterates using the command `semilogy`.

2. Let $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{F})$ be given.

(a) If $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$, show that $\boldsymbol{A}^\ell\boldsymbol{v} = \lambda^\ell\boldsymbol{v}$ for $\ell = 2, 3, \dots$.

(b) If $\boldsymbol{A}^M = \boldsymbol{0}_n$ for some $M \geq 2$ ($\boldsymbol{A}$ is nilpotent), show that $\text{trace}(\boldsymbol{A}) = 0$.

(c) If $\boldsymbol{A}$ is nilpotent, show that $\boldsymbol{A}$ is not invertible.

3. Find the algebraic and geometric multiplicities of all the eigenvalues for the matrix,

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

4. Suppose that $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{F})$. If $\lambda \in \sigma(\boldsymbol{A})$, show that:

(a) $a\lambda \in \sigma(a\boldsymbol{A})$

(b) $1 + a\lambda \in \sigma(\boldsymbol{I}_n + a\boldsymbol{A})$.

5. If $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}^n$, find:

(a) $\sigma(\boldsymbol{x}\boldsymbol{y}^{\mathrm{H}})$

(b) $\sigma(\boldsymbol{I}_n + \boldsymbol{x}\boldsymbol{y}^{\mathrm{H}})$

(c) $\det(\boldsymbol{I}_n + \boldsymbol{x}\boldsymbol{y}^{\mathrm{H}})$.

6. Suppose that $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{M}_n(\mathbb{F})$ are similar, i.e., $\boldsymbol{A} = \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P}$ for some invertible matrix $\boldsymbol{P}$. Show that for any integer k, $\boldsymbol{A}^k = \boldsymbol{P}^{-1}\boldsymbol{B}^k\boldsymbol{P}$.

7. Suppose that $\boldsymbol{A} \in \mathcal{M}_n(\mathbb{F})$ is semisimple with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

(a) Let

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{F}_n[x]$$

be any polynomial. Prove the *Semisimple Spectral Mapping Theorem*: the eigenvalues of

$$p(\boldsymbol{A}) = a_0 \boldsymbol{I}_n + a_1 \boldsymbol{A} + a_2 \boldsymbol{A}^2 + \cdots + a_n \boldsymbol{A}^n$$

are $\{p(\lambda_1), p(\lambda_2), \ldots, p(\lambda_n)\}$.

(b) Show that $p_{\boldsymbol{A}}(\boldsymbol{A}) = \boldsymbol{0}_n$, where $p_{\boldsymbol{A}}(\lambda)$ is the characteristic polynomial for the matrix $\boldsymbol{A}$.