Math 355 Homework Problems #6

MATRIX ANALYSIS AND APPLIED LINEAR ALGEBRA, by C. Meyer

1. Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{F}^n$, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^{\mathrm{H}}\boldsymbol{y}$. Let a norm on $\mathbb{F}^n$ induced by this inner product be denoted $\|\cdot\|_2$,

$$\|\boldsymbol{x}\|_2^2 = \langle \boldsymbol{x}, \boldsymbol{x} \rangle.$$

A matrix $\boldsymbol{Q} \in \mathcal{M}_n(\mathbb{F})$ is an orthonormal matrix if the columns of $\boldsymbol{Q}$ form an orthonormal basis for $\mathbb{F}^n$. Show that:

(a) $\boldsymbol{Q}^{\mathrm{H}}\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^{\mathrm{H}} = \boldsymbol{I}_n$

(b) $\|\boldsymbol{Q}\boldsymbol{x}\|_2 = \|\boldsymbol{x}\|_2$ (the linear transformation preserves norm)

(c) $\langle \boldsymbol{Q}\boldsymbol{x}, \boldsymbol{Q}\boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$ (the linear transformation preserves angle)

2. Let the inner product on $\mathbb{R}_5[x]$ be given by

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x.$$

(a) Find the length of x.

(b) Find the angle between x and x^5.

3. Let

$$S = \operatorname{Span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \right\} \subset \mathcal{M}_2(\mathbb{F}),$$

and let $\mathcal{M}_2(\mathbb{F})$ have the Frobenius inner product, $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \operatorname{trace}(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{B})$.

(a) Find an orthonormal basis for S.

(b) Find the orthogonal projection of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ onto S.

4. Consider the matrix,

$$\boldsymbol{A} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

(a) Find the QR-factorization.

(b) Find the orthogonal projection matrix, $\boldsymbol{P}_{\mathrm{Col}(\boldsymbol{A})}$.

5. Let the inner product on $\mathcal{M}_n(\mathbb{F})$ be the Frobenius inner product, $\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \operatorname{trace}(\boldsymbol{A}^{\mathrm{H}}\boldsymbol{B})$. Let $\boldsymbol{E}_{jk} = \boldsymbol{e}_j\boldsymbol{e}_k^{\mathrm{H}} \in \mathcal{M}_n(\mathbb{F})$ for $j, k = 1, \dots, n$ be the rank-one matrix with a one in the jk entry, and zero everywhere else. Show that:

(a) $\langle \boldsymbol{E}_{jk}, \boldsymbol{E}_{\ell m} \rangle = \begin{cases} 0, & (\ell, m) \neq (j, k) \\ 1, & (\ell, m) = (j, k) \end{cases}$

(b) $\{\boldsymbol{E}_{jk}\}$ is an orthonormal basis for $\mathcal{M}_n(\mathbb{F})$.